On differentiability properties of Lipschitz functions on a Banach space with a Lipschitz uniformly Gâteaux differentiable bump function

L. Zajíček

Abstract. We improve a theorem of P.G. Georgiev and N.P. Zlateva on Gâteaux differentiability of Lipschitz functions in a Banach space which admits a Lipschitz uniformly Gâteaux differentiable bump function. In particular, our result implies the following theorem: If d is a distance function determined by a closed subset A of a Banach space X with a uniformly Gâteaux differentiable norm, then the set of points of $X \setminus A$ at which d is not Gâteaux differentiable is not only a first category set, but it is even σ-porous in a rather strong sense.

Keywords: Lipschitz function, Gâteaux differentiability, uniformly Gâteaux differentiable, bump function, Banach-Mazur game, σ-porous set

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1. Introduction

In [8] I formulated without a proof a theorem (Theorem 4) which asserts that if a Banach space X admits a Lipschitz bump function which is uniformly differentiable in each direction, then each Lipschitz function of a certain type is Gâteaux differentiable at all points of a residual set. As an easy consequence of this theorem the following result (Corollary 3 of [8]) was stated.

Theorem A. *Let X be a Banach space with a uniformly Gâteaux differentiable norm. Then, for an arbitrary closed set A, the distance function $d(x) = dist(x, A)$ is Gâteaux differentiable at each point of a residual subset of X.*

Unfortunately, when after some time a sketch of the proof of the first mentioned theorem (Theorem 4 of [8]) was written down, it appeared that it contains a gap.

However, Theorem A was obtained by P. Georgiev (see the last note in [3] and [5]). Moreover, P. Georgiev has proved [4] a result (which also implies Theorem A) on differentiability properties of general Lipschitz functions on a Banach space X which admits a uniformly Gâteaux differentiable norm. Namely, he proved that any such space X is a Λ-space (in the terminology of [12], see Definition 1 below). A similar result was obtained in [6] also under a slightly weaker assumption that X admits a Lipschitz uniformly Gâteaux differentiable bump function. (Note that

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the main result of the preprint [13] by Wee-Kee Tang says that the above "slightly weaker assumption" is in fact an equivalent one.)

Recently I have observed that the gap in my original proof can be filled and that this modified proof gives also the mentioned results of [4] and [6]. In the present article this modified proof is given. There are two reasons for it:

(a) The proof is simpler and more elementary than these of [4] and [6]; it uses no smooth variational principle but instead of it one simple lemma (Lemma 1 below).

(b) Our proof gives also, via a recent result of M. Zelený [11] on a modification of the Banach-Mazur game, an improvement of results of [4] and [6]. Namely, it gives that the corresponding exceptional set is not only of the first category, but it is small in a more restrictive sense — it is σ-globally very porous.

To formulate the result precisely, we need some definitions. The definition of a Λ-space in [12] and [2] is based on a notion of a "subgradient". To distinguish this (very weak) notion of subgradient from others, we will use in the article the name (WD)-subgradient (weak Dini subgradient).

Definition 1. (i) Let X be a Banach space and let f be a locally Lipschitz function on X. We shall say that $x^* \in X^*$ is a (WD)-subgradient of f at $x \in X$ if

$$D_v^+ f(x) := \overline{\lim}_{h \to 0+} \frac{f(x+hv) - f(x)}{h} \geq (v, x^*) \ \text{ for every } \ v \in X.$$

(ii) A Banach space X is said to be a Λ-space, if each Lipschitz function f on X has a (WD)-subgradient at each point x of a residual subset of X.

Remark 1. (a) Of course, each (WD)-subgradient lies in the Clarke's subdifferential $\partial f(x)$.

(b) Let f be a Lipschitz function on X which has all one-sided directional derivatives at a point $x \in X$. Suppose further that both f and $-f$ have a (WD)-subgradient at x. Then it is not difficult to prove that f is Gâteaux differentiable at x. (It is clearly sufficient to suppose only that f has a (WD)-subgradient if we know that f has all (two-sided) directional derivatives at x.)

