Nonlinear homogeneous eigenvalue problem in R^N: nonstandard variational approach

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Abstract. The nonlinear eigenvalue problem for p-Laplacian

$$\begin{cases} -\operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u) = \lambda g(x)|u|^{p-2}u \text{ in } \mathbb{R}^N, \\ u > 0 \text{ in } \mathbb{R}^N, \lim\limits_{|x|\to\infty} u(x) = 0, \end{cases}$$

is considered. We assume that $1 < p < N$ and that g is indefinite weight function. The existence and $C^{1,\alpha}$-regularity of the weak solution is proved.

Keywords: eigenvalue, the p-Laplacian, indefinite weight, regularity

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1. Introduction

We consider the nonlinear eigenvalue problem

$$\begin{cases} -\operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u) = \lambda g(x)|u|^{p-2}u \text{ in } \mathbb{R}^N, \\ u > 0 \text{ in } \mathbb{R}^N, \lim\limits_{|x|\to\infty} u(x) = 0, \end{cases} \tag{1.1}$$

where $1 < p < N$, g is a function that changes sign, i.e. g is an indefinite weight function, a is a positive and bounded function and λ is a real parameter.

The aim of this work is to prove the existence and $C^{1,\alpha}$ regularity of the weak solution of (1.1). In comparison to similar results we use a *nonstandard variational approach* — we do not minimize a Reyleigh-type quotient.

Let us note that this work was motivated by recent work [3] in which the following nonhomogeneous eigenvalue problem was considered:

$$\begin{cases} -\operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u) = \lambda f(x,u) \text{ in } \mathbb{R}^N, \\ u > 0 \text{ in } \mathbb{R}^N \text{ and } \lim\limits_{|x|\to\infty} u(x) = 0, \end{cases}$$

where f is a Carathéodory function satisfying the condition $0 \le f(x,t) \le g(x)|t|^\gamma$ with $p < \gamma < p^* = \frac{Np}{N-p}$ and g satisfying suitable integrability assumptions.

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Modifying the approach from [3] we can deal with our problem and to get (λ, u) satisfying (1.1).

In this paper we will use the following notation: $L^p := L^p(\mathbb{R}^N)$ denotes the usual Lebesgue space with the norm $\|\cdot\|_p$, $W^{1,p} := W^{1,p}(\mathbb{R}^N)$ is the usual Sobolev space and $D(\mathbb{R}^N) := C_0^\infty(\mathbb{R}^N)$ is the space of all functions with compact support in $\mathbb{R}^N$ with continuous derivatives of all orders.

2. Preliminaries, hypotheses and formulation of the main result

We assume that $a = a(x)$ is a measurable function such that

$$0 < a_0 \le a(x) \in L^\infty, \tag{2.1}$$

g is an indefinite weight function satisfying:

(g_1) there exists an open subset $\Omega \neq \emptyset$ of $\mathbb{R}^N$ such that

$$g(x) > 0 \text{ a.e. in } \Omega;$$

(g_2) there exists a real number $\delta, 0 < \delta < \infty$ such that

$$g \in L^{\frac{N}{p}} \cap L^{\frac{N}{p}+\delta}.$$

Let us consider the function space: $X := \{u \in L^{p^*}; \nabla u \in (L^p)^N\}$ equipped with the norm $\|u\| := \left(\int a(x)|\nabla u|^p\, dx\right)^{\frac{1}{p}}$. (Here and henceforth the integrals are taken over $\mathbb{R}^N$ unless otherwise specified.) Then X is a reflexive Banach space.

Using (2.1) and Sobolev inequality (see [1]) we conclude that there is a constant $C_1 > 0$ such that

$$\|u\|_{p^*} \le C_1 \|u\| \tag{2.2}$$

holds for all $u \in X$.

Definition 2.1. A *weak solution* of (1.1) is a pair (λ, u) such that $\lambda > 0$, $u \in X$, $u \neq 0$ and

$$\int a(x)|\nabla u|^{p-2}\nabla u \nabla v\, dx = \lambda \int g(x)|u|^{p-2}uv\, dx \tag{2.3}$$

for all $v \in X$. In this case u is called an *eigenfunction* corresponding to the *eigenvalue* $\lambda > 0$.

Let us remark that under the assumptions (2.1) and (g_2) the integrals in (2.3) are well defined.

