Indices of Orlicz spaces and some applications

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Abstract. We study connections between the Boyd indices in Orlicz spaces and the growth conditions frequently met in various applications, for instance, in the regularity theory of variational integrals with non-standard growth. We develop a truncation method for computation of the indices and we also give characterizations of them in terms of the growth exponents and of the Jensen means. Applications concern variational integrals and extrapolation of integral operators.

Keywords: Boyd indices, Orlicz spaces, Simonenko indices, non-standard growth conditions, variational integrals, interpolation, extrapolation

Classification: Primary 46E30; Secondary 26A12, 35A15, 35B10, 42B20, 46E35

1. Introduction

The aim of this paper is to establish connections between the Boyd indices of Orlicz spaces and the growth conditions on Young functions appearing in the theory of non-linear b.v.p. with non-standard growth and in the interpolation and extrapolation theory in Lebesgue and Orlicz spaces.

From large number of relevant references dealing with various sorts of indices in Orlicz and also more general r.i. spaces we recall the *Matuszewska-Orlicz indices* in [17], the Boyd indices [4], [5], [6], the *Zippin indices* [28], Maligranda [16] with many further references. The detailed exposition in the general setting of r.i. spaces and also in Orlicz spaces can be found in Bennett and Sharpley [3]. The Boyd indices in Lorentz-Orlicz spaces have been studied in Montgomery-Smith [19]. It is well known that all the indices mentioned above coincide in Orlicz spaces so that henceforth we shall speak only about the Boyd indices.

The definition of the Boyd indices is very simple, nevertheless, a particular computation might be extremely difficult. This paper is intended also as a contribution to development of effective methods of establishing their values.

In Sections 2 and 3 we present estimates for the Boyd indices of a Young function Φ in terms of the growth conditions

$$p\Phi(t) \le t\Phi'(t) \le q\Phi(t), \qquad t \in \mathbb{R}^1, \tag{1.1}$$

* This work has been partly performed as a part of a National Research Project and partly supported by G.N.A.F.A.

** Partly supported by Grant No. 201/96/0431 of GA ČR.

giving birth to the Simonenko indices. Of crucial importance is the following theorem, linking the reciprocals $i(\Phi)$ and $I(\Phi)$ of the Boyd indices of a Young function Φ with the Simonenko indices $p(\Lambda)$ and $q(\Lambda)$ of equivalent Young functions. We shall agree that from now on all Young functions and their complementary functions satisfy the Δ_2-condition.

Theorem 1.1. *Let Φ be a Young function. Then*

$$i(\Phi) = \sup_{\Lambda\sim\Phi} p(\Lambda) \quad \textit{and} \quad I(\Phi) = \inf_{\Lambda\sim\Phi} q(\Lambda).$$

As observed above, the definition of the Boyd indices does not often represent an efficient tool for computation and one has to find another way. The following theorems give an answer in this direction. We develop a truncation technique, which itself represents another means for dealing with the Boyd indices of Young functions, particularly in the cases when the limits r_0 and r_∞ in Theorem 1.3 do not exist (cf. Example 5.8); in Section 4 we prove the following theorem:

Theorem 1.2. *Let Φ be a Young function. Put*

$$F_\Phi(t) = \frac{t\Phi'(t)}{\Phi(t)}, \qquad t > 0,$$

and let us define

$$[F_\Phi]_\mu(t) = \max(F_\Phi(t), \mu) \quad \textit{and} \quad [F_\Phi]^\mu(t) = \min(F_\Phi(t), \mu), \qquad t > 0.$$

Then

$$i(\Phi) \geq \sup\{\mu > 0\,;\ \int_0^\infty ([F_\Phi]_\mu(s) - F_\Phi(s))\,\frac{ds}{s} < \infty\}$$

and

$$I(\Phi) \leq \inf\{\mu > 0\,;\ \int_0^\infty ([F_\Phi]^\mu(s) - F_\Phi(s))\,\frac{ds}{s} < \infty\}.$$

If there exist

$$r_0 = \lim_{t\to 0} \frac{t\Phi'(t)}{\Phi(t)} \quad \textit{and} \quad r_\infty = \lim_{t\to\infty} \frac{t\Phi'(t)}{\Phi(t)},$$

then

$$i(\Phi) = \sup\{\mu > 0\,;\ \int_0^\infty ([F_\Phi]_\mu(s) - F_\Phi(s))\,\frac{ds}{s} < \infty\} \tag{1.2}$$

and

$$I(\Phi) = \inf\{\mu > 0\,;\ \int_0^\infty ([F_\Phi]^\mu(s) - F_\Phi(s))\,\frac{ds}{s} < \infty\}. \tag{1.3}$$

When proving Theorem 1.2 we also obtain the following estimates, which have been established in Fiorenza and Krbec [8], by a different method.

Theorem 1.3. *Let Φ be a Young function. Then*

$$i(\Phi) \le \min\Bigl(\limsup_{t\to 0} \frac{t\Phi'(t)}{\Phi(t)}\,,\ \limsup_{t\to\infty} \frac{t\Phi'(t)}{\Phi(t)}\Bigr),$$
$$i(\Phi) \ge \min\Bigl(\liminf_{t\to 0} \frac{t\Phi'(t)}{\Phi(t)}\,,\ \liminf_{t\to\infty} \frac{t\Phi'(t)}{\Phi(t)}\Bigr),$$

and

$$I(\Phi) \le \max\Bigl(\limsup_{t\to 0} \frac{t\Phi'(t)}{\Phi(t)}\,,\ \limsup_{t\to\infty} \frac{t\Phi'(t)}{\Phi(t)}\Bigr),$$
$$I(\Phi) \ge \max\Bigl(\liminf_{t\to 0} \frac{t\Phi'(t)}{\Phi(t)}\,,\ \liminf_{t\to\infty} \frac{t\Phi'(t)}{\Phi(t)}\Bigr).$$

Hence if there exist

$$r_0 = \lim_{t\to 0} \frac{t\Phi'(t)}{\Phi(t)} \quad \textit{and} \quad r_\infty = \lim_{t\to\infty} \frac{t\Phi'(t)}{\Phi(t)},$$

then $i(\Phi) = \min(r_0, r_\infty)$ and $I(\Phi) = \max(r_0, r_\infty)$.