Definition 2. Let P be a metric space and $M \subset P$. We say that

(i) M is globally very porous if there exists $c > 0$ such that for every open ball $B(a,r)$ there exists an open ball $B(b,cr) \subset B(a,r) \setminus M$ and

(ii) M is σ-globally very porous if it is a countable union of globally very porous sets.

Remark 2. Each globally very porous set is clearly nowhere dense and each σ-globally very porous set is clearly of the first category. It is not difficult to prove that in each Banach space there exists a first category set which is not σ-globally very porous. (Corresponding more difficult results concerning the weaker notion of a σ-porous set are proved in [10] in the case of a Banach space and stated in [9] in the case of an arbitrary topologically complete space without isolated points.)

Definition 3. (i) Let X be a Banach space and $\|.\|$ be a norm on X. We say that $\|.\|$ is a uniformly Gâteaux differentiable norm (a UG-differentiable norm) if, for each $v \in X$, $\|v\| = 1$, the limit

$$\lim_{t \to 0} \frac{\|x + th\| - \|x\|}{t}$$

exists and is uniform on $\{x \in X : \|x\| = 1\}$.

(ii) Let X be a Banach space and let f be a real function on X. We say that f is a uniformly Gâteaux differentiable (UG-differentiable) bump function if f is a nonzero Gâteaux differentiable function with a bounded support and if, for each $v \in X$, $\|v\| = 1$, the limit

$$\lim_{t \to 0} \frac{f(x + tv) - f(x)}{t}$$

is uniform on X.

Now we can formulate our main result.

Theorem 1. *Let X be a Banach space which admits a Lipschitz UG-differentiable bump function and let f be a real Lipschitz function on X. Then f is (WD)-differentiable at all points of X except those which belong to a σ-globally very porous set.*

Remark 3. (a) It is well known and easy to prove that if a Banach space admits an equivalent uniformly Gâteaux differentiable norm then it admits a Lipschitz UG-differentiable bump function. By [13], the converse implication is also true.

(b) Some facts about spaces which admit a UG-differentiable norm can be found in [1].

An easy consequence of Theorem 1 is the following result which improves Theorem A.

Theorem 2. *Let X be a Banach space with a uniformly Gâteaux differentiable norm. Then, for an arbitrary closed set A, the distance function $d(x) = dist(x, A)$ is Gâteaux differentiable at all points of $X \setminus A$ except those which belong to a σ-globally very porous set.*

It is well-known (cf. e.g. [7, Proposition 2]) that, in a strictly convex Banach space X, the fact that the distance function $dist(x, A)$ is Gâteaux differentiable at x implies that the metric projection

$$P_A(x) := \{y \in A : \|x - y\| = dist(x, A)\}$$

is not multivalued (i.e., it is an empty set or a singleton). Consequently Theorem 2 immediately implies the following result.

Corollary 1. *Let X be a Banach space with a norm which is simultaneously strictly convex and UG-differentiable and let $A \subset X$ be a closed set. Then the set of points $x \in X$ at which the metric projection $P_A(x)$ is multivalued is σ-globally very porous.*

Now we shall describe the mentioned result of M. Zelený which gives a characterization of σ-globally very porous sets in a Banach space X based on a modification of the Banach-Mazur game. We shall call this game GVP-game here (GVP is for "globally very porous"); in [11] another terminology is used.

Two players play the GVP-game corresponding to a set $M \subset X$ and a sequence of positive numbers $(c_n)_1^\infty$ as follows:

In his first move the first player chooses an open ball $U_1 = B(x_1, \rho_1)$, then the second player chooses a ball $V_1 = B(y_1, r_1) \subset U_1$, the first player chooses a ball $U_2 = B(x_2, \rho_2) \subset V_1$ and so on. The second player wins if

$$\bigcap_{n=1}^{\infty} V_n \cap M = \emptyset \quad \text{and}$$
$$r_n > c_n \rho_n \quad \text{for each positive integer } n.$$

M. Zelený [11, Corollary of Theorem 2] has proved the following result.

Theorem Z. *A subset M of a Banach space X is σ-globally very porous if and only if there exists a sequence of positive numbers $(c_n)_1^\infty$ such that the second player has a winning strategy in the GVP-game corresponding to M and $(c_n)_1^\infty$.*

2. Lemmas

In the following, $B(x, r)$ and $\overline{B}(x, r)$ are open and closed balls with center x and radius r, respectively. If h is a real function on a Banach space X, then $h'(x, v) := \lim_{t \to 0} \frac{h(x+tv)-h(x)}{t}$ is the two-sided derivative of h at x in the direction v. We say that f is an L-Lipschitz function, if f is a Lipschitz function with Lipschitz constant L.