The main result of our paper is the following

Theorem 2.1. *Let us assume* (2.1), (g_1) *and* (g_2). *Then the problem* (1.1) *has a positive eigenvalue* $\lambda > 0$ *and a corresponding eigenfunction* $u \in X$, $u > 0$ *in* $\mathbb{R}^N$ *and* $\lim_{|x|\to\infty} u(x) = 0$. *Moreover, the eigenvalue* λ *is simple, isolated and unique in the following sense: if* $\tilde{\lambda} \neq \lambda$ *is a positive eigenvalue of* (1.1) *and* $\tilde{u}$ *is a corresponding eigenfunction then* $\tilde{u}$ *changes sign in* $\mathbb{R}^N$.

Corollary. *Let the assumptions of Theorem* 2.1 *be satisfied and moreover, let* $a \in C^1(\mathbb{R}^N)$. *Then the assertion of Theorem* 2.1 *holds with* $u \in C^{1,\alpha}(B_R(0))$, *for any* $R > 0$ *and* $\alpha = \alpha(R) \in]0, 1[$.

Remark 2.1. Similar results were proved in papers [2], [5] and [6]. However, different (more restrictive) assumptions on the weight function g and different methods were used in these papers. On the other hand, our result does not contain any information about "higher" eigenvalues of (1.1).

3. Proof of Theorem 2.1 and of Corollary

Proposition 3.1. *Assume* (2.1), (g_1) *and* (g_2). *Then the problem* (1.1) *has a weak solution* $(\lambda, u), u \in X$ *and* $\lambda > 0$, *such that* $u \not\equiv 0$ *and* $u \geq 0$ *in* $\mathbb{R}^N$.

Proof: The proof follows the lines of Theorem 3.1 in [3]. Since the character of our problem is different from that considered in [3], we give the proof in detail here for the reader's convenience. Let $\alpha \in]1, p[$ be fixed and consider the following functional:

$$J = \frac{\int g(x)|u|^p\,dx}{\|u\|^\alpha + \|u\|^{p^*}}\,.$$

It is easy to see that J is well-defined for any $u \in X$, $u \not\equiv 0$. Due to (2.2) and the Hölder inequality we have

$$\int g(x)|u|^p\,dx \leq \int |g(x)||u|^p\,dx \leq \|g\|_{\frac{N}{p}}\|u\|^p_{p^*} \leq C_1^p\|g\|_{\frac{N}{p}}\|u\|^p. \tag{3.1}$$

Since $\alpha < p < p^*$ then $\|u\|^p \leq \|u\|^\alpha + \|u\|^{p^*}$ for all $u \in X$, so

$$J(u) \leq \frac{\int g(x)|u|^p\,dx}{\|u\|^p} \leq C_1^p\|g\|_{\frac{N}{p}}.$$

Then there exists a constant $s_1 < \infty$ ($s_1 = C_1^p\|g\|_{\frac{N}{p}}$) such that $J(u) \leq s_1$ holds for all $u \in X$, $u \not\equiv 0$. Thus $s := \sup_{\substack{u\in X\\ u\not\equiv 0}} J(u)$ is a real number.

Lemma 3.1. *There exist* $s_0 \in]0, s[$ *and a sequence* $(u_n)_{n=1}^\infty \subset X$, $u_n \geq 0$ *such that* $s_0 \leq J(u_n) \leq s$ *holds for all* n, *and* $\lim_{n\to\infty} J(u_n) = s$. *Furthermore,* $\int g|u_n|^p\,dx \to \int g|u|^p\,dx$, *as* $n \to \infty$ *and* $J(u) = s$ *for some* $u \in X$.

Proof of Lemma 3.1: Let Ω be from (g_1) and choose $\varphi_0 \in D(\mathbb{R}^N)$ such that $\operatorname{supp}\varphi_0 \subset\subset \Omega$ and $\sup_{x\in\mathbb{R}^N} \varphi_0(x) > 0$. Set $s_0 = \frac{1}{2}J(\varphi_0)$. Then $s_0 =$

