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On CCC boolean algebras and partial orders

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Abstract. We partially strengthen a result of Shelah from [Sh] by proving that if $\kappa = \kappa^\omega$ and P is a CCC partial order with e.g. $|P| \le \kappa^{+\omega}$ (the ω^{th} successor of κ) and $|P| \le 2^\kappa$ then P is κ-linked.

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Shelah has proved in [Sh] that if κ is a cardinal with $\kappa^\omega = \kappa$ then every CCC boolean algebra B with $|B| \le \kappa^+$ is κ-centered. Equivalently, this means that every CCC compact Hausdorff space X of weight $w(X) \le \kappa^+$ has density $d(X) \le \kappa$.

Since $w(X) \le 2^{d(X)}$ is always valid for a compact T_2 space X, it is natural to raise the question whether κ^+ could be replaced by 2^κ in the above result. Shelah mentions in [Sh] without proof that, at least consistently, this cannot be done. Moreover, we have recently shown in [HJSz] that there is, in ZFC, a compact CCC Hausdorff space of density ω_2 and weight 2^{ω_2}. Thus if $2^\omega = \omega_1$ and $2^{\omega_1} = 2^{\omega_2} = \omega_3$ this yields a CCC compact T_2 space of weight $\omega_3 = 2^{\omega_1}$ with density greater than ω_1, or equivalently a CCC boolean algebra of size $\omega_3 = 2^{\omega_1}$ that is not ω_1-centered.

Our aim in this note is to show that some strengthenings of Shelah's result are nonetheless provable for higher successors of κ. Let us recall for this purpose that a subset A of a partially ordered set $\langle P, \le\rangle$ is said to be *linked* if for any $p, q \in A$ there is $r \in P$ with $r \le p, q$, i.e. any two members of A are compatible. We say that P is κ-linked if it is the union of κ many linked subsets and we write

$$\operatorname{link}(P) = \min\{\kappa \ge \omega : P \text{ is } \kappa\text{-linked}\}.$$

If B is a boolean algebra then, of course, we put $\operatorname{link}(B) = \operatorname{link}(B^+)$.

We have, implicitly, referred above to the fact that any κ-centered boolean algebra B satisfies $|B| \le 2^\kappa$. In fact, the following stronger result is easily provable.

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Lemma 1. *If B is a boolean algebra then $|B| \le 2^{\mathrm{link}(B)}$.*

Proof: Let $B^+ = \bigcup\{A_\alpha : \alpha \in \kappa\}$ where each A_α is linked. By Zorn's lemma, we may actually assume that A_α is a maximal linked subset of B^+ for all $\alpha \in \kappa$. Given $b \in B^+$, let

$$I_b = \{\alpha \in \kappa : b \in A_\alpha\}.$$

Clearly, it suffices to show that if $b \ne b'$ then $I_b \ne I_{b'}$.

Assume that $b - b' \ne 0$ and fix $\alpha \in \kappa$ with $b - b' \in A_\alpha$. Then $b' \notin A_\alpha$ since A_α is linked, while $b \in A_\alpha$ follows from the maximality of A_α. Thus we see that $I_b \ne I_{b'}$. □

We may now formulate a partial strengthening of Shelah's result as follows.

Theorem 2. *Let $\kappa = \kappa^\omega$ and B be a CCC boolean algebra with $|B| \le 2^\kappa$ and also satisfying the following condition $(*)$:*

- $(*)$ *for every cardinal μ if $\kappa < \mu < |B|$ and $\mathrm{cf}(\mu) = \omega$ then $\mu^\omega = \mu^+$ and $\square_\mu$ hold.*

  Then B is κ-linked.

The proof of this theorem is based on the following lemma.

Lemma 3. *Assume $\kappa^\omega = \kappa$ and that B is a boolean algebra which can be written as*

$$B = \bigcup\{X_\alpha : \alpha \in \lambda\},$$

where $\lambda \le 2^\kappa$ and $\{X_\alpha : \alpha \in \lambda\}$ is an increasing and continuous sequence of subsets of B such that for each $\alpha \in \lambda$ we have $X_\alpha = \bigcup\{B_\alpha^n : n \in \omega\}$ with every B_α^n being a subalgebra of B that is complete and κ-linked. Then B is also κ-linked.