Lemma 1. *Let h be a L-Lipschitz function defined on a Banach space X such that $h(0) = p > 0$ and h vanishes on $X \setminus B(0, 1)$. Suppose that $a \in X$ and $\tau > 0$ are given; put*

$$h^*(x) = h_{a,\tau}(x) = \tau h\Big(\frac{x-a}{\tau}\Big).$$

Further suppose that $K < p$ and a K-Lipschitz function f on $\overline{B}(a, \tau)$ are given; denote

$$c = \frac{p-K}{2L}\,. \tag{1}$$

Then for each $\delta > 0$ there exist a real number y and $z \in B(a,\tau)$ such that

$$h^*(x) + y \leq f(x) \quad \text{for each} \quad x \in \overline{B}(a,\tau), \tag{2}$$

$$f(z) < h^*(z) + y + \delta \quad \text{and} \tag{3}$$

$$B(z, c\tau) \subset B(a,\tau). \tag{4}$$

Proof: At first we observe that h^* is also L-Lipschitz since

$$|h^*(x) - h^*(y)| \leq \tau L \|\frac{x-a}{\tau} - \frac{y-a}{\tau}\| = L\|x-y\|.$$

Now suppose that $\delta > 0$ is given; we can suppose that

$$\delta < \frac{(p-K)\tau}{2}.$$

Since both h^* and f are bounded on $\overline{B}(a,\tau)$, we can put $y := \inf\{f(x) - h^*(x) : x \in \overline{B}(a,\tau)\}$; we see that the condition (2) is satisfied. Obviously there exists $z \in \overline{B}(a,\tau)$ such that (3) holds. To prove (4), suppose on the contrary that there exists a point $v \in B(z, c\tau) \setminus B(a,\tau)$. Then

$$\begin{gathered} \tau p = h^*(a) = h^*(a) - h^*(v) = (h^*(a) - h^*(z)) + (h^*(z) - h^*(v)) \leq \\ (f(a) - y) - (f(z) - y - \delta) + (h^*(z) - h^*(v)) \leq \\ |f(a) - f(z)| + \delta + |h^*(z) - h^*(v)| < \\ K\tau + \frac{(p-K)\tau}{2} + Lc\tau = \tau p, \end{gathered}$$

which is a contradiction. □

We will need also the following geometrically obvious lemma.

Lemma 2. *Let h and $h^* = h_{a,\tau}$ be as in Lemma 1. Further suppose that h is differentiable at all points in the direction $v \in X$. Let $\epsilon > 0$, $\delta > 0$ and*

$$|\frac{h(p+tv) - h(p)}{t} - h'(p,v)| < \varepsilon \quad \text{whenever} \quad p \in X \quad \text{and} \quad 0 < |t| \leq \delta.$$

Then h^* is also differentiable at all points in the direction v and

$$|\frac{h^*(q+sv) - h^*(q)}{s} - (h^*)'(q,v)| < \varepsilon \quad \text{whenever} \quad q \in X \quad \text{and} \quad 0 < |s| \leq \tau\delta.$$

3. Proofs of Theorems

Proof of Theorem 1: Suppose that f is K-Lipschitz and choose a $p > K$. Since X admits a uniformly Gâteaux differentiable Lipschitz bump function b it is easy to show that there exists $L > 0$ and a uniformly Gâteaux differentiable function h on X which meets the assumptions from Lemma 1 (we can easily find h in the form $h(x) = \alpha b(\beta x - y)$ for some real numbers α, β and $y \in X$). Define c by (1). Let M be the set of those points at which f is not (WD)-subdifferentiable. By Theorem Z it is sufficient to prove that the second player has a winning strategy in the the GVP-game corresponding to M and $(c_n)_1^\infty$, where $c_n = \frac{c}{2n^2}$. We shall show that the following strategy does the job:

Suppose the first player chose an open ball $U_n = B(a_n, \tau_n)$ in his n-th move. In our strategy we apply Lemma 1 to f, $a = a_n$, $\tau = \tau_n$, $\delta = \frac{c\tau_n}{n^2}$; choose corresponding $y = y_n$, $z = z_n$ and define $V_n := B(z_n, \frac{c\tau_n}{n^2})$ as the n-th move of the second player.