$\frac{1}{2}\frac{\int\limits_\Omega g|\varphi_0|^p\,dx}{\|\varphi_0\|^\alpha+\|\varphi_0\|^{p^*}} > 0$ and $s_0 < s$. Let $(u_n)_{n=1}^\infty \subset X$, $u_n \neq 0$, be a sequence such that $J(u_n) \to s$ as $n \to \infty$. Since $s_0 < s$ we can choose $(u_n)_{n=1}^\infty$ such that $J(u_n) \geq s_0$ for all n and due to the equality $J(u) = J(|u|)$ we may assume that $u_n \geq 0$. Then (3.1) implies that there exists s_1 such that

$$s_0(\|u_n\|^\alpha + \|u_n\|^{p^*}) \leq s_1\|u_n\|^p$$

holds for all n; so we can find real numbers $0 < \delta_1 < \delta_2$ such that

$$\delta_1 \leq \|u_n\| \leq \delta_2 \tag{3.2}$$

hold for all n, and this implies that $(u_n)_{n=1}^\infty$ is bounded in X. Due to the reflexivity of X we may assume without loss of generality that for some $u \in X$ we have $u_n \to u$ weakly in X and pointwise a.e. in $\mathbb{R}^N$. (Remark that for any bounded open set $B \subset \mathbb{R}^N$ we have for $u \in X$:

$$\Big(\int\limits_B |u|^{p^*}\,dx\Big)^{\frac{1}{p^*}} \leq \|u\|_{p^*} \leq C_1\|u\|$$

and

$$\Big(\int\limits_B |\nabla u|^p\,dx\Big)^{\frac{1}{p}} \leq \|\nabla u\|_p \leq C_1'\|u\|$$

and then

$$\|u\|_{W^{1,p}(B)} \leq C\|u\|$$

holds for all $u \in X$. The compact imbedding $W^{1,p}(B) \hookrightarrow\hookrightarrow L^p(B)$ then implies that $u_{n_k} \to u$ in $L^p(B)$ and hence pointwise a.e.). This implies that $u \geq 0$ a.e. in $\mathbb{R}^N$. Using the Hölder inequality, for all $0 \leq R \leq \infty$ and all n, we have

$$\begin{aligned}\Big|\int\limits_{|x|\geq R} g(x)|u_n|^p\,dx\Big| &\leq \int\limits_{|x|\geq R} |g(x)||u_n|^p\,dx \\ &\leq \Big(\int\limits_{|x|\geq R} |g(x)|^{\frac{N}{p}}\,dx\Big)^{\frac{p}{N}}\Big(\int\limits_{|x|\geq R} |u_n|^{p^*}\,dx\Big)^{\frac{N-p}{N}} \\ &\leq C_2\Big(\int\limits_{|x|\geq R} |g(x)|^{\frac{N}{p}}\,dx\Big)^{\frac{p}{N}},\end{aligned}$$

where C_2 is a constant independent of R and n. The same holds also for u:

$$\Big|\int\limits_{|x|\geq R} g(x)|u|^p\,dx\Big| \leq C_3\Big(\int\limits_{|x|\geq R} |g(x)|^{\frac{N}{p}}\,dx\Big)^{\frac{p}{N}}.$$

Since $g \in L^{\frac{N}{p}}$, we have $\lim\limits_{R\to\infty} \int\limits_{|x|\geq R} |g|^{\frac{N}{p}}\, dx = 0$, which implies that, for any $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that

$$\Bigg| \int\limits_{|x|\geq R_\varepsilon} g(x)|u|^p\, dx \Bigg| \leq \varepsilon$$

and

$$\Bigg| \int\limits_{|x|\geq R_\varepsilon} g(x)|u_n|^p\, dx \Bigg| \leq \varepsilon$$

hold for all n.

On the other hand, using the Rellich-Kondrachov theorem (see [1]), and the continuity of the Nemytskii operator we prove, for $\varepsilon > 0$ fixed that:

$$\int\limits_{|x|<R_\varepsilon} g(x)|u_{n_k}|^p\, dx \to \int\limits_{|x|<R_\varepsilon} g(x)|u|^p\, dx \text{ as } n \to \infty.$$

Indeed, let us consider the function $F(x,t) := g(x)|t|^p$, then

$$|F(x,t)| = |g||t|^p < \frac{|g|^{\frac{N}{p}+\delta}}{\frac{N}{p}+\delta} + \frac{|t|^m}{\frac{m}{p}},$$

for all $t \in \mathbb{R}$ and a.e. x in $B_\varepsilon := \{x \in \mathbb{R}^N; |x| < R_\varepsilon\}$, where $m := p\left(\frac{N}{p}+\delta\right)'$ and the dash denotes the exponent conjugate.