Besides that we also pursue the problem of relations between the indices of Φ and the (Jensen) integral mean $\mathcal{M}_\Phi(f) = \Phi^{-1}\bigl(\frac{1}{|\Omega|}\int_\Omega \Phi(f)\,dx\bigr)$, studied in Fiorenza [7] in connection with (1.1) and we get still another characterization of the indices in Theorem 3.7. A straightforward application of this theorem yields an alternative proof and actually an improvement of Migliaccio's theorem (see [18]) on extrapolation of reverse Hölder's inequality.

In the concluding Section 5 we present several further applications. Since in general $p(\Phi) \le i(\Phi) \le I(\Phi) \le q(\Phi)$ and any of these inequalities can be sharp it often occurs that conditions in terms of the Simonenko indices are more restrictive. On the other hand, many particular problems have been dealt with, from one reason or another, with use of the growth exponents. A general common feature of the approach offered by Theorem 1.1 is use of the Boyd indices in claims while sticking to the Simonenko indices in proofs. This can be done, for instance, when studying regularity properties of (Q-quasi)minima of

$$I(\Omega, v) = \int_\Omega \Phi(Dv) w\,dx, \tag{1.4}$$

where Ω is a bounded open subset of $\mathbb{R}^n$, $v = (v_1, \dots, v_N) \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$, w is an A_p weight and Φ satisfies (1.1). Let $m_0 = \inf\{m \ge 1;\ w \in A_m\}$.

Sbordone [24] proved that the condition

$$\frac{nm_0 q}{nm_0 + q} < p \le q < nm_0 \tag{1.5}$$

implies that $\Phi^r(|Du|)$ is locally integrable in Ω with some $r > 1$. We shall show that p and q in (1.5) can be replaced by the lower and the upper index of Φ, respectively. An analogous approach yields a variant of Harnack's inequality for non-negative (Q-quasi)minima of a non-weighted version of (1.4) for scalar functions v, studied under the condition (1.1) by Moscariello [20].

We are also concerned with extrapolation of inequalities for integral operators, we formulate and prove a generalization of Simonenko's extrapolation theorem [26].

At the end of the paper we present examples of particular Young functions, showing how our results can be used for computations of the Boyd indices when the definition seems to be of no practical use.

2. Preliminaries

Let us fix notation and recall basic concepts. For our purposes, a *Young function* will be any non-negative, even, convex function $\Phi : \mathbb{R}^1 \to \mathbb{R}^1$ such that Φ is (strictly) increasing on $[0,\infty)$, and $\lim_{t\to 0} \Phi(t)/t = 0$, $\lim_{t\to\infty} \Phi(t)/t = \infty$. To avoid non-important technicalities we shall suppose that Φ' exists everywhere in $\mathbb{R}^1$.

A Young function Φ satisfies the Δ_2*-condition* ($\Phi \in \Delta_2$) if there is $c > 0$ such that $\Phi(2t) \le c\Phi(t)$ for all $t \in \mathbb{R}^1$.

Let Φ be a Young function. Then $\widetilde{\Phi}(s) = \sup\{|st| - \Phi(t);\ t \in \mathbb{R}^1\}$, $s \in \mathbb{R}^1$, is the so called *complementary function with respect to* Φ. The function Φ is said to be *equivalent* to another Young function Ψ (we shall write $\Phi \sim \Psi$) if there is $c > 0$ such that $\Psi(c^{-1}t) \le \Phi(t) \le \Psi(ct)$, $t \in \mathbb{R}^1$, with some $c > 0$.

Let $\Phi \in \Delta_2$ be a Young function and let us define

$$h_\Phi(\lambda) = \sup_{t>0} \frac{\Phi(\lambda t)}{\Phi(t)}, \qquad \lambda > 0.$$

The numbers

$$i(\Phi) = \lim_{\lambda\to 0_+} \frac{\log h_\Phi(\lambda)}{\log\lambda} = \sup_{0<\lambda<1} \frac{\log h_\Phi(\lambda)}{\log\lambda} \tag{2.1}$$

and

$$I(\Phi) = \lim_{\lambda\to\infty} \frac{\log h_\Phi(\lambda)}{\log\lambda} = \inf_{1<\lambda<\infty} \frac{\log h_\Phi(\lambda)}{\log\lambda} \tag{2.2}$$

are called the *lower index of* Φ and the *upper index of* Φ, respectively. Sometimes these indices are called the *fundamental indices of* Φ.

The numbers $i(\Phi)$ and $I(\Phi)$ are reciprocals of the Boyd indices (see Boyd [4], [5], [6]). The right wing equalities in (2.1) and (2.2) follow from known properties of subadditive functions (since $\log h_\Phi$ enjoys this property), see, e.g. [12], and $I(\Phi) < \infty$ by virtue of the assumption $\Phi \in \Delta_2$.

Always $1 \le i(\Phi) \le I(\Phi)$ and it is $i(\Phi) > 1$ if and only if the complementary function $\widetilde{\Phi}$ satisfies the Δ_2-condition. The couples $i(\widetilde{\Phi})$ and $I(\Phi)$, and $I(\widetilde{\Phi})$ and

$i(\Phi)$ behave similarly as conjugate exponents of power functions (see, e.g. [5], [6], [12]), namely, $i(\widetilde{\Phi}) = I(\Phi)/(I(\Phi)-1)$ and $I(\widetilde{\Phi}) = i(\Phi)/(i(\Phi)-1)$.

Throughout the paper we shall assume that all Young functions under consideration satisfy the Δ_2-condition together with their complementary functions.

We observe that the Boyd indices turned out to be particularly useful in the theory of classical operators in Orlicz spaces. The well-known Muckenhoupt theorem [21] gives a characterization of weights w in $\mathbb{R}^n$ for which the Hardy maximal operator takes $L^p(w)$ boundedly into $L^p(w)$, namely, $w \in A_p$, the Muckenhoupt class. This has been extended to the context of reflexive Orlicz spaces (that is, $\Phi, \widetilde{\Phi} \in \Delta_2$) by Kerman and Torchinsky [13]. They showed that the maximal operator is modularly continuous if and only if w belongs to the Muckenhoupt class $A_{i(\Phi)}$.

Some more connections between the indices and the theory of classical operators in Orlicz spaces can be found in Kokilashvili and Krbec [14, Chapters 2, 3].

