On the product of a compact space with an hereditarily absolutely countably compact space

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Abstract. We show that the product of a compact, sequential T_2 space with an hereditarily absolutely countably compact T_3 space is hereditarily absolutely countably compact, and further that the product of a compact T_2 space of countable tightness with an hereditarily absolutely countably compact ω-bounded T_3 space is hereditarily absolutely countably compact.

Keywords: compact, countably compact, absolutely countably compact, hereditarily absolutely countably compact, ω-bounded, countable tightness, sequential space

Classification: 54D20, 54B10, 54D55

Introduction and preliminary

Recently Matveev in [Mat1], [Mat2] introduced a new property called absolute countable compactness (acc) which is stronger than countable compactness. He also introduced the related property hereditarily absolutely countably compact (hacc). Matveev ([Mat1]) proved that the acc and hacc properties are not necessarily preserved by products with compact space (see [Mat1, Example 2.2 and Remark 5.3]). Matveev proved however that if Y is compact and first countable, and X is an acc or hacc T_2 space, then $X \times Y$ is acc or hacc (see [Mat1, Theorem 2.3 or 5.4]), and he raised the following two questions

Question 1 [Mat1]. *Is $X \times Y$ acc provided Y is a compact space with countable tightness and X is an acc space?*

Question 2 [Mat1]. *Is $X \times Y$ hacc provided Y is a compact space with countable tightness and X is an hacc space?*

Vaughan ([Vau2]) proved that the product of a compact sequential T_2 space with an acc T_3 space is acc. With the previous result and the following well known Balogh's theorem, Vaughan gave an affirmative answer to Question 1 in models of the proper forcing axiom [PFA] (Question 1 remains open in ZFC).

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Theorem 1 [Bal]**.** [PFA] *Every compact Hausdorff space of countable tightness is sequential.*

In this paper we augment Vaughan's proof ([Vau2]) to show that the product of a compact sequential T_2 space with an hacc T_3 space is hacc. Then, with this result and Balogh's theorem, we can give an affirmative answer to Question 2 provided we assume the proper forcing axiom (Question 2 remains open in ZFC).

We also consider further conditions under which the product of a compact space with an hacc space is hacc. Vaughan ([Vau2]) proved that the product of a compact T_2 space of countable tightness with an acc ω-bounded T_3 space is acc and that the product of a compact T_2 space of countable density-tightness (defined below) with an acc T_3 space of countable density-tightness is acc. In this paper we consider "analogs" of the previous results for hacc spaces, that is we prove that the product of a compact T_2 space of countable tightness with an hacc ω-bounded T_3 space is hacc and that the product of a compact T_2 space of countable tightness with an acc T_3 space of countable tightness is hacc. Further, we prove that the product of a compact T_2 space of countable density-tightness with an hacc T_3 space of countable density-tightness need not be hacc.

Concerning Question 2, we note that if in some model of set theory there is a counterexample $X \times Y$ to Question 2, then Y must be a compact non-sequential space with countable tightness, and X must be an hacc space which is not ω-bounded and does not have countable tightness or, equivalently, there exists a closed subset of X which does not have countable density-tightness.

Recall the following definitions:

Definition 1 [Eng]**.** *A space X is called* countably compact *provided every countable open cover of X has a finite subcover.*

Note that a characterization of countable compactness (see [Eng, 3.12.22(d)]) states that a T_2 space X is countable compact iff for every open cover $\mathcal{U}$ of X there exists a finite set $F \subset X$ such that $St(F,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap F \neq \varnothing\} = X$.

Definition 2 [Mat1]**.** *A space X is said to be* absolutely countably compact (acc) *provided for every open cover $\mathcal{U}$ of X and every dense $D \subset X$, there exists a finite set $F \subset D$ such that $St(F,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap F \neq \varnothing\} = X$.*

Matveev ([Mat1]) noted that every compact space is acc and every acc T_2 space is countably compact; he proved that every countably compact space with countable density-tightness (defined below) is acc. Further Vaughan ([Vau2]) proved that every countably compact, orthocompact space (defined below) is acc.

Matveev ([Mat1]) demonstrated that acc is not hereditary with respect to closed subsets, even with respect to regular closed subsets. Then he introduced the following definition.

Definition 3 [Mat1]**.** *A space X is said to be* hereditarily absolutely countably compact (hacc) *if all closed subspaces of X are acc.*

We also recall some other definitions. A space X has countable tightness provided that whenever $A \subset X$ and $x \in \overline{A}$ there exists a countable $C \subset A$ such that $x \in \overline{C}$. Using different terminology, Matveev introduced ([Mat1]) the notion of countable density-tightness.

