Sets of extended uniqueness and σ-porosity

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Abstract. We show that there exists a closed non-σ-porous set of extended uniqueness. We also give a new proof of Lyons' theorem, which shows that the class of $H^{(n)}$-sets is not large in U_0.

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Let us recall several basic notions. The symbol $\mathbb{T}$ stands for the interval $[0, 2\pi]$ with 0 and 2π identified. A complex Borel measure μ on $\mathbb{T}$ is said to be *Rajchman,* if $\lim_{|n|\to+\infty} |\hat{\mu}(n)| = 0$, where $\hat{\mu}(n) = \int e^{-inx}\,\mathrm{d}\mu$, $n \in \mathbb{Z}$. A set $P \subset \mathbb{T}$ is called a set of *extended uniqueness* if for every positive Rajchman measure μ we have $\mu(P) = 0$. We denote by U_0 the class of closed sets of extended uniqueness. We say that a class $\mathcal{B} \subset U_0$ is *large in* U_0 if complex Borel measure μ is Rajchman if and only if $\mu(P) = 0$ for every $P \in \mathcal{B}$. See [KL] for details.

Let (P, ρ) be a metric space. The open ball with the center $x \in P$ and the radius $r > 0$ is denoted by $B(x, r)$. Let $M \subset P$, $x \in P$, $R > 0$. Then we define

$$\gamma(x, R, M) = \sup\{r > 0;\ \text{for some } z \in P,\ B(z, r) \subset B(x, R) \setminus M\},$$
$$p(x, M) = \limsup_{R\to 0+} \frac{\gamma(x, R, M)}{R}.$$

A set $M \subset P$ is said to be *porous* if $p(x, M) > 0$ for every $x \in M$. A countable union of porous sets is called *σ-porous* set. The class of all closed σ-porous subsets of $\mathbb{T}$ is denoted by $\mathcal{P}_\sigma$.

The notion of σ-porosity was introduced by E.P. Dolzhenko ([D]) to describe certain class of exceptional sets, which appears in the study of boundary behaviour of complex functions. There are many other results describing sets of exceptional points in terms of σ-porous sets (cf. [Z_2]).

Each σ-porous subset of $\mathbb{R}$ is clearly meager. Using Lebesgue density theorem we can prove that each σ-porous set has Lebesgue measure zero. On the other hand there exists a meager non-σ-porous set with Lebesgue measure zero ([Z_1]). As for the sets of extended uniqueness, Borel ones have also Lebesgue measure zero

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and are meager. The first fact is well-known and the second one was obtained by Debs and Saint-Raymond ([DSR]) as a solution of a longstanding open problem.

Our main goal is to show that meagerness in Debs–Saint-Raymond's result cannot be replaced by σ-porosity.

We will give a new proof of Lyons' theorem concerning largeness of the class of all $H^{(n)}$-sets in U_0. (See [KL] for the definition of $H^{(n)}$-sets.) We will use the result from [Š], which shows that each $H^{(n)}$-set is σ-porous. This result is unfortunately unpublished, but there exists a manuscript in English. See also [Z$_3$].

We start with the following lemma.

Lemma. *There exists a Borel measure μ on $[0, 2\pi]$ such that*

(i) *μ is not Rajchman,*
(ii) *for every σ-porous set P we have $\mu(P) = 0$.*

We will need the following theorem to prove our Lemma.

Theorem A ([T]). *Let μ be a Borel measure on $S \subset \mathbb{R}$ fulfilling the following conditions:*

(i) *There exists $d > 1$ such that*

$$\sum_{\substack{I \text{ is bounded and} \\ \text{contiguous to } \overline{S}}} \mu(d \star I) < +\infty.$$

(ii) *There exist $c > 1$, $C > 0$ and $\delta > 0$ such that $\mu(c \star I) \leq C\mu(I)$ for every interval I with the length less than δ and with the center in S.*
(iii) *All countable sets are μ-null.*

Then $\mu(P) = 0$, whenever P is σ-porous subset of $\mathbb{R}$.