In many applications (calculus of variations, interpolation etc.) it is useful to assume that the Young function in question is in a certain sense between two powers t^p and t^q and an appropriate quantitative analysis is needed. At one hand, $\Phi \in \Delta_2$ is equivalent to existence of $p_0, p_1 \in [1,\infty)$, $p_0 \le p_1$, such that

$$\Phi(\lambda t) \le C \max(\lambda^{p_0}, \lambda^{p_1})\, \Phi(t), \qquad \lambda, t \ge 0, \tag{2.3}$$

(see Gustavsson and Peetre [12]) and this in turn gives

$$\Phi(\lambda t) \ge C^{-1} \min(\lambda^{p_0}, \lambda^{p_1})\, \Phi(t), \qquad \lambda, t \ge 0, \tag{2.4}$$

and one can see that sup of those p_0 and inf of those p_1 such that (2.3) and (2.4) hold equals to $i(\Phi)$ and $I(\Phi)$, respectively. We observe that the growth estimates (2.3) and/or (2.4) immediately give the formulas (cf. [16])

$$i(\Phi) = \sup\{p\,;\ \inf_{\substack{u>0\\ \lambda\ge 1}} \lambda^{-p}\frac{\Phi(\lambda u)}{\Phi(u)} > 0\}, \qquad I(\Phi) = \inf\{p\,;\ \sup_{\substack{u>0\\ \lambda\ge 1}} \lambda^{-p}\frac{\Phi(\lambda u)}{\Phi(u)} < \infty\}.$$

On the other hand, a control of a Young function $\Phi \in C^1$ by power functions frequently used, for instance, in connections with applications to PDEs (see, e.g. [9], [24], [25], [20]) has the form (1.1), which can be equivalently written as

$$\frac{\Phi(t)}{t^p} \nearrow \quad \text{and} \quad \frac{\Phi(t)}{t^q} \searrow \quad \text{on} \ \ (0,\infty). \tag{2.5}$$

To distinguish the couples of exponents p, q in (1.1) belonging to different Young functions we shall sometimes write $p = p_\Phi$ and $q = q_\Phi$. The numbers p and q will be called *growth exponents* of Φ.

Let us consider the *Simonenko indices*, see [26], that is, the best p and q such that (1.1) holds:

$$p(\Phi) = \inf_{t>0} \frac{t\Phi'(t)}{\Phi(t)} \quad \text{and} \quad q(\Phi) = \sup_{t>0} \frac{t\Phi'(t)}{\Phi(t)}.$$

It is known ([15, Theorem 5.1]) that $\Phi, \widetilde{\Phi} \in \Delta_2$ if and only if $1 < p(\Phi) \le q(\Phi) < \infty$.

We conclude this section with a survey of useful properties of the lower, upper and Simonenko indices. Not all of them are used in the sequel, but they are listed for completeness. Their straightforward proofs are omitted.

Proposition 2.1. *For $r > 0$ let $e_r(t) = |t|^r$. Then*

$$\begin{aligned}
p(\tilde{\Phi}) &= \frac{q(\Phi)}{q(\Phi)-1}, & q(\tilde{\Phi}) &= \frac{p(\Phi)}{p(\Phi)-1},\\
i(\tilde{\Phi}) &= \frac{I(\Phi)}{I(\Phi)-1}, & I(\tilde{\Phi}) &= \frac{i(\Phi)}{i(\Phi)-1},\\
p(\Phi^{-1}) &= \frac{1}{q(\Phi)}, & q(\Phi^{-1}) &= \frac{1}{p(\Phi)},\\
i(\Phi^{-1}) &= \frac{1}{I(\Phi)}, & I(\Phi^{-1}) &= \frac{1}{i(\Phi)},\\
p(\Phi \circ \Psi) &\ge p(\Phi)p(\Psi), & q(\Phi \circ \Psi) &\le q(\Phi)q(\Psi),\\
p(e_r \circ \Phi) &= p(\Phi)r, & q(e_r \circ \Phi) &= q(\Phi)r,\\
p(\Phi \circ e_r) &= p(\Phi)r, & q(\Phi \circ e_r) &= q(\Phi)r,\\
i(\Phi \circ \Psi) &\ge i(\Phi)i(\Psi), & I(\Phi \circ \Psi) &\le I(\Phi)I(\Psi),\\
i(e_r \circ \Phi) &= i(\Phi)r, & I(e_r \circ \Phi) &= I(\Phi)r,\\
i(\Phi \circ e_r) &= i(\Phi)r, & I(\Phi \circ e_r) &= I(\Phi)r,\\
p(\Phi \cdot \Psi) &\ge p(\Phi) + p(\Psi), & q(\Phi \cdot \Psi) &\le q(\Phi) + q(\Psi),\\
p(\Phi \cdot e_r) &= p(\Phi) + r, & q(\Phi \cdot e_r) &= q(\Phi) + r,\\
i(\Phi \cdot \Psi) &\ge i(\Phi) + i(\Psi), & I(\Phi \cdot \Psi) &\le I(\Phi) + I(\Psi),\\
i(\Phi \cdot e_r) &= i(\Phi) + r, & I(\Phi \cdot e_r) &= I(\Phi) + r.
\end{aligned}$$

3. Characterizations of indices by growth exponents and integral means

The following results are stated for both the lower and the upper indices. Nevertheless, the proofs are analogous or could be deduced thanks to a certain "duality" between $i(\Phi)$ and $I(\Phi)$, therefore we shall restrict ourselves only to the proofs of the part concerning the lower indices.

We start with a quantitative relation between (2.3) and (2.5) (see Gustavsson and Peetre [12] and Persson [22]) that will be useful in the sequel:

Proposition 3.1. *Let Φ be a Young function. Then the following statements are equivalent*

(i) *There are $1 \le p_0, p_1 < \infty$ such that $\Phi(\lambda t) \le C \max(\lambda^{p_0}, \lambda^{p_1})\, \Phi(t)$ for all $\lambda, t \ge 0$, with C independent of λ and t;*

(ii) *Φ is equivalent to a function Θ such that $\Theta(t)/t^{p_0} \nearrow$ and $\Theta(t)/t^{p_1} \searrow$ on* $(0,\infty)$.