This is a winning strategy. In fact, suppose that a play at which the second player has used the above strategy is over and $x \in \bigcap_{n=1}^\infty V_n$. Let x_n^* be the Gâteaux derivative of h_{a_n,τ_n} at the point z_n. Since all h_{a_n,τ_n} are L-Lipschitz, $\|x_n^*\| \le L$ and the Alaoglu-Bourbaki theorem implies that we can choose an $x^* \in X^*$ which is a w^*-cluster point of the sequence (x_n^*). Now it is sufficient to show that x^* is a (WD)-subgradient of f at the point x.

To this end choose an arbitrary $v \in X$, $\|v\| = 1$, and put

$$t_n = cn^{-1}\tau_n.$$

Since clearly $t_n \to 0$, it is sufficient to prove that

$$\overline{\lim}_{n\to\infty} \frac{f(x + t_n v) - f(x)}{t_n} \ge (v, x^*). \tag{5}$$

To prove (5), choose arbitrarily $\varepsilon > 0$ and a natural number n_0. Now we can choose $n > n_0$ such that

$$\left|\frac{h(p + tv) - h(p)}{t} - h'(p, v)\right| < \varepsilon \quad \text{whenever} \quad p \in X \quad \text{and} \quad 0 < t \le \frac{c}{n}, \tag{6}$$

$$(2K + 1)n^{-1} < \varepsilon, \qquad \text{and} \tag{7}$$

$$|(v, x^*) - (v, x_n^*)| < \varepsilon. \tag{8}$$

Then, since f is K-Lipschitz and $x \in V_n$, we have

$$f(x + t_n v) - f(x) \ge f(z_n + t_n v) - f(z_n) - \frac{2Kc\tau_n}{n^2}. \tag{9}$$

The choice of z_n and t_n implies that $z_n + t_n v \in U_n$ (since $B(z_n, c\tau_n) \subset U_n$ by (4)) and (we use (2) and (3))

$$f(z_n + t_n v) - f(z_n) \ge h^*(z_n + t_n v) - h^*(z_n) - c\tau_n n^{-2}, \text{ where } h^* = h_{a_n,\tau_n}. \tag{10}$$

On account of Lemma 2 and (6) we obtain that

$$|(v, x_n^*) - \frac{h^*(z_n + t_n v) - h^*(z_n)}{t_n}| < \varepsilon. \tag{11}$$

Since $t_n = c\tau_n n^{-1}$, (9), (10), (11), (7) and (8) give

$$\begin{gathered}\frac{f(x + t_n v) - f(x)}{t_n} \geq \frac{f(z_n + t_n v) - f(z_n)}{t_n} - \frac{2K}{n} \geq \\ \frac{h^*(z_n + t_n v) - h^*(z_n)}{t_n} - n^{-1} - 2Kn^{-1} \\ \geq (v, x_n^*) - 2\varepsilon \geq (v, x^*) - 3\varepsilon.\end{gathered}$$

Thus we have proved (5) and the proof is complete. □

Proof of Theorem 2: By Theorem 3 of [7] the one-sided derivative $d'_+(x, v) = \lim_{h\to 0+} \frac{d(x+hv)-d(x)}{h}$ exists for all $x \in X \setminus A$ and $v \in X$. Since d is 1-Lipschitz on $X \setminus A$, it can be extended to a 1-Lipschitz function d^* on X. By Remark 3 (a) we can apply Theorem 1 to d^* and $-d^*$. Then we obtain, on account of Remark 1 (b), the statement of the theorem. □

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Department of Mathematical Analysis, Charles University, Sokolovská 83, 186 00 Praha 8, Czech Republic

E-mail: Zajicek@karlin.mff.cuni.cz

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