Proof of Lemma: We use a modification of the construction from [T]. Let R be a closed (open) bounded interval and $k > 0$. Then $k \star R$ denotes the closed (open) interval with the same center as R has and with k times greater length. Let $(k_n)_{n=1}^{+\infty}$ be an increasing sequence of natural numbers. We divide closed bounded interval R into 2^{k_n+2} many closed subintervals with the same length and with pairwise disjoint interiors. Let $\mathcal{R}_n(R)$ be the set of all intervals mentioned above without these intervals, which contain the center of the interval R. We define sets of closed intervals as follows:

$$\mathcal{R}_0 = \{[0, 2\pi]\}, \qquad \mathcal{R}_n = \bigcup \{\mathcal{R}_n(R);\ R \in \mathcal{R}_{n-1}\}.$$

We define inductively a function $\tau : \bigcup_{n=0}^{+\infty} \mathcal{R}_n \to [0, 1]$ such that $\tau([0, 2\pi]) = 1$ and for every $n \in \mathbb{N}$ and for all intervals $R \in \mathcal{R}_n$, $R' \in \mathcal{R}_{n-1}$ with $R \subset R'$ we put

$$\tau(R) = \begin{cases} \alpha\, 2^{-2k_n-1}\tau(R'), & \text{for } R \subset 2^{-k_n} \star R', \\ 3\alpha\, 2^{-k_n-k-2}\tau(R'), & \text{for } \operatorname{Int} R \subset 2^{-k+1} \star R' \setminus 2^{-k} \star R',\ k \in \{2, \dots, k_n\} \\ 3\beta\, 2^{-k_n-3}\tau(R'), & \text{for } \operatorname{Int} R \subset R' \setminus \frac{1}{2} \star R', \end{cases}$$

where $\alpha = \frac{4}{7}$ and $\beta = \frac{8}{7}$. Since

$$\sum_{R \in \mathcal{R}_n,\, R \subset R'} \tau(R) = \tau(R') \quad \text{for every} \quad R' \in \mathcal{R}_{n-1},$$

there exists Borel measure μ such that $\operatorname{supp} \mu \subset S = \bigcap_{n=0}^{+\infty} \bigcup \{R;\, R \in \mathcal{R}_n\}$ and $\mu(I) = \tau(I)$, whenever $I \in \bigcup_{n=0}^{+\infty} \mathcal{R}_n$.

Observe the following fact:

($\star$) for every $n \in \mathbb{N}$ and $K, L \in \mathcal{R}_n$, $\partial K \cap \partial L \neq \emptyset$ we have $\mu(K) \geq \frac{1}{4}\mu(L)$.

At first we show that $\mu(P) = 0$ for each σ-porous set P. It is sufficient to show that μ fulfills the conditions (i), (ii) and (iii) from Theorem A.

Ad (i): Putting $d = 2$ we obtain

$$\begin{aligned}\sum_{n=1}^{+\infty} \sum_{R \in \mathcal{R}_{n-1}} \mu(2 \star (2^{-k_n - 1} \star \operatorname{Int} R)) &= \sum_{n=1}^{+\infty} \sum_{R \in \mathcal{R}_{n-1}} \alpha 2^{-2k_n} \mu(R) \\ &= \sum_{n=1}^{+\infty} \alpha 2^{-2k_n} \leq \alpha \sum_{n=1}^{+\infty} 2^{-2n} < +\infty.\end{aligned}$$

Ad (ii): We will show that this condition is fulfilled for $c = 2$, $C = 148$ and $\delta = 4\pi$. Let J be an interval with the center $x \in S$ such that the length of J is less than 4π. Let $n \in \mathbb{N}$ be the smallest natural number such that there exists intervals $R' \in \mathcal{R}_n$, $R \in \mathcal{R}_{n-1}$ such that $x \in R' \subset J \cap R$. Let $Q = J \cap \bigcup_{k=0}^{k_n+1} \partial(2^{-k} \star R)$. We distinguish the two cases.