Lemma 3.2. *Let a Young function Φ satisfy* (1.1). *Then $p \le p(\Phi) \le i(\Phi) \le I(\Phi) \le q(\Phi) \le q$.*

Proof: Given $\lambda \in (0,1)$ we have by Proposition 3.1 and (2.3) that $\Phi(\lambda t) \le c\lambda^p \Phi(t)$ for every $p \le p(\Phi)$. Hence $h_\Phi(\lambda) \le c\lambda^p$ and

$$\frac{\log h_\Phi(\lambda)}{\log \lambda} \ge p + \frac{\log c}{\log \lambda},$$

implying that $i(\Phi) \ge p$. □

Example 3.3. In general $p(\Phi) \ne i(\Phi)$. Put

$$\Phi(t) = \begin{cases} t^2 & \text{if } t \in [0,1), \\ 2t-1 & \text{if } t \in [1,2), \\ t^2/2+1 & \text{if } t \in [2,\infty), \end{cases}$$

and $\Phi(t) = \Phi(-t)$ for $t < 0$. Then Φ is a Young function with a continuous derivative. An elementary calculation yields that $p(\Phi) = 4/3$ and $q(\Phi) = 2$. On the other hand, by virtue of Theorem 1.3 we get $i(\Phi) = 2$. Let us observe that $t^2/2 \le \Phi(t) \le t^2$, $t \in \mathbb{R}^1$, and $4/3 = p(\Phi) \ne p(t^2) = 2$.

Remark 3.4. When dealing only with the Boyd indices, then the assumption $\Phi \in C^1$ often is not a restriction. Indeed, if $\Phi \in \Delta_2$, then

$$\Theta(t) = \int_0^{|t|} \frac{\Phi(s)}{s}\, ds, \qquad t \in \mathbb{R}^1,$$

is a continuously differentiable Young function equivalent to Φ and such that $\Theta \in \Delta_2$, and $\widetilde{\Theta} \in C^1$. Moreover, if (1.1) holds, then the same is true for Θ. Also, $i(\Phi) = i(\Theta)$, and, if $\widetilde{\Phi} \in \Delta_2$, then $\widetilde{\Theta} \in \Delta_2$. This is a consequence of invariance of these properties with respect to the equivalence of Young functions, namely, if $\Phi \in \Delta_2$ and $\Phi \sim \Psi$, then $\Psi \in \Delta_2$, and if $\Phi \sim \Psi$, then $i(\Phi) = i(\Psi)$ and $I(\Phi) = I(\Psi)$.

Remark 3.5. Let us observe, however, that $\Phi \sim \Psi$ does not generally imply that $p(\Phi) = p(\Psi)$. For instance, a simple computation shows that the Simonenko indices of Φ from Example 3.3 and the corresponding Θ from Remark 3.4 do not coincide. This indicates that relations of $p(\Phi)$ and $q(\Phi)$ at one hand and the Boyd indices at the other hand are of a rather delicate nature.

Given a Young function Φ, let us consider the class of equivalent Young functions. By Lemma 3.2 we know that

$$p(\Lambda) \le i(\Lambda) = i(\Phi) \quad \text{and} \quad I(\Phi) = I(\Lambda) \le q(\Lambda) \quad \text{for all } \Lambda \sim \Phi. \tag{3.1}$$

We are now in a position to prove Theorem 1.1

PROOF OF THEOREM 1.1: By (3.1) we need only to prove that $\sup_{\Lambda\sim\Phi} p(\Lambda) \geq i(\Phi)$. Let $\varepsilon > 0$. Then (see, e.g. [14, Chapter I]) there is C_ε such that

$$\Phi(\lambda t) \leq C_\varepsilon \max\left(\lambda^{i(\Phi)-\varepsilon}, \lambda^{I(\Phi)+\varepsilon}\right)\Phi(t), \qquad \lambda, t \geq 0.$$

By Proposition 3.1 there is an even function Θ_ε with growth exponents $i(\Phi) - \varepsilon$ and $I(\Phi) + \varepsilon$, and equivalent to Φ. Moreover, Θ_ε can be chosen in such a way that it is also a Young function. Indeed, multiplying (1.1), with Θ_ε instead of Φ by t^{-1} and integrating over $(0, s)$, $s > 0$, we arrive at

$$(i(\Phi) - \varepsilon)\Lambda_\varepsilon(s) \leq s\Lambda'_\varepsilon(s) \leq (I(\Phi) + \varepsilon)\Lambda_\varepsilon(s), \qquad s > 0,$$

where

$$\Lambda_\varepsilon(t) = \int_0^{|t|} \frac{\Theta_\varepsilon(s)}{s}\, ds, \qquad t \in \mathbb{R}^1,$$

(cf. Remark 3.4). Therefore the functions Φ and Λ_ε are such that

$$\frac{\Lambda_\varepsilon(t)}{t^{i(\Phi)-\varepsilon}} \nearrow \quad \text{and} \quad \frac{\Lambda_\varepsilon(t)}{t^{I(\Phi)+\varepsilon}} \searrow \quad \text{on} \ \ (0,\infty), \tag{3.2}$$

and, on the other hand, are equivalent because both are equivalent to Θ_ε. But (3.2) implies immediately that $i(\Phi) - \varepsilon \leq p(\Lambda_\varepsilon) \leq \sup_{\Lambda\sim\Phi} p(\Lambda)$ and therefore the proof is complete. $\square$

We shall finish this section with another characterization using the Jensen means

$$\mathcal{M}_\Phi(f) = \Phi^{-1}\left(\frac{1}{|\Omega|}\int_\Omega \Phi(f)\, dx\right),$$

where Ω is a bounded open subset of $\mathbb{R}^n$. Let

$$\mathcal{M}_r(f) = \left(\frac{1}{|\Omega|}\int_\Omega |f|^r\, dx\right)^{1/r}.$$

In [7] there is proved that the growth conditions (1.1) imply the existence of positive constants c_1, c_2 such that

$$c_1\mathcal{M}_p(f) \leq \mathcal{M}_\Phi(f) \leq c_2\mathcal{M}_q(f). \tag{3.3}$$

This can be slightly improved as follows:

Proposition 3.6. *The inequality* (3.3) holds with every $0 < p < i(\Phi)$ and every $I(\Phi) < q < \infty$.

Proof: Let $p < i(\Phi)$. By virtue of Theorem 1.1 there exists $\Lambda \sim \Phi$ such that $p(\Lambda) > p$. Hölder's inequality and the fact that $\Lambda^{-1} \sim \Phi^{-1}$ yield

$$\mathcal{M}_p(f) \le \mathcal{M}_{p(\Lambda)}(f) \le c' \mathcal{M}_\Lambda(f) \le c'' \mathcal{M}_\Phi(f).$$

□

A natural question arises, namely, whether the inequality (3.3) can be used for another characterization of the indices. We shall show that this is indeed the case.