Proof: Let us start by defining for any $b \in B$ and $\langle \alpha, n\rangle \in \lambda \times \omega$ the element $\pi_\alpha^n(b)$ of B_α^n by

$$\pi_\alpha^n(b) = \bigwedge\{a \in B_\alpha^n : b \le a\}.$$

This is always possible since B_α^n is complete.

Then we define $\sigma(b)$ as the set of those $\alpha \in \lambda$ for which there is some $b_\alpha \in X_\alpha$ such that $b \le b_\alpha$ and for every $c \in X_\beta$ with $\beta < \alpha$ and $b < c$ we have $c - b_\alpha \ne 0$.

We claim that $\sigma(b)$ is a countable subset of λ. Indeed, let us assume indirectly that $|\sigma(b)| \ge \omega_1$ and let $\{\alpha_\xi : \xi \in \omega_1\}$ enumerate in the increasing order the first ω_1 members of $\sigma(b)$. For each $\xi \in \omega_1$, since $\alpha_\xi \in \sigma(b)$ there is some $b_\xi \in X_{\alpha_\xi}$ such that $b \le b_\xi$ and $c - b_\xi \ne 0$ whenever $b \le c$ and $c \in \bigcup\{X_\beta : \beta \in \alpha_\xi\}$.

If $\alpha = \bigcup\{\alpha_\xi : \xi \in \omega_1\}$ then there is $n \in \omega$ such that the set

$$z = \{\xi \in \omega_1 : b_\xi \in B_\alpha^n\}$$

is uncountable and hence $\{\alpha_\xi : \xi \in z\}$ is cofinal in α. Now, $\{b_\xi : \xi \in z\} \subset B_\alpha^n$, hence we have $c = \bigwedge\{b_\xi : \xi \in z\} \in B_\alpha^n$, as B_α^n is complete. By continuity, there is some $\beta < \alpha$ with $c \in X_\beta$. But there is some $\xi \in z$ with $\beta < \alpha_\xi$ as well, and then $b \le c \le b_\xi$ contradicts the choice of b_ξ.

Next, given two elements $a, b \in B^+$ we call them "connected", and denote this by $a \sim b$, if for each $\alpha \in \sigma(a) \cap \sigma(b)$ and for every $n \in \omega$ we have

$$\pi^n_\alpha(a) \wedge \pi^n_\alpha(b) \neq 0.$$

We prove then that $a \sim b$ implies $a \wedge b \neq 0$.

Indeed, if $a \wedge b = 0$ then let α be the smallest cardinal for which there is some $c \in X_\alpha$ that separates a and b, i.e. $a \le c$ and $b \wedge c = 0$. We first show that

$$\alpha \in \sigma(a) \cap \sigma(b).$$

That $\alpha \in \sigma(a)$ is witnessed by c, because if $\beta < \alpha$ and $d \in X_\beta$ with $a \le d$ then d cannot separate a and b by the minimality of α, hence $d \wedge b \neq 0$, consequently $d - c \neq 0$. Similarly, we can show that $-c$ witnesses $\alpha \in \sigma(b)$. Let us now choose $n \in \omega$ such that $c \in B^n_\alpha$ (and so $-c \in B^n_\alpha$). Then $a \le c$ implies $\pi^n_\alpha(a) \le c$ and similarly $b \le -c$ implies $\pi^n_\alpha(b) \le -c$, consequently $\pi^n_\alpha(a) \wedge \pi^n_\alpha(b) = 0$, showing that a and b are not connected.

Given $\langle \alpha, n \rangle \in \lambda \times \omega$ let

$$\mathcal{L}^n_\alpha = \{L^n_\alpha(\nu) : \nu \in \kappa\}$$

be a family of linked subsets of B^n_α with $(B^n_\alpha)^+ = \bigcup \mathcal{L}^n_\alpha$, i.e. $\mathcal{L}^n_\alpha$ shows the κ-linkedness of B^n_α.