Hence the Nemytskii operator N_F associated with F is continuous from $L^m(B_\varepsilon)$ in $L^1(B_\varepsilon)$. Note that

$$\frac{N}{p} < \frac{N}{p} + \delta \;\text{ implies }\; \left(\frac{N}{p}+d\right)' < \left(\frac{N}{p}\right)' = \frac{p^*}{p}$$

and hence $m < p^*$. Then from imbeddings

$$X \hookrightarrow W^{1,p}(B_\varepsilon) \hookrightarrow\hookrightarrow L^M(B_\varepsilon)$$

we conclude that $N_F(u_n) \to N_F(u)$ in $L^1(B_\varepsilon)$, i.e.

$$\int\limits_{|x|<R_\varepsilon} g(x)|u_n|^p\, dx \to \int\limits_{|x|<R_\varepsilon} g(x)|u|^p\, dx \quad\text{as}\quad n \to \infty.$$

Finally,

$$\left|\int g(x)(|u_n|^p - |u|^p)\,dx\right| \le \left|\int\limits_{|x|\ge R_\varepsilon} g(x)|u_n|^p\,dx\right| + \left|\int\limits_{|x|\ge R_\varepsilon} g(x)|u|^p\,dx\right| + $$
$$+\left|\int\limits_{B_\varepsilon} g(x)(|u_n|^p - |u|^p)\,dx\right| \le 3\varepsilon$$

for n large enough, which implies that

$$\int g(x)|u_n|^p\,dx \to \int g(x)|u|^p\,dx \quad \text{as} \quad n \to \infty. \tag{3.3}$$

Since we have $J(u_n) \ge s_0$ for all n, then

$$\int g(x)|u_n|^p\,dx \ge s_0(\|u_n\|^\alpha + \|u_n\|^{p^*}).$$

Due to (3.2) we have,

$$\int g(x)|u_n|^p\,dx \ge s_0(\delta_1^\alpha + \delta_1^{p^*})$$

and (3.3) implies

$$\int g(x)|u^p|\,dx \ge s_0(\delta_1^\alpha + \delta_1^{p^*}) > 0$$

and, therefore, $u \not\equiv 0$ in $\mathbb{R}^N$. From the uniform boundedness principle, we obtain

$$\|u\|^\alpha + \|u\|^{p^*} \le \liminf(\|u_n\|^\alpha + \|u_n\|^{p^*})$$

and so

$$\begin{aligned} s = \limsup J(u_n) &= \limsup\left(\frac{\int g(x)|u_n|^p\,dx}{\|u_n\|^\alpha + \|u_n\|^{p^*}}\right) \\ &= \limsup\left(\frac{1}{\|u_n\|^\alpha + \|u_n\|^{p^*}}\right)\int g(x)|u|^p\,dx \\ &\le \frac{\int g(x)|u|^p\,dx}{\liminf(\|u_n\|^\alpha + \|u_n\|^{p^*})} \le \frac{\int g(x)|u|^p\,dx}{\|u\|^\alpha + \|u\|^{p^*}} = J(u) \end{aligned}$$

and, consequently $J(u) = s$. The lemma is proved. □

Now, we prove that u is an eigenfunction corresponding to a positive eigenvalue $\lambda > 0$. Since $u \not\equiv 0$ in $\mathbb{R}^N$ then for any fixed $v \in X$ we can find $\varepsilon_0 = \varepsilon_0(v) > 0$

such that $\|u+\varepsilon v\| > 0$ holds for all $\varepsilon \in\,]-\varepsilon_0,\varepsilon_0[$. We consider the function $\eta :]-\varepsilon_0,\varepsilon_0[\, \to \mathbb{R}$ defined as follows:

$$\eta(\varepsilon) = J(u+\varepsilon v).$$

The function $F(\varepsilon) = \int a(x)|\nabla u + \varepsilon\nabla v|^p\,dx = \|u+\varepsilon v\|^p$ is differentiable and

$$F'(\varepsilon) = p\int a(x)|\nabla u + \varepsilon\nabla v|^{p-2}(\nabla u + \varepsilon\nabla v)\nabla v\,dx.$$

Since $\|u+\varepsilon v\| > 0$ on $]-\varepsilon_0,\varepsilon_0[$ then the same is true for $F(\varepsilon)$, i.e. $F(\varepsilon) > 0$ on $]-\varepsilon_0,\varepsilon_0[$, and, therefore, the function $G(\varepsilon) := \|u+\varepsilon v\|^\alpha = (F(\varepsilon))^{\frac{\alpha}{p}}$ is differentiable and

$$G'(\varepsilon) = \frac{\alpha}{p}F'(\varepsilon)(F(\varepsilon))^{\frac{\alpha}{p}-1}.$$