Theorem 3.7. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open subset of $\mathbb{R}^n$. Then*

$$i(\Phi) = \sup\{r \ge 1\,;\ \mathcal{M}_\Phi(f) \ge c\mathcal{M}_r(f) \quad \textit{for some } c > 0\}, \tag{3.4}$$

$$I(\Phi) = \inf\{s \ge 1\,;\ \mathcal{M}_\Phi(f) \le c\mathcal{M}_s(f) \quad \textit{for some } c > 0\}. \tag{3.5}$$

Proof: By Proposition 3.6 we already know that

$$i(\Phi) \le \sup\{r \ge 1\,;\ \mathcal{M}_\Phi(f) \ge c\mathcal{M}_r(f) \quad \text{for some } c > 0\}.$$

Let us assume that there are $\varepsilon > 0$, $c > 0$ such that

$$\mathcal{M}_\Phi(f) \ge c\mathcal{M}_{i(\Phi)+\varepsilon}(f). \tag{3.6}$$

Put $g(t) = (f(t))^{i(\Phi)+\varepsilon}$, $\Psi(t) = \Phi(t^{1/(i(\Phi)+\varepsilon)})$, then $\Psi^{-1}(s) = (\Phi^{-1}(s))^{i(\Phi)+\varepsilon}$. From (3.6) we get that there is $c_1 > 0$ such that

$$\Psi\left(\frac{1}{|\Omega|}\int_\Omega |f|\,dx\right) \le c_1\left(\frac{1}{\Omega}\int_\Omega \Psi(c_1 f)\,dx\right),$$

that is, according to [14, Lemma 1.1.1] the function Ψ is pseudoconvex in the sense that there is a convex function ω and a constant $c_2 > 0$ such that $\omega(t) \le \Psi(t) \le c_2\omega(c_2 t)$ for all $t \ge 0$. It is easy to see that the respective indices of ω and Ψ must coincide, in particular, $i(\Psi) \ge 1$. Our assumptions give $i(\Psi) = i(\Phi)/(i(\Phi)+\varepsilon) < 1$ which is a contradiction. Hence (3.4) holds. □

The preceding theorem yields an alternative proof and actually a mild improvement of a theorem due to Migliaccio [18] on the extrapolation of the reverse Hölder inequality.

Corollary 3.8. *Let $p < i(\Phi)$, $\Omega \subset \mathbb{R}^n$ a bounded open set, and*

$$\frac{1}{|\Omega|}\int_\Omega \Phi(w)\,dx \le b\Phi\Big(\frac{1}{|\Omega|}\int_\Omega w\,dx\Big), \tag{3.7}$$

then

$$\frac{1}{|\Omega|}\int_\Omega w^p\,dx \le c(b,p)\Big(\frac{1}{|\Omega|}\int_\Omega w\,dx\Big)^p.$$

Proof: According to (3.4) we have

$$\Big(\frac{1}{|\Omega|}\int_\Omega w^p\,dx\Big)^{1/p} \le c(p)\Phi^{-1}\Big(\frac{1}{|\Omega|}\int_\Omega \Phi(w)\,dx\Big) \le \frac{c(b,p)}{|\Omega|}\int_\Omega w\,dx$$

and we are done. We observe that the assumption (3.7) is weaker than the original one, when (3.7) is required for εw with every $\varepsilon > 0$. □

4. Proofs of formulas for the Boyd indices

Proofs of Theorems 1.2, 1.3: Let us assume first that r_0 and r_∞ exist. We will divide the proof into the following steps:

Step 1. $i(\Phi) \le \min(r_0, r_\infty)$.

Step 2. $\min(r_0, r_\infty) \le \sup\{\mu > 0\,;\ \int_0^\infty([F_\Phi]_\mu(s) - F_\Phi(s))\,ds/s < \infty\}$.

Step 3. $\sup\{\mu > 0\,;\ \int_0^\infty([F_\Phi]_\mu(s) - F_\Phi(s))\,ds/s < \infty\} \le i(\Phi)$.

Step 1. We shall make use of Theorem 1.1. Let $\Lambda \sim \Phi$ be a Young function. Then $\Lambda(t) = a(t)\Phi(t)$, where $0 < m \le a(t) \le M < \infty$ for all $t > 0$ with m and M independent of t. We have

$$F_\Lambda(t) = \frac{ta'(t)}{a(t)} + F_\Phi(t), \qquad t > 0,$$

and let us point out that

$$\liminf_{t\to 0} F_a(t) \le 0. \tag{4.1}$$

Indeed, assuming that $\liminf_{t\to 0} F_a(t) > \varepsilon > 0$, there is $\delta > 0$ such that $F_a(t) > \varepsilon$ for all $t \in (0,\delta)$ and therefore

$$M \ge a(\delta) - a(0) = \int_0^\delta a'(t)\,dt > \varepsilon\int_0^\delta a(t)\,\frac{dt}{t} > \varepsilon m\int_0^\delta \frac{dt}{t} = \infty.$$

By virtue of (4.1) we have

$$p_\Lambda \le \liminf_{t\to 0} F_\Lambda(t) \le \liminf_{t\to 0} F_a(t) + \lim_{t\to 0} F_\Phi(t) \le r_0,$$
$$i(\Phi) = \sup_{\Lambda\sim\Phi} p_\Lambda \le r_0.$$

The proof of $\liminf_{t\to\infty} F_a(t) \le 0$ is analogous. We arrive at $i(\Phi) \le \min(r_0, r_\infty)$.

Step 2. We claim that

$$\mu < \min(r_0, r_\infty) \Rightarrow \mu \le \sup\{\mu > 0\,;\ \int_0^\infty ([F_\Phi]_\mu(s) - F_\Phi(s))\,\frac{ds}{s} < \infty\}$$

and therefore that supremum is greater than or equal to $\min(r_0, r_\infty) = i(\Phi)$. Indeed, given $\varepsilon > 0$, put $\mu = \min(r_0, r_\infty) - \varepsilon$. Then there is $0 < \delta_1 < \delta_2$ such that $F_\Phi(t) \ge \mu$ for all $t \in (0, \delta_1) \cup (\delta_2, \infty)$ and we have $[F_\Phi]_\mu(s) - F_\Phi(s) = 0$ for all $s \in (0, \delta_1) \cup (\delta_2, \infty)$, hence

$$\int_0^\infty ([F_\Phi]_\mu(s) - F_\Phi(s))\,\frac{ds}{s} < \infty.$$

Step 3. Let $\mu > 0$ be such that the integral in (1.3) is finite and put

$$a(t) = \exp\biggl(\int_1^t ([F_\Phi]_\mu(s) - F_\Phi(s))\,\frac{ds}{s}\biggr), \qquad t \ge 0.$$

By the definition of $[F_\Phi]_\mu$ we have $a \in L_\infty$ and therefore the function

$$\Psi(t) = \int_0^{|t|} \frac{a(s)\Phi(s)}{s}\,ds, \qquad t \in \mathbb{R}^1,$$

is a Young function equivalent to Φ. Further,

$$p(\Psi) \ge \inf_{t>0}\biggl[\frac{t(a(t)\Phi(t))'}{a(t)\Phi(t)}\biggr] = \inf_{t>0}\Bigl[F_\Phi(t) + t(\log a)'(t)\Bigr] = \inf_{t>0}\Bigl[[F_\Phi]_\mu(t)\Bigr] \ge \mu$$

which yields $\mu \le p(\Psi) \le i(\Psi) = i(\Phi)$.