Consider ${}^{\lambda\times\omega}\kappa$ as a power of the discrete space $D(\kappa)$ of size κ with the countable support product topology. It is well-known (see e.g. [EK]) that since $\kappa = \kappa^\omega$ and $|\lambda \times \omega| = \lambda \le 2^\kappa$ this space has a dense subset $H \subset {}^{(\lambda\times\omega)}\kappa$ with $|H| = \kappa$.

For any $b \in B^+$ let s_b be the function with domain $\sigma(b) \times \omega$ and having for any $\alpha \in \sigma(b)$ and $n \in \omega$ the value

$$s_b(\alpha, n) = \min\{\nu \in \kappa : \pi^n_\alpha(b) \in L^n_\alpha(\nu)\}.$$

Then s_b determines a basic open set in the above mentioned countable support product space, hence there is some $h \in H$ with $s_b \subset h$ as H is dense in this space. In other words, if we set for $h \in H$

$$L_h = \{b \in B^+ : s_b \subset h\},$$

then

$$B^+ = \bigcup\{L_h : h \in H\},$$

hence we shall be done if we can show that L_h is linked for every $h \in H$.

This, in turn, follows from the following observation: any two members of L_h are connected. Indeed, if $a, b \in L_h$ then $s_a \cup s_b \subset h$, in particular the functions

s_a and s_b are compatible. But this means that for any $\alpha \in \sigma(a) \cap \sigma(b)$ and $n \in \omega$ we have

$$s_a(\alpha, n) = s_b(\alpha, n) = h(\alpha, n) = \nu,$$

hence both $\pi^n_\alpha(a)$ and $\pi^n_\alpha(b)$ belong to $L^n_\alpha(\nu)$, i.e. $\pi^n_\alpha(a) \wedge \pi^n_\alpha(b) \neq 0$. $\square$

The proof of Lemma 3 has thus been completed, and we can now return to that of Theorem 2.

Proof of Theorem 2: We do induction on $\lambda = |B|$. Of course, we may assume that $\kappa < \lambda$. So let us assume that $\lambda > \kappa$ is given and Theorem 2 holds for $|B| < \lambda$. We will distinguish three cases.

Case 1. $\mathrm{cf}(\lambda) = \omega$. Now we can write $B = \bigcup\{B_n : n \in \omega\}$, where B_n is a subalgebra of B with $|B| < \lambda$ for each $n \in \omega$. Then B is κ-linked because so is every B_n.

Case 2. $\mathrm{cf}(\lambda) > \omega$ and $\mu < \lambda$ implies $\mu^\omega < \lambda$. Clearly, in this case we have $\lambda^\omega = \lambda$, hence the completion of B also has cardinality λ, hence we may actually assume that B is complete. Standard arguments, using that B is CCC and $\mu^\omega < \lambda$ for $\mu < \lambda$, then imply that we can write B in the form

$$B = \bigcup\{B_\alpha : \alpha \in \lambda\},$$

where $\{B_\alpha : \alpha \in \lambda\}$ is an increasing and continuous sequence of subalgebras of B such that $|B_\alpha| < \lambda$, moreover B_α is complete whenever $\mathrm{cf}(\alpha) > \omega$. Note that this automatically implies that for $\mathrm{cf}(\alpha) = \omega$ the subalgebra B_α is the union of countably many complete subalgebras of B, hence with $B_\alpha = X_\alpha$ all the assumptions of Lemma 2 are clearly satisfied. Consequently, B is κ-linked.

Case 3. $\lambda = \mu^+$ with $\mathrm{cf}(\mu) = \omega$. (Note that, by $(*)$, this must occur if neither Case 1 nor Case 2 applies.) Again, by $\lambda^\omega = \lambda$ we may assume that B is also complete. Let us then index the members of B by the ordinals below λ, i.e. set $B = \{b_\xi : \xi \in \lambda\}$.

Since $(*)$ also implies $\square_\mu$, let us fix a corresponding $\square$-sequence $\langle C_\alpha : \alpha \in \lambda' \rangle$, that is for each limit ordinal $\alpha \in \lambda$ then C_α is a closed unbounded subset of α such that $|C_\alpha| < \mu$ and $C_\beta = \beta \cap C_\alpha$ whenever $\beta \in C'_\alpha$.