(1) The number of elements of Q is less or equal to 1. It implies that

$$2^{-k_n - 1} \star R \cap J = \emptyset. \tag{$\star\star$}$$

Let $K_1, \dots, K_p$ be these intervals from the set $\mathcal{R}_n$, which are contained in J and $L_1, \dots, L_q$ be these intervals from $\mathcal{R}_n$, which intersect the set $S \cap 2 \star J$. Thus we have $S \cap 2 \star J \subset \bigcup_{i=1}^{q} L_i$. Let $r_n \in \mathbb{R}$ be the length of the intervals from $\mathcal{R}_n$. The length of the interval J is at most $(p+2)r_n$. (We used the fact ($\star\star$).) It implies that the length of $2 \star J$ is at most $(2p+4)r_n$. Therefore $q \leq 2p + 5$. Now fix $j \in \{1, \dots, p\}$ and $i \in \{1, \dots, q\}$. We distinguish the following possibilities.

(a) Suppose that $L_i \subset R$. Let $x \in 2^{-l} \star R \setminus 2^{-l-1} \star R$, $l \in \mathbb{N} \cup \{0\}$. If $\operatorname{dist}(\operatorname{center}(R), L_i) \leq \operatorname{dist}(\operatorname{center}(R), K_j)$, then $\mu(K_j) \geq \mu(L_i)$. Suppose that $\operatorname{dist}(\operatorname{center}(R), L_i) > \operatorname{dist}(\operatorname{center}(R), K_j)$. We have $L_i \subset 2^{-l+1} \star R \cap R$ and $K_j \cap 2^{-l-2} \star R = \emptyset$. From the fact ($\star$) we obtain that $\mu(K_j) \geq \frac{1}{16}\mu(L_i)$.

(b) Suppose that $L_i \not\subset R$. Then there exists an interval $\widetilde{R} \in \mathcal{R}_{n-1}$ such that $\partial R \cap \partial \widetilde{R} \neq \emptyset$ and $L_i \subset \widetilde{R}$. From ($\star$) we have $\mu(R) \geq \frac{1}{4}\mu(\widetilde{R})$. Observing $K_j \cap \frac{1}{4} \star R = \emptyset$ we can conclude $\mu(K_j) \geq \frac{1}{16}\mu(L_i)$.

We have proved that

$$\min\{\mu(K_1),\dots,\mu(K_p)\} \geq \frac{1}{16}\max\{\mu(L_1),\dots,\mu(L_q)\}.$$

It gives

$$\frac{\mu(2\star J)}{\mu(J)} \leq \frac{16(2p+5)}{p} \leq 112.$$

(2) The number of elements of Q is greater or equal to 2. Let k be the smallest natural number from the set $\{1,2,\dots,k_n+1\}$ such that $J\cap\partial(2^{-k+1}\star R)\neq\emptyset$. Then we have

$$\mu(J) \geq \frac{1}{2}\mu(2^{-k+1}\star R\setminus 2^{-k}\star R).$$

We also have $2\star J\subset 2^{-k+3}\star R$ and therefore

$$\begin{aligned}\mu(2\star J) &\leq \mu(2^{-k+3}\star R\setminus 2^{-k+2}\star R)+\mu(2^{-k+2}\star R\setminus 2^{-k+1}\star R)\\ &\quad+\mu(2^{-k+1}\star R\setminus 2^{-k}\star R)+\mu(2^{-k}\star R)\\ &\leq (16\cdot 4+4\cdot 2+1+1)\mu(2^{-k+1}\star R\setminus 2^{-k}\star R).\end{aligned}$$

It gives that

$$\frac{\mu(2\star J)}{\mu(J)} \leq 148.$$

Ad (iii): This condition is clearly fulfilled.