If one of the above limits does not exist, Theorem 1.2 follows from Step 3, and Theorem 1.3 follows by replacing r_0, r_∞ by $\limsup_{t\to 0}$ and $\limsup_{t\to\infty} F_\Phi(t)$, respectively, in the proof of Step 1, and by $\liminf_{t\to 0} F_\Phi(t)$ and $\liminf_{t\to\infty} F_\Phi(t)$, respectively, in the proof of Step 2.

In accordance with our previous agreement we omit the analogous proof of the part of the theorems concerning $I(\Phi)$. $\square$

5. Applications

As we have observed in Introduction we are going to present some applications of the developed theory, following the general pattern, namely replacement of the growth exponents by the Boyd indices in the respective claims.

Let Φ be a Young function such that

$$p\Phi(t) \le t\Phi'(t) \le q\Phi(t), \qquad t \in \mathbb{R}^1, \tag{5.1}$$

with some $1 < p \le q < \infty$. Further, let

$$\mathcal{I}(\Omega, v) = \mathcal{I}_\Phi(\Omega, v) = \int_\Omega \Phi(|Dv|)\,dx, \tag{5.2}$$

where Ω is a bounded open set of $\mathbb{R}^n$ and $v = (v_1, \dots, v_N)$.

A function $u \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ is a *minimizer* of $\mathcal{I}$ if

$$\mathcal{I}(\operatorname{supp}\varphi, u) \le \mathcal{I}(\operatorname{supp}\varphi, u + \varphi)$$

for every $\varphi \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ supported in Ω.

Let $Q \ge 1$. A function $u \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ is a *Q-quasiminimizer* of $\mathcal{I}$ if

$$\mathcal{I}(\operatorname{supp}\varphi, u) \le Q\mathcal{I}(\operatorname{supp}\varphi, u + \varphi)$$

for every $\varphi \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ supported in Ω.

Regularity of minimizers for the functionals in (5.2) have been studied in Fusco and Sbordone [9]. They proved that if u is a minimizer of $\mathcal{I}$ then the condition (5.1) together with

$$q < p^* = \frac{np}{n-p} \quad \text{if } p < n \tag{5.3}$$

guarantee that there is $r > 1$ such that $\Phi(|Du|) \in L^r_{\mathrm{loc}}(\Omega)$. Note that they use the Young function from Example 5.8 to illustrate the result.

The minima of functionals of type (5.2) have been studied by Giaquinta and Giusti [10], Giaquinta and Modica [11], Sbordone [23] under the condition

$$c_1 t^p - c_2 \le \Phi(t) \le c_3(1 + t^q), \qquad t > 0, \tag{5.4}$$

with $p = q$. It is easy to see that (5.1) implies (5.4), nevertheless, as observed in [9] the growth exponents from (5.1) need not be necessarily the best ones for which (5.4) holds; if $\overline{p}$ and $\overline{q}$ are the best possible exponents from (5.4), then $p \le \overline{p} \le \overline{q} \le q$ and in general any of these inequalities can be sharp.

More generally, let $w \in A_m$ with some $m > 1$ and consider

$$\mathcal{J}(\Omega, v) = \mathcal{J}_\Phi(\Omega, v) = \int_\Omega \Phi(|Dv|) w\,dx. \tag{5.5}$$

Let

$$m_0 = \inf\{m \ge 1\,;\ w \in A_m\}. \tag{5.6}$$

Sbordone [24] has proved that the condition

$$\frac{nm_0 q}{nm_0 + q} < p \le q < nm_0 \tag{5.7}$$

implies that $\Phi^r(|Du|)$ is locally integrable in Ω with some $r > 1$.

The condition (5.3) can be rewritten as

$$\frac{1}{p} - \frac{1}{q} < \frac{1}{n} \qquad \text{if } p < n$$

so that (5.3) is actually a condition on the distance of the reciprocals of the growth exponents p and q; the situation is similar as to the left wing inequality in (5.7). Recalling Lemma 3.2 we see that the conditions

$$I(\Phi) < (i(\Phi))^* = \frac{ni(\Phi)}{n - i(\Phi)} \tag{5.8}$$

if $i(\Phi) < n$, and

$$\frac{nm_0 I(\Phi)}{nm_0 + I(\Phi)} < i(\Phi) \le I(\Phi) < nm_0 \tag{5.9}$$

are weaker than (5.3) and (5.7), respectively.

As noticed in [9] the regularity statement for (5.2) holds for Q-quasiminimizers, too. In fact, giving a closer look at the proof in [9] one can see that the original assumption about u is only used for proving the Caccioppoli type inequality

$$\int_{B_{R/2}} \Phi(|Du|)\, dx \le c \int_{B_{R/2}} \Phi\left(\frac{u - u_R}{R}\right) dx, \tag{5.10}$$

where $B_R \subset\subset \Omega$ is any ball of the radius R. Going along the lines of the proof it becomes clear that the inequality (5.10) with a possibly different constant c can be proved without any extra effort if u is a Q-quasiminimizer, too.

We can now present several applications.

Theorem 5.1.