Using $\mathrm{cf}(\mu) = \omega$ we may write $\mu = \sum\{\mu_n : n \in \omega\}$ with $\mu_n < \mu$ for each $n \in \omega$, moreover every ordinal $\beta \in \lambda$ can be written as $\beta = \bigcup\{S^n_\beta : n \in \omega\}$ with $|S^n_\beta| \leq \mu_n$ for all $n \in \omega$. Next, if $\alpha \in \lambda'$ is a limit ordinal then we set $T^n_\alpha = \bigcup\{S^n_\beta : \beta \in C_\alpha\}$. Then we have $|T^n_\alpha| \leq |C_\alpha| \cdot \mu_n < \mu$.

It is clear from $(*)$ that μ must be ω-inaccessible, i.e. $\varrho < \mu$ implies $\varrho^\omega < \mu$. This and the fact that B is CCC imply that for any subset $A \subset B$ if $|A| < \mu$ then $|\mathrm{gen}(A)| < \mu$ as well, where $\mathrm{gen}(A)$ denotes the *complete* subalgebra of B generated by A. In particular, we always have

$$|\mathrm{gen}(\{b_\xi : \xi \in S^n_\beta\})| < \mu$$

and

$$|\operatorname{gen}(\{b_\xi : \xi \in T^n_\alpha\})| < \mu$$

for any $\beta \in \lambda$, $\alpha \in \lambda'$, $n \in \omega$. Consequently, if we let D denote the set of those limit ordinals $\delta \in \lambda$ that satisfy both

$$\{\eta : b_\eta \in \operatorname{gen}(\{b_\xi : \xi \in S^n_\beta\})\} \subset \delta$$

for each $\langle \beta, n \rangle \in \delta \times \omega$ and

$$\{\eta : b_\eta \in \operatorname{gen}(\{b_\xi : \xi \in T^n_\alpha\})\} \subset \delta$$

for all limit ordinals $\alpha \in \delta$ and $n \in \omega$, then D is closed and unbounded in λ.

Let $D = \{\delta_\nu : \nu \in \lambda\}$ be the increasing (and continuous) enumeration of D and set for each $\nu \in \lambda$

$$X_\nu = \{b_\xi : \xi \in \delta_\nu\}.$$

We claim that $\{X_\nu : \nu \in \lambda\}$ satisfies the conditions of Lemma 3. That it forms an increasing and continuous sequence with B as its union is obvious. To see the rest, it will clearly suffice to show that each X_ν is the union of countably many complete subalgebras of B, for (by the inductive hypothesis) they must all be κ-linked.

Here we have to distinguish two cases. First, if $\operatorname{cf}(\delta_\nu) = \omega$ then we may choose ordinals $\{\beta_i : i \in \omega\} \subset \delta_\nu$ with $\delta_\nu = \bigcup\{\beta_i : i \in \omega\}$ and observe that from

$$\beta_i = \bigcup\{S^n_{\beta_i} : n \in \omega\}$$

and from $\delta_\nu \in D$ we have

$$X_\nu = \bigcup\{\operatorname{gen}(\{b_\xi : \xi \in S^n_{\beta_i}\}) : \langle n, i \rangle \in \omega^2\}.$$

Secondly, if $\operatorname{cf}(\delta_\nu) > \omega$ then we have

$$X_\nu = \bigcup\{\operatorname{gen}(\{b_\xi : \xi \in T^n_{\delta_\nu}\}) : n \in \omega\}.$$

Indeed, this follows from the fact that if $\operatorname{cf}(\delta_\nu) > \omega$ then $\alpha, \beta \in C'_{\delta_\nu}$ with $\alpha \in \beta$ imply $T^n_\alpha \subset T^n_\beta$, moreover we also have

$$T^n_{\delta_\nu} = \bigcup\{T^n_\alpha : \alpha \in C'_{\delta_\nu}\}$$

and

$$\delta_\nu = \bigcup\{T^n_{\delta_\nu} : n \in \omega\}.$$

The proof is now completed, since we have shown that Lemma 3 can be applied to $\{X_\nu : \nu \in \lambda\}$ and consequently B is κ-linked. $\square$

Let us recall now the well-known fact that if P is a CCC partial ordering then its completion B is a CCC boolean algebra with $|B| \geq |P|^\omega$ (see e.g. [K, II. 3.3]). Consequently we immediately obtain the following equivalent formulation of Theorem 2.