At $\varepsilon = 0$ we have

$$G'(0) = \alpha\|u\|^{\alpha-p}\int a(x)|\nabla u|^{p-2}\nabla u\nabla v\,dx.$$

The same remains true for the function $H(\varepsilon) = \|u+\varepsilon v\|^{p^*}$. Hence

$$H'(0) = p^*\|u\|^{p^*-p}\int a(x)|\nabla u|^{p-2}\nabla u\cdot\nabla v\,dx.$$

Thus η is differentiable on $]-\varepsilon_0,\varepsilon_0[$. Since 0 is a maximum of η, we have $\eta'(0) = 0$, which implies that

$$\int a(x)|\nabla u|^{p-2}\nabla u\nabla v\,dx = \lambda\int g(x)|u|^{p-2}v\,dx$$

holds for all $v \in X$, where

$$\lambda = \frac{p(\|u\|^\alpha + \|u\|^{p^*})}{(p^*\|u\|^{p^*-p} + \alpha\|u\|^{\alpha-p})\int g(x)|u|^p\,dx}.$$

Proposition 3.2. *Let $u \in X$ be a weak solution for* (1.1) *such that $u \not\equiv 0$, $u \geq 0$ a.e. in $\mathbb{R}^N$. Then $u \in L^r$ for all $p^* \leq r \leq \infty$.*

PROOF: We use Nash-Moser bootstrap iterations similarly as in [3]. For $M > 0$ define $v_M(x) = \inf\{u(x), M\}$ and let choose $v = v_M^{kp+1}$ (for some $k > 0$) as a test function in (2.3). Then it is easy to see that $v \in X \cap L^\infty$ and that

$$\int a(x)|\nabla u|^{p-2}\nabla u\nabla(v_M^{\kappa p+1})\,dx = \lambda\int g(x)|u|^{p-2}uv_M^{\kappa p+1}\,dx.$$

On one hand, due to (2.2) we have

$$
\begin{aligned}
&\int a(x)|\nabla u|^{p-2}\nabla u\nabla(v_M^{kp+1})\,dx = (kp+1)\int a(x)|\nabla u|^{p-2}\nabla u\nabla v_M v_M^{kp}\,dx\\
&\geq (kp+1)\int a(x)|\nabla v_M|^p\ v_M^{kp}\,dx = \frac{kp+1}{(k+1)^p}\int a(x)|\nabla(v_M^{k+1})|^p\,dx\\
&\geq \frac{1}{C_1^p}\frac{kp+1}{(k+1)^p}\left(\int v_M^{(k+1)p^*}\,dx\right)^{\frac{p}{p^*}}.
\end{aligned}
\tag{3.4}
$$

On the other hand,

$$
\begin{aligned}
&\int g(x)|u|^{p-2}uv_M^{kp+1}\,dx = \int g(x)u^{p-1}v_M^{kp+1}\,dx\\
&\leq \int |g(x)|u^{p-1}v_M^{kp+1}\,dx \leq \int |g(x)|u^{(1+k)p}\,dx\\
&\leq \|g\|_{(\frac{N}{p}+\delta)}\left(\int u^{(k+1)q}\,dx\right)^{\frac{p}{q}},
\end{aligned}
\tag{3.5}
$$

where $q = p\left(\frac{N}{p}+\delta\right)'$. From (3.4) and (3.5) we obtain

$$
\frac{1}{C_1^p}\frac{kp+1}{(k+1)^p}\left(\int v_M^{(k+1)p^*}\,dx\right)^{\frac{p}{p^*}} \leq \lambda\|g\|_{\left(\frac{N}{p}+\delta\right)}\left(\int u^{(k+1)q}\,dx\right)^{\frac{p}{q}}.
$$

Then there exists a constant $C_3 > 0$, $C_3 = \lambda C_1^p\|g\|_{\left(\frac{N}{p}+\delta\right)}$ such that

$$
\left(\int v_M^{(k+1)p^*}\,dx\right)^{\frac{p}{p^*}} \leq C_3\frac{(k+1)^p}{(kp+1)}\left(\int u^{(k+1)q}\,dx\right)^{\frac{p}{q}},
$$

i.e.