Definition 4 [Vau2]. *The* density-tightness *of a space X, denoted $d_t(X)$, is the smallest infinite cardinal κ such that for every dense subset $D \subset X$ and every $x \in X$ there exists a subset E of D such that $|E| \leq \kappa$ and $x \in \overline{E}$.*

Further X is called *orthocompact* provided for every open cover $\mathcal{U}$ there exists an open refinement $\mathcal{V}$ such that for every $\mathcal{V}' \subset \mathcal{V}$, we have $\bigcap\{V \in \mathcal{V}' : x \in V\}$ is open for each $x \in X$. A set $A \subset X$ is called *sequentially closed* if and only if A contains all limits of all sequences from A; X is called a *sequential space* provided every sequentially closed set is closed (every sequential space has countable tightness, see [Eng, 1.7.13(c)]). X is called *ω-bounded* if every countable subset is contained in a compact set.

Further we will use the following standard notation: for a set D, $[D]^{<\omega}$ denotes the set of all finite subsets of D and $[D]^{\omega}$ the set of all finite or countable subsets of D. If D is a subset of a topological space X, the *$\aleph_0$-closure of D* (see [Arh]) is the set $[D]_{\aleph_0} = \bigcup\{\overline{M} : M \in [D]^{\omega}\}$.

1. The product of a compact sequential T_2 space with an hacc T_3 space is hacc.

Theorem 1.1. *If Y is a compact sequential T_2 space, and X is an hacc T_3 space, then $X \times Y$ is hacc.*

Proof: We proceed similarly to the corresponding proof of Vaughan ([Vau2, Theorem 1.2]). By contradiction, suppose there exists a closed non acc subset F of $X \times Y$, i.e., there exist a closed $F \subset X \times Y$, an open cover $\mathcal{U}$ of F and a dense subset D of F such that for all $B \in [D]^{<\omega}$ we have that $St(B,\mathcal{U}) \not\supset F$. Proceeding as in [Vau2, Theorem 1.2], we conclude that the closed sets $F_B = \pi_Y(F \setminus St(B,\mathcal{U}))$ form a filter base on Y, where π_Y is the projection on Y. Hence by compactness there exists

$$y \in \cap\{F_B : B \in [D]^{\omega}\}.$$

Since $X \times \{y\}$ is homeomorphic to the hacc space X, and F is closed in $X \times Y$, then $(X \times \{y\}) \cap F$ is acc.

As in [Vau2, Theorem 1.2], there exists an open set $V \subset X$ such that $(V \times \{y\}) \cap F \neq \varnothing$ and

$$(\overline{V} \times \{y\}) \cap [D]_{\aleph_0} = \varnothing. \tag{1}$$

Let $Z = \pi_Y((\overline{V} \times Y) \cap [D]_{\aleph_0})$. Again as in [Vau2, Theorem 1.2], Z is sequentially closed in Y and, since Y is sequential, Z is closed in Y. Note that $\pi_Y((V \times Y) \cap D) \subset \pi_Y((\overline{V} \times Y) \cap [D]_{\aleph_0}) = Z \subset \pi_Y((\overline{V} \times Y) \cap F)$. Since D is dense in F and

$(V \times Y) \cap F$ is a nonempty open subset of F, we have that $(V \times Y) \cap D$ is dense in $(V \times Y) \cap F$. As π_Y is a continuous mapping, we have that $\overline{\pi_Y((V \times Y) \cap D)}$ is dense in $\pi_Y((V \times Y) \cap F)$, then $\pi_Y((V \times Y) \cap F) \subseteq \overline{\pi_Y((\overline{V} \times Y) \cap [D]_{\aleph_0})} = \overline{Z} = Z$. Since $y \in \pi_Y((V \times \{y\}) \cap F)$, we have that $y \in Z = \pi_Y((\overline{V} \times Y) \cap [D]_{\aleph_0})$. Then $(\overline{V} \times \{y\}) \cap [D]_{\aleph_0} \neq \varnothing$, but this contradicts (1), and completes the proof. □

Corollary 1.1. [PFA] *If Y is a compact T_2 space with countable tightness, and X is an hacc T_3 space, then $X \times Y$ is hacc.*

2. The product of a compact T_2 space of countable tightness with an hacc ω-bounded T_3 space is hacc.

Theorem 2.1. *$X \times Y$ is hacc provided Y is a compact T_2 space of countable tightness and X is an hacc, ω-bounded T_3 space.*