Now we show that μ is not a Rajchman measure. Fix $n\in\mathbb{N}$. The intervals from $\mathcal{R}_n$ have the length r_n. The number $\frac{2\pi}{r_n}$ is clearly natural. We have

$$\begin{aligned}\int_0^{2\pi}\cos\frac{2\pi}{r_n}x\,\mathrm{d}\mu &= \sum_{R\in\mathcal{R}_n}\int_R\cos\frac{2\pi}{r_n}x\,\mathrm{d}\mu \geq \sum_{R\in\mathcal{R}_n}\left(\frac{1}{2}\mu(R\setminus\frac{3}{4}\star R)-\mu(\frac{1}{2}\star R)\right)\\ &= \sum_{R\in\mathcal{R}_n}\left(\frac{1}{2}3\beta 2^{-k_{n+1}-3}2^{k_{n+1}}\mu(R)-(\mu(R)-3\beta 2^{-k_{n+1}-3}2^{k_{n+1}+1}\mu(R))\right)\\ &= \sum_{R\in\mathcal{R}_n}\left(\frac{3\beta}{16}-(1-\frac{3\beta}{4})\right)\mu(R)=\frac{15\beta}{16}-1>0.\end{aligned}$$

It implies that μ is not Rajchman. □

The fundamental theorem concerning largeness in U_0 is due to Lyons and reads as follows.

Theorem ([L]). *The class U_0 is large in U_0.*

Now we are able to prove the main result of this paper.

Theorem. *There exists a closed non-σ-porous set of extended uniqueness.*

Proof: Suppose that $U_0 \subset \mathcal{P}_\sigma$. Then the measure μ from Lemma must be Rajchman according to the previous Theorem. This contradiction proves our Theorem. $\square$

Theorem ([L]). *The class $\bigcup_{n=1}^{+\infty} H^{(n)}$ is not large in U_0.*

Proof: According to [Š] we have $\bigcup_{n=1}^{+\infty} H^{(n)} \subset \mathcal{P}_\sigma$. We also have that $\bigcup_{n=1}^{+\infty} H^{(n)} \subset U_0$ (cf. [KL]). The class $\bigcup_{n=1}^{+\infty} H^{(n)}$ is not large since $\mathcal{P}_\sigma \cap U_0$ is not large as Lemma shows. $\square$

Remark The question, whether $\mathcal{P}_\sigma \subset U_0$, has the negative answer too (cf. [Z_2]). The Salem-Zygmund theorem gives that there exists a symmetric perfect set of constant ratio of dissection, which is not the set of extended uniqueness (cf. [KL]). But it is easy to see that this set is porous (cf. [Z_2]). This answers the question, which was posed in [BKR].

Remark Let us note that there exists also a closed non-σ-porous set of uniqueness, but the proof of this result is much more complicated than the proof for sets of extended uniqueness and uses a completely different method. The proof will appear in a subsequent paper.

References

[BKR] Bukovský L., Kholshchevnikova N.N., Repický M., *Thin sets of harmonic analysis and infinite combinatorics*, Real Analysis Exchange **20.2** (1994-95), 454–509.

[DSR] Debs G., Saint-Raymond J., *Ensembles boréliens d'unicité et d'unicité au sens large*, Ann. Inst. Fourier (Grenoble) **37** (1987), 217–239.

[D] Dolzhenko E.P., *Boundary properties of arbitrary functions* (in Russian), Izv. Akad. Nauk. SSSR Ser. Mat. **31** (1967), 3–14.

[KL] Kechris A., Louveau A., *Descriptive Set Theory and the Structure of Sets of Uniqueness*, Cambridge U. Press, Cambridge, 1987.

[L] Lyons R., *The size of some classes of thin sets*, Studia Math. **86.1** (1987), 59–78.

[T] Tkadlec J., *Construction of a finite Borel measure with σ-porous sets as null sets*, Real Analysis Exchange **12** (1986-87), 349–353.

[Š] Šleich P., *Sets of type $H^{(s)}$ are σ-bilaterally porous*, unpublished.

[Z_1] Zajíček L., *Sets of σ-porosity and sets of σ-porosity(q)*, Časopis Pěst. Mat. **101** (1976), 350–359.

[Z_2] Zajíček L., *Porosity and σ-porosity*, Real Analysis Exchange **13** (1987-88), 314–350.

[Z_3] Zajíček L., *A note on σ-porous sets*, Real Analysis Exchange **17.1** (1991-92), 18.

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