$$
\|v_M\|_{(k+1)p^*} \leq C_4^{\frac{1}{k+1}}\left[\frac{k+1}{(kp+1)^{\frac{1}{p}}}\right]^{\frac{1}{k+1}}\|u\|_{(k+1)q},
\tag{3.6}
$$

where $C_4 = C_3^{\frac{1}{p}} > 0$. Since $u \in X$, it follows from (2.2) that $u \in L^{p^*}$. Then we can choose $k = k_1$ in (3.6) such that $(k_1+1)q = p^*$ i.e. $k_1 = \frac{p^*}{q} - 1$. Then we have

$$
\|v_M\|_{(k_1+1)p^*} \leq C_4^{\frac{1}{k_1+1}}\left[\frac{k_1+1}{(k_1p+1)^{\frac{1}{p}}}\right]^{\frac{1}{k_1+1}}\|u\|_{p^*}.
$$

But, $\lim_{M\to\infty} v_M(x) = u(x)$ and the Fatou lemma implies

$$\|u\|_{(k_1+1)p^*} \leq C_4^{\frac{1}{k_1+1}} \left[\frac{k_1+1}{(k_1p+1)^{\frac{1}{p}}}\right]^{\frac{1}{k_1+1}} \|u\|_{p^*}.$$

Then $u \in L^{(k_1+1)p^*}$, and we can choose $k = k_2$ in (3.6) such that $(k_2+1)q = (k_1+1)p^*$ i.e. $k_2 = \frac{(p^*)^2}{q^2} - 1$. Repeating the same argument we get

$$\|u\|_{(k_2+1)p^*} \leq C_4^{\frac{1}{k_2+1}} \left[\frac{k_2+1}{(k_2p+1)^{\frac{1}{p}}}\right]^{\frac{1}{k_2+1}} \|u\|_{(k_1+1)p^*}.$$

By induction

$$\|u\|_{(k_n+1)p^*} \leq C_4^{\frac{1}{k_n+1}} \left[\frac{k_n+1}{(k_np+1)^{\frac{1}{p}}}\right]^{\frac{1}{k_n+1}} \|u\|_{(k_{n-1}+1)p^*}$$

holds for all $n \in \mathbb{N}$, where $k_n = \left(\frac{p^*}{q}\right)^n - 1$. Then

$$\|u\|_{(k_n+1)p^*} \leq C_4^{\sum_{j=1}^{n} \frac{1}{k_j+1}} \prod_{j=1}^{n} \left[\frac{k_j+1}{(k_jp+1)^{\frac{1}{p}}}\right]^{\frac{1}{k_j+1}} \|u\|_{p^*}.$$

But $\lim_{y\to\infty} \left[\frac{y+1}{(yp+1)^{\frac{1}{p}}}\right]^{\frac{1}{\sqrt{y+1}}} = 1$ and $\left[\frac{y+1}{(yp+1)^{\frac{1}{p}}}\right]^{\frac{1}{\sqrt{y+1}}} > 1$ for all $y > 0$. Then there exists a constant $C_5 > 0$ such that

$$1 < \left[\frac{k_n+1}{(k_np+1)^{\frac{1}{p}}}\right]^{\frac{1}{\sqrt{k_n+1}}} < C_5$$

holds for all $n \in \mathbb{N}$, and therefore,

$$\|u\|_{(k_n+1)p^*} \leq C_4^{\sum_{j=1}^{n} \frac{1}{k_j+1}} C_5^{\sum_{j=1}^{n} \frac{1}{\sqrt{k_j+1}}} \|u\|_{p^*},$$

since

$$\begin{cases} \frac{1}{k_j+1} = \left(\frac{q}{p^*}\right)^j, \frac{q}{p^*} < 1, \\ \frac{1}{\sqrt{k_j+1}} = \left(\sqrt{\frac{q}{p^*}}\right)^j, \sqrt{\frac{q}{p^*}} < 1. \end{cases}$$

Then we conclude that there exists a constant $C_6 > 0$ such that

$$\|u\|_{(k_n+1)p^*} \leq C_6\|u\|_{p^*}$$

holds for all $n \in \mathbb{N}$. Since $k_n \to \infty$ as $n \to \infty$, we get $u \in L^\infty$ and by interpolation $u \in L^r$ for all $r \in \left[\frac{Np}{N-p}, \infty\right]$. This completes the proof of Proposition 3.2. $\square$