Theorem 2′. *Let $\kappa = \kappa^\omega$ and P be a CCC partial ordering such that $|P| \le 2^\kappa$, moreover if $\kappa < \mu < |P|$ and $\mathrm{cf}\,(\mu) = \omega$ then $\mu^\omega = \mu^+$ and $\Box_\mu$ holds. Then P is κ-linked.*

Note that if 2^κ is a finite successor of κ, i.e. $2^\kappa < \kappa^{+\omega}$, then the latter condition is automatically satisfied.

Now, if $\mathcal{G}$ is any graph and $Q(\mathcal{G})$ is the partial order of finite $\mathcal{G}$-independent sets (see e.g. [HJSz]) then it is easy to see that $Q(\mathcal{G})$ is κ-linked if and only if it is κ-centered. Consequently, if e.g. $\kappa = \kappa^\omega$ and $2^\kappa < \kappa^{+\omega}$ and $\mathcal{G}$ is a graph for which $Q(\mathcal{G})$ is CCC and $|\mathcal{G}| \le 2^\kappa$ then $Q(\mathcal{G})$ must be κ-centered. In particular, we obtain the following result which shows that the use of hypergraphs, as opposed to just ordinary graphs, was essential in [HJSz] in producing ZFC examples of CCC partial orders with prescribed centeredness.

Corollary 4. *Let $\kappa = \kappa^\omega < \lambda < 2^\lambda = 2^\kappa < \kappa^{+\omega}$. Then there is no CCC partial order P with $\mathrm{link}\,(P) = \lambda$. In particular, there is no graph $\mathcal{G}$ such that $Q(\mathcal{G})$ is CCC and $\mathrm{cent}\,(\mathcal{G}) = \mathrm{cent}\,(Q(\mathcal{G})) = \lambda$.*

Proof: Assume, indirectly, that $\mathrm{link}\,(P) = \lambda$. Then for the completion B of P we also have $\mathrm{link}\,(B) = \lambda$, hence by Lemma 1 we have $|B| \le 2^\lambda = 2^\kappa < \kappa^{+\omega}$. Consequently Theorem 2 applies to B and thus we have

$$\mathrm{link}\,(B) = \mathrm{link}\,(P) \le \kappa < \lambda,$$

which is a contradiction. $\Box$

As a particular case, we get for instance that $2^\omega = \omega_1$ and $2^{\omega_1} = 2^{\omega_2} = \omega_3$ imply that there is no CCC partial order of linkedness ω_2, in particular there is no graph $\mathcal{G}$ for which $Q(\mathcal{G})$ is CCC and $\mathrm{cent}\,(\mathcal{G}) = \omega_2$.

Of course, Corollary 4 remains valid if instead of $2^\kappa < \kappa^{+\omega}$ we only assume the weaker condition that $\kappa < \mu < 2^\kappa$ and $\mathrm{cf}\,(\mu) = \omega$ imply both $\mu^\omega = \mu^+$ and $\Box_\mu$.

Next we are going to examine the naturally arising question whether condition $(*)$ in Theorem 2 (or the corresponding condition in Theorem 2′) is essential. The answer to this question is "yes", and it necessarily involves large cardinals. Indeed, it is well-known that the existence of a cardinal $\mu > 2^\omega$ for which $\mathrm{cf}\,(\mu) = \omega$ but either $\mu^\omega \ne \mu^+$ or $\Box_\mu$ fails implies the consistency of e.g. measurable cardinals.