(i) *Let Φ be a Young function and let $u \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ be a Q-quasiminimizer of* (5.2)*. Suppose that* (5.8) *holds provided $i(\Phi) < n$. Then there is $r > 1$ such that $\Phi^r(|Du|) \in L^1_{\mathrm{loc}}(\Omega)$.*

(ii) *Let Φ be a Young function and let $u \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ be a Q-quasiminimizer of* (5.5) *with $w \in A_m$ for some $m > 1$. Let m_0 be the critical index from* (5.6)*. If* (5.9) *holds, then there is $r > 1$ such that $\Phi^r(|Du|) \in L^1_{\mathrm{loc}}(\Omega)$.*

In the paper by Moscariello [20], the condition (5.3) has been also used to prove a Harnack type inequality for scalar-valued Q-quasiminimizers of (5.2), namely, if $u \ge 0$ is a Q-quasiminimizer of (5.2) and $B_R \subset\subset \Omega$ is a ball, then for every $\sigma \in (0,1)$ there exists a constant $C = C(p, q, Q, n, \sigma)$ such that

$$\sup_{x \in B_{\sigma R}} u(x) \le C \inf_{x \in B_{\sigma R}} u(x).$$

(See Di Benedetto and Trudinger [2] for the original L^p-setting.)

This result can also be formulated in terms of the Boyd indices:

Theorem 5.2. *Let $n > 1$ and suppose that* (5.8) *holds provided $i(\Phi) < n$. Let B_R be a ball whose closure is contained in Ω. If $u \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^1)$ is a non-negative Q-quasiminimizer of* (5.2)*, then for every $\sigma \in (0,1)$ there is $C = C(p, q, Q, n, \sigma)$ such that*

$$\sup_{x \in B_{\sigma R}} u(x) \leq C \inf_{x \in B_{\sigma R}} u(x).$$

Proofs of Theorems 5.1, 5.2: We restrict ourselves to the proof of Theorem 5.1 (i). After that it will be clear how to proceed in the remaining cases.

Let u be a Q-quasiminimizer of (5.2) with Φ satisfying the condition (5.8). Choose $\varepsilon > 0$ in such a way that

$$I(\Phi) + \varepsilon \leq (i(\Phi) - \varepsilon)^*. \tag{5.11}$$

We know (see the proof of Theorem 1.1) that there is a Young function $\Lambda_\varepsilon \sim \Phi$ such that

$$p(\Lambda_\varepsilon) \geq i(\Phi) - \varepsilon, \qquad q(\Lambda_\varepsilon) \leq I(\Phi) + \varepsilon. \tag{5.12}$$

Then the functional

$$\mathcal{I}_{\Lambda_\varepsilon}(\Omega, v) = \int_\Omega \Lambda_\varepsilon(|Dv|)\, dx$$

is of type (5.2) and by virtue of (5.11) and (5.12) the growth exponents of Λ_ε satisfy inequality (5.3). As Q-quasiminima of Λ_ε are Q-quasiminima of Φ (with possibly different Q) and vice versa we are done. □

Remark 5.3. The calculus of the indices $i(\Phi)$ and $I(\Phi)$ can be difficult in particular cases, for instance, when Theorems 1.2, 1.3 are not sufficient. On the other hand, the behaviour of Φ near the origin, although relevant for $i(\Phi)$, does not in fact play any role in Theorem 5.1 (cf. the final remarks in [9] and [24]). This suggests another couple of indices. If we put

$$\widetilde{p} = \widetilde{p}_\Phi = \liminf_{t \to \infty} \frac{t\Phi'(t)}{\Phi(t)}, \qquad \widetilde{q} = \widetilde{q}_\Phi = \limsup_{t \to \infty} \frac{t\Phi'(t)}{\Phi(t)},$$

then $p_\Phi \leq \widetilde{p}_\Phi \leq \widetilde{q}_\Phi \leq q_\Phi$, and therefore the conditions

$$\widetilde{p}_\Phi \leq \widetilde{q}_\Phi < (\widetilde{p}_\Phi)^* = \frac{n\widetilde{p}_\Phi}{n - \widetilde{p}_\Phi} \tag{5.13}$$

and

$$\frac{nm_0\widetilde{q}_\Phi}{nm_0 + \widetilde{q}_\Phi} < \widetilde{p}_\Phi \leq \widetilde{q}_\Phi < nm_0 \tag{5.14}$$

are weaker than (5.3) and (5.7), respectively.

It is possible to prove a result better than Theorem 5.1. To this goal we shall need the following special construction:

Lemma 5.4. *Let Φ be a Young function such that*

$$1 < \ell \leq \liminf_{t\to\infty} \frac{t\Phi'(t)}{\Phi(t)}.$$

Then there is a Young function Ψ such that

$$c_1\Phi(t) - c_2 \leq \Psi(t) \leq c_3\Phi(t) + c_4, \qquad t \in \mathbb{R}^1, \tag{5.15}$$

for some $c_1, c_2, c_3, c_4 > 0$ independent of t, and

$$i(\Psi) = \ell, \qquad I(\Psi) \leq \limsup_{t\to\infty} \frac{t\Phi'(t)}{\Phi(t)}. \tag{5.16}$$

Proof: *Step* 1. We show that there is a Young function G and constants $c_1', c_2' > 0$ such that

$$c_1'\Phi(t) - c_2' \leq G(t) \leq \Phi(t), \qquad t \in \mathbb{R}^1, \tag{5.17}$$

and

$$q_G > \ell. \tag{5.18}$$

If $q_\Phi > \ell$ it suffices to put $G = \Phi$ and we are done. If this is not the case, let $c > 0$ be such that

$$\frac{2\Phi'(2)}{\Phi(2) - c} > \ell, \qquad \frac{\Phi(2) - c}{\Phi(1) + \Phi'(1)} < 1,$$

and define

$$G(t) = \begin{cases} \dfrac{\Phi(2) - c}{\Phi(1) + \Phi'(1)}\,\Phi(t) & \text{if } |t| \leq 1, \\ \dfrac{\Phi(2) - c}{\Phi(1) + \Phi'(1)}\,[\Phi'(1)(|t| - 1) + \Phi(1)] & \text{if } 1 < |t| < 2, \\ \Phi(t) - c & \text{if } |t| \geq 2. \end{cases}$$

It is easy to check that G satisfies (5.17) and (5.18).