Proposition 3.3. *Let $u \in X$, $u \geq 0$ and $u \not\equiv 0$ be a weak solution of* (1.1)*. Then $u > 0$ in $\mathbb{R}^N$ and*

$$\lim_{|x|\to\infty} u(x) = 0.$$

Proof: The positivity of u follows from the weak Harnack type inequality proved in [10, Theorem 1.1]. More precisely, due to Proposition 3.2 we have $u \in L^\infty$, and using Theorem 1.1 of [10] there exists a constant $C_R > 0$ such that

$$\max_{K(R)} u(x) \leq C_R \min_{K(R)} u(x), \tag{3.7}$$

where $K(R)$ denotes the cube in $\mathbb{R}^N$ of side R and center 0 whose sides are parallel to the coordinate axes. Let $D \subset \mathbb{R}^N$ be such that $|D| \neq 0$ and $u \equiv 0$ a.e. in D. Then there exists $R_0 > 0$ such that $|D \cap K(R_0)| \neq 0$ (otherwise $D = \bigcup_{R\in Q} (D \cap K(R))$ will be of measure zero). Thus $0 \leq \max_{K(R)} u(x) \leq C_R \min_{K(R)} u(x) = 0$ holds for all $R > R_0$ which implies $u \equiv 0$ in $K(R)$. Hence $u = 0$ a.e. in $\mathbb{R}^N$, a contradiction. Thus $u > 0$ in $\mathbb{R}^N$. Finally, let $B_r(x)$ denote the ball centered at $x \in \mathbb{R}^N$ with radius $r > 0$. Then by Theorem 1 of [8], for some $C = C(N,p) > 0$ we obtain an estimate:

$$\|u\|_{L^\infty(B_1(x))} \leq C \, \|u\|_{L^{p^*}(B_2(x))}$$

independently of $x \in \mathbb{R}^N$. Hence the decay of u follows. □

Proposition 3.4. *The value of $\lambda > 0$ (and $u > 0$) is independent of the choice of $\alpha \in \,]1,p[$ at the beginning of the proof of Proposition* 3.1*. Namely, λ is simple, isolated and if $\tilde{\lambda}$ is a positive eigenvalue of* (1.1) *and $\tilde{u}$ is corresponding eigenfunction then $\tilde{u}$ changes sign in $\mathbb{R}^N$.*

Proof: The simplicity of λ follows from the proof of Lemma 3.1 in [7] adapted for $\Omega = \mathbb{R}^N$. Remaining two facts (i.e. λ is isolated and unique positive eigenvalue having eigenfunctions which do not change sign) follow from the proof of Lemma 2.3 in [4]. In particular, this implies that λ and u are independent of α. □

The assertion of Theorem 2.1 follows now from Propositions 3.1–3.4. The assertion of Corollary follows directly from the regularity result [9].

References

[1] Adams R.A., *Sobolev Spaces*, Academic Press, 1975.

[2] Allegretto W., Huang Y.X., *Eigenvalues of the indefinite weight p-Laplacian in weighted spaces*, Funkc. Ekvac. **38** (1995), 233–242.

[3] Drábek P., *Nonlinear eigenvalue for p-Laplacian in* $\mathbb{R}^N$, Math. Nach **173** (1995), 131–139.

[4] Drábek, Huang Y.X., *Bifurcation problems for the p-Laplacian in* $\mathbb{R}^N$, to appear in Trans. of AMS.

[5] Fleckinger J., Manasevich R.F., Stavrakakis N.M., de Thelin F., *Principal eigenvalues for some quasilinear elliptic equations on* $\mathbb{R}^N$, preprint.
[6] Huang Y.X., *Eigenvalues of the p-Laplacian in* $\mathbb{R}^N$ *with indefinite weight*, Comment. Math. Univ. Carolinae **36** (1995), 519–527.
[7] Lindqvist P., *On the equation* $\operatorname{div}(|\nabla u|^{p-2}\nabla u)+\lambda|u|^{p-2}u=0$, Proc. Amer. Math. Society **109** (1990), 157–164.
[8] Serin J., *Local behavior of solutions of quasilinear equations*, Acta Math. **111** (1964), 247–302.
[9] Tolkdorf P., *Regularity for a more general class of quasilinear elliptic equations*, J. Diff. Equations **51** (1984), 126–150.
[10] Trudinger N.S., *On Harnack type inequalities and their applications to quasilinear elliptic equations*, Comm. Pure. Appl. Math. **20** (1967), 721–747.

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