Proof: Also in this case, we proceed similarly to the corresponding proof of Vaughan ([Vau2, Theorem 1.4]). The beginning of the proof repeats the first part of the proof of Theorem 1.1, with the modification (as in [Vau2, Theorem 1.4]) that the sets F_B need not be closed. We get that there exists

$$y \in \cap\{\overline{F_B} : B \in [D]^\omega\}$$

and an open set $V \subset X$ such that $(V \times \{y\}) \cap F \neq \varnothing$ and

$$(\overline{V} \times \{y\}) \cap [D]_{\aleph_0} = \varnothing.$$

Now we show that

$$y \in \overline{\pi_Y((V \times Y) \cap D)}.$$

Since D is dense in F, we have that $\pi_Y((V \times Y) \cap D)$ is dense in $\pi_Y((V \times Y) \cap F)$ and then $\pi_Y((V \times Y) \cap F) \subset \overline{\pi_Y((V \times Y) \cap D)}$. Hence, as $(V \times \{y\}) \cap F \neq \varnothing$ and $y \in \pi_Y((V \times \{y\}) \cap F)$, we have that $y \in \overline{\pi_Y((V \times Y) \cap D)}$.

Proceeding as in [Vau2, Theorem 1.4], we obtain the desired conclusion. □

Vaughan ([Vau2]) obtained that the product of a compact T_2 space of countable tightness with a countably compact *GO*-space (generalized ordered spaces, i.e., spaces which are subspaces of linearly ordered topological spaces; see [FL]) is acc. By Theorem 2.1, we have the following result concerning hacc (and then acc).

Corollary 2.1. *$X \times Y$ is hacc provided Y is a compact T_2 space of countable tightness and X is a countably compact GO-space.*

Proof: Since every countably compact *GO*-space is ω-bounded ([GFW, Theorem 3]), and every *GO*-space is orthocompact ([FL, 5.23]) we have that X is an ω-bounded, orthocompact space; further, as every countably compact, orthocompact space is acc ([Vau2]) and orthocompactness and countable compactness are hereditary with respect to closed subsets, we have that X is hacc. Then, by Theorem 2.1, $X \times Y$ is hacc. □

3. The product of a compact T_2 space of countable tightness with an acc T_3 space of countable tightness is hacc.

We have the following result:

Proposition 3.1. *$X \times Y$ is hacc provided Y is a compact T_2 space with countable tightness and X is an acc T_3 space with countable tightness.*

Proof: It is well-known that $X \times Y$ is countably compact. By Malyhin's Theorem (see [Mal]), $X \times Y$ has countable tightness. Both properties, countable compactness and countable tightness, are hereditary with respect to closed subsets. Thus by Matveev's Theorem (see [Mat1, Theorem 1.8]), $X \times Y$ is hacc. $\square$

While trying to answer to the following question: *do there exist a space X and a compact space Y such that X is hacc, $X \times Y$ is acc, but $X \times Y$ is not hacc*?, we obtained the following example which answer this question and also demonstrates why some assumptions in Theorems 1.1 and 2.1 cannot be weakened.

Example 3.1. *The product of an hacc Tychonoff space of countable density-tightness with a compact T_2 space of countable density-tightness need not be hacc.*

Consider the Franklin-Rajagopalan spaces (see [Eng, 3.12.17(d)]) $X = T \cup Z$ and $Y = T \cup Z'$, where T is homemorphic to ω with discrete topology, Z is homemorphic to the ordinal $\mathbf{t}$ (see [vD], [Vau1] or [Eng, 3.12.17(d)] where it is denoted δ) with order topology and Z' is homemorphic to $\mathbf{t}+1$ with order topology. We have that $d_t(X) = \omega$ and $d_t(Y) = \omega$ because both X and Y contain countable dense subsets of isolated points (so, of course, $d_t(X \times Y) = \omega$ and then $X \times Y$ is acc ([Mat1, Lemma 1.7])). Now we show that X is hacc. Let F be a closed subspace of X. Then $F = F_T \cup F_Z$ where $F_T = \overline{F \cap T}$ and $F_Z = \overline{F \setminus F_T}$. F_T has countable density-tightness because T is a countable set of isolated points; further, since Z is a linearly ordered topological space, then F_Z is a GO-space. Then, F_T and F_Z are acc ([Mat1, Lemma 1.7] and [Vau2, Corollary 1.7], respectively). Since T is open in X and F is closed in X, we have that F_T and F_Z are regular closed in F. So, F is written as union of regular closed acc spaces; then, from Proposition 4.3 in [Mat1], it follows that F is acc. So, X is hacc. Further Y is compact, but $X \times Y$ is not hacc because it contains a closed copy of $\mathbf{t} \times (\mathbf{t} + 1)$ that is not acc (see [Bon, Theorem 1.2]). It is worth mentioning that under the assumption $\mathbf{t} = \omega_1$, the space X is even first-countable.

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