Example 5. If there is supercompact cardinal then it is consistent to have a model W of ZFC in which $2^\omega = \omega_1$, $2^{\omega_1} = \omega_{\omega+1} = \lambda$ and there is a graph $\mathcal{G} = \langle \lambda, E \rangle$ of chromatic number ω_2 such that $Q(\mathcal{G})$ is CCC. In particular, we have then that $|Q(\mathcal{G})| = 2^{\omega_1}$ but

$$\mathrm{link}\,(\mathcal{G}) = \mathrm{cent}\,(\mathcal{G}) \ge \mathrm{chr}\,(\mathcal{G}) > \omega_1!$$

Proof: In [HJSh, 4.6 and 4.7] it was shown that the existence of a supercompact cardinal implies the consistency of *GCH* with the existence of a stationary set

$S \subset \lambda$ and a sequence $\langle A_\alpha : \alpha \in S\rangle$ such that $\bigcup A_\alpha = \alpha$, $tpA_\alpha = \omega_1$ and $|A_\alpha \cap A_\beta| < \omega$ if $\{\alpha, \beta\} \in [S]^2$, moreover that GCH plus the existence of such a sequence $\langle A_\alpha : \alpha \in S\rangle$ imply the existence of a graph $\mathcal{G} = \langle \lambda, E\rangle$ such that $\mathrm{chr}(\mathcal{G}) = \omega_2$ and $[\omega, \omega]$ does not embed into $\mathcal{G}$. A closer look at the proof of 4.7 will reveal that from GCH we only need CH and $\diamondsuit(S)$ to obtain this graph $\mathcal{G}$. Consequently, if we start with a ground model V satisfying GCH and having the above mentioned stationary set $S \subset \lambda$ of ω_1-limits and the ω-almost disjoint sequence $\langle A_\alpha : \alpha \in S\rangle$ and then we add λ-many Cohen subsets of ω_1 to V, i.e. we set $W = V^{\mathcal{F}n(\lambda;\omega_1)}$, then we have such a graph in the extension W as well.

Indeed, that S remains stationary and CH holds in W are standard. To show that $\diamondsuit(S)$ will also be valid in W, we can use, in V, $\diamondsuit(S)$ together with the facts that $\mathcal{F}n(\lambda;\omega_1)$ has the ω_2-CC and $|\mathcal{F}n(\lambda;\omega_1)| = \lambda = \lambda^{\omega_1}$ to "capture" all nice names of subsets of λ in W (see [K]).

Consequently, we shall be done if we can show that $Q(\mathcal{G})$ is CCC for every graph $\mathcal{G}$ that does not embed the complete bipartite graph $[\omega, \omega]$. $\square$

Lemma 6. *If $\mathcal{G} = \langle \kappa, E\rangle$ is a graph such that $[\omega, \omega]$ does not embed into $\mathcal{G}$ then $Q(\mathcal{G})$ is CCC.*

PROOF: Assume, indirectly, that there is a pairwise incompatible collection $X \in [Q(\mathcal{G})]^{\omega_1}$. By the usual Δ-system and counting arguments we may assume that $X = \{x_\alpha : \alpha \in \omega_1\}$ with $x_\alpha \cap x_\beta = \emptyset$ and $|x_\alpha| = n$ for $\{\alpha, \beta\} \in [\omega_1]^2$. Let $x_\alpha = \{\zeta_i^{(\alpha)} : i \in n\}$.

We can now define a partition

$$p : \omega \times (\omega_1 \setminus \omega) \longrightarrow n \times n$$

such that if $\langle k, \alpha\rangle \in \omega \times (\omega_1 \setminus \omega)$ and $p(k, \alpha) = \langle i, j\rangle$ then $\{\zeta_i^{(k)}, \zeta_j^{(\alpha)}\} \in E$, for this is exactly what the incompatibility of x_k and x_α means. Applying to this partition p the Erdös-Rado polarized partition relation

$$\binom{\omega_1}{\omega} \longrightarrow \binom{\omega_1, \omega}{\omega, \omega}^{1,1},$$

or rather its easy consequence

$$\binom{\omega_1}{\omega} \longrightarrow \binom{\omega}{\omega}^{1,1}_{n^2}$$

then yields infinite sets $A \subset \omega$ and $B \subset \omega_1 \setminus \omega$ and a fix pair $\langle i, j\rangle \in n \times n$ such that $\{\zeta_i^{(k)}, \zeta_j^{(\alpha)}\} \in E$ whenever $\langle k, \alpha\rangle \in A \times B$, hence we obtain that $[\omega, \omega]$ embeds into $\mathcal{G}$, and this is a contradiction. $\square$

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