Step 2. We shall construct Ψ. Choose $t_0 > 0$ such that $t_0G'(t_0)/G(t_0) > \ell$ and put

$$\Psi(t) = \begin{cases} \dfrac{G(t_0)}{t_0^\ell}|t|^\ell & \text{if } |t| \leq t_0, \\ G(t) & \text{if } |t| > t_0. \end{cases}$$

Then Ψ is a Young function; its convexity follows from

$$\lim_{t\to t_0-} \Psi'(t) = \frac{\ell G(t_0)}{t_0^\ell}\, t_0^{\ell-1} = \frac{\ell G(t_0)}{t_0} < G'(t_0) \le \lim_{t\to t_0+} G'(t).$$

Moreover, $G(t) - G(t_0) \le \Psi(t) \le G(t) + G(t_0)$, $t \in \mathbb{R}^1$, hence

$$\frac{\Phi(2) - c}{\Phi(1) + \Phi'(1)}\, \Phi(t) - G(t_0) \le \Psi(t) \le G(t) + G(t_0) \le \Phi(t) + \Phi(t_0), \qquad t \in \mathbb{R}^1,$$

and (5.15) is proved.

Step 3. We shall prove (5.16). By Theorem 1.3 we have

$$\begin{aligned} i(\Psi) &\le \min\left(\limsup_{t\to 0} \frac{t\Psi'(t)}{\Psi(t)}\,, \limsup_{t\to\infty} \frac{t\Psi'(t)}{\Psi(t)}\right) = \min\left(\ell\,, \limsup_{t\to\infty} \frac{tG'(t)}{G(t)}\right) \\ &= \min\left(\ell\,, \limsup_{t\to\infty} \frac{t\Phi'(t)}{\Phi(t)}\right) = \ell. \end{aligned}$$

On the other hand,

$$\begin{aligned} i(\Psi) &\ge \min\left(\liminf_{t\to 0} \frac{t\Psi'(t)}{\Psi(t)}\,, \liminf_{t\to\infty} \frac{t\Psi'(t)}{\Psi(t)}\right) = \min\left(\ell\,, \liminf_{t\to\infty} \frac{tG'(t)}{G(t)}\right) \\ &= \min\left(\ell\,, \liminf_{t\to\infty} \frac{t\Phi'(t)}{\Phi(t)}\right) = \ell. \end{aligned}$$

The estimate for $I(\Psi)$ follows directly from Theorem 1.3 because $\Psi = G$ near infinity, and G and Φ differ by a constant on $(2, \infty)$. $\square$

Similarly, one can prove the “dual” statement:

Lemma 5.5. *Let Φ be a Young function such that*

$$\limsup_{t\to\infty} \frac{t\Phi'(t)}{\Phi(t)} \le \ell < \infty.$$

Then there is a Young function Ψ such that

$$c_1\Phi(t) - c_2 \le \Psi(t) \le c_3\Phi(t) + c_4, \qquad t \in \mathbb{R}^1, \tag{5.19}$$

for some $c_1, c_2, c_3, c_4 > 0$ independent of t, and

$$I(\Psi) = \ell, \qquad i(\Psi) \ge \liminf_{t\to\infty} \frac{t\Phi'(t)}{\Phi(t)}. \tag{5.20}$$

Combining Lemma 5.4 with the foregoing considerations we arrive at

Theorem 5.6. *The assertions* (i) *and* (ii) *in Theorem* 5.1 *hold also under the conditions* (5.13) *with* $\widetilde{p}_\Phi < n$ *and* (5.14), *respectively.*

Next we shall pay attention to one of the interesting pioneering results due to Simonenko [26] on extrapolation of integral operators with a kernel, which enables very simply and naturally to carry over for instance the theorems on singular integrals from L^p to L_Φ spaces with finite indices. The original assumption in [26] for the claim in the following theorem reads $1 < p(\Phi) \le q(\Phi) < q$.

Theorem 5.7. *Let*

$$Kf(x) = \int_\Omega A(x,y)f(y)\,dy$$

and let us suppose that K *is continuous in* $L^q(\Omega)$. *Let* $B(x_0,r)$ *denote a ball centered at* x_0 *and of the radius* r. *Assume that*

$$\sup \int_{(B(x_0,r)^c)\cap\Omega} |A(x,y) - A(x,y')|\,dx < \infty,$$

where the sup is taken over all $y, y' \in B(x_0, 2r)$, $x_0 \in \mathbb{R}^n$, $r > 0$, *and* $(B(x_0,r))^c$ *denotes the complement of* $B(x_0,r)$. *If* Φ *is a Young function such that* $1 < i(\Phi) \le I(\Phi) < q$, *then* K *is continuous in* $L_\Phi(\Omega)$.

Proof: It is clear that the function Φ can be substituted by an equivalent Young function in the claim and at the same time we see that this cannot be done in the assumption. Nevertheless, arguing as in the proof of Theorem 1.1, we find an equivalent Young function Λ_ε whose Simonenko index $q(\Phi)$ is arbitrarily near to $I(\Phi)$ and we apply Simonenko's theorem from [26] to K in L_{Λ_ε}. □

One can think about possible applications in the above described spirit to other problems, too, since the growth condition (1.1) appears also in connections with areas which we have not touched here. For instance, (1.1) plays a major role in the recent paper by Aïssaoui [1] on Bessel potentials in Orlicz spaces.

Example 5.8. The following examples illustrate possible behaviour of Young functions. The verification is a matter of simple calculation.

(1) Let $\Phi(t) = e|t|^3$ if $|t| < e$ and $\Phi(t) = t^{4+\sin\log\log t}$ if $|t| \ge e$ (cf. [9]). Then $p(\Phi) = 4 - \sqrt{2}$, $q(\Phi) = 4 + \sqrt{2}$, further, (5.4) holds with $p = 3$ and $q = 5$. At the same time, using just the definition, it is likely extremely difficult to find the lower and the upper indices. Further, the limit r_∞ does not exist so that Theorem 1.3 cannot be used. Nevertheless, invoking Theorem 1.2, we simply obtain that $i(\Phi)$ and $I(\Phi)$ coincide with $p(\Phi)$ and $q(\Phi)$, respectively. Indeed, every truncation of F_Φ by $\mu > 4 - \sqrt{2}$ makes the integral in (4.1) infinite; similarly for the upper index.

(2) Put $\Phi(0) = 0$ and $\Phi(t) = |t|^r \exp(\sqrt{1 + s\log^+|t|})$ otherwise, $r \ge 1$, $s > 0$ (Talenti [27]). Then $i(\Phi) = I(\Phi) = r$ simply by Theorem 1.3. As to the Simonenko indices we have $p(\Phi) = r$ and $q(\Phi) = r + s/2$. Notice that $q(\Phi)$ depends on s while $I(\Phi)$ does not.

Acknowledgment. This paper was finished when the first author visited Institute of Mathematics of the Czech Academy of Sciences in Prague. He would like to thank for support and hospitality.

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(*Received* October 24, 1996)