On a d-parameter ergodic theorem for continuous semigroups of operators satisfying norm conditions

Shigeru Hasegawa, Ryotaro Sato

Abstract. A continuous multiparameter version of Chacon's vector valued ergodic theorem is proved.

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1. Introduction and the theorem

Let X be a reflexive Banach space with norm $|\cdot|$ and (Ω, Σ, μ) be a σ-finite measure space. For $1 \le p \le \infty$, let $L_p(\Omega; X) = L_p((\Omega, \Sigma, \mu); X)$ denote the usual Banach space of all X-valued strongly measurable functions f on Ω with the norm given by

$$\|f\|_p = \Big(\int |f|^p \, d\mu \Big)^{1/p} < \infty \quad \text{if } \ 1 \le p < \infty,$$
$$\|f\|_\infty = \operatorname{ess\,sup}\{|f(\omega)| : \omega \in \Omega\} < \infty \quad \text{if } \ p = \infty.$$

Let $d \ge 1$ be an integer, and let $T_i = \{T_i(t) : t > 0\}$, $1 \le i \le d$, be strongly continuous one-parameter semigroups of linear contractions in $L_1(\Omega; X)$ such that all the operators $T_i(t)$ are also bounded linear operators in $L_\infty(\Omega; X)$. Thus T_i, $1 \le i \le d$, can be considered to be strongly continuous one-parameter semigroups of bounded linear operators in $L_p(\Omega; X)$ for each $1 < p < \infty$, by the Riesz convexity theorem. In this paper we shall assume that there are strongly continuous one-parameter semigroups $P_i = \{P_i(t) : t > 0\}$, $1 \le i \le d$, of positive linear contractions in $L_1(\Omega; \mathbf{R})$, $\mathbf{R}$ being the real numbers, such that

(i) for all $f \in L_1(\Omega; X)$ and $t > 0$,

$$|T_i(t)f(\omega)| \le P_i(t)|f|(\omega) \ \text{ a.e. on } \Omega, \tag{1}$$

(ii) for all $f \in L_1(\Omega; \mathbf{R}) \cap L_\infty(\Omega; \mathbf{R})$ and $\alpha > 0$,

$$\|A_\alpha(P_i)f\|_\infty \le K\|f\|_\infty < \infty, \tag{2}$$

where

$$A_\alpha(P_i)f = \frac{1}{\alpha} \int_0^\alpha P_i(t)f \, dt \ \text{ for } \ f \in L_1(\Omega; \mathbf{R}).$$

Under this hypothesis we will prove the following multiparameter pointwise ergodic theorem for $T_1, \dots, T_d$.

Theorem. *If the semigroups $T_1, \dots, T_d$ commute and the semigroups $P_1, \dots, P_d$ are both L_1 and L_∞ contraction semigroups, or if the semigroups $P_1, \dots, P_d$ commute, then for every $f \in L_1(\Omega; X)$ the limit*

$$q\text{–}\lim_{\alpha\to\infty} \alpha^{-d} \int_0^\alpha \cdots \int_0^\alpha T_1(t_1)T_2(t_2)\dots T_d(t_d)f\, dt_1 \dots dt_d$$

exists a.e. on Ω, where $q\text{–}\lim_{\alpha\to\infty}$ means that the limit is taken as α tends to infinity along a countable dense subset of the positive real numbers.

This theorem may be considered to be a continuous multiparameter version of Chacon's vector valued ergodic theorem ([2]). See also [4]. Here of course the authors think that it is more natural to ask whether the conclusion of the theorem holds without assuming the existence of such positive semigroups $P_1, \dots, P_d$, when the semigroups $T_1, \dots, T_d$ commute and they are both L_1 and L_∞ contraction semigroups. But we failed to have an idea for its proof.

2. A lemma

Let $T_1, \dots, T_d$ and $P_1, \dots, P_d$ be the same as in the preceding section. By letting $T_i(0) = P_i(0) = I$ (the identity operator) for each $1 \le i \le d$, we can obviously extend T_i and P_i to the one-parameter semigroups $\tilde{T}_i$ and $\tilde{P}_i$ defined on the interval $[0, \infty)$, respectively. Let us suppose the semigroups $T_1, \dots, T_d$ commute, and define

$$\tilde{T}(t) = \tilde{T}_1(t_1)\tilde{T}_2(t_2)\dots\tilde{T}_d(t_d) \quad \text{for} \quad t = (t_1, \dots, t_d) \in \mathbf{R}_d^+, \tag{3}$$

where

$$\mathbf{R}_d^+ = \{t = (t_1, \dots, t_d) : t_i \ge 0,\ 1 \le i \le d\}.$$

Then $\tilde{T} = \{\tilde{T}(t) : t \in \mathbf{R}_d^+\}$ becomes a d-parameter semigroup of linear contractions in $L_1(\Omega; X)$ such that it is strongly continuous on the interior $\mathbf{P}_d = \{t = (t_1, \dots, t_d) : t_i > 0,\ 1 \le i \le d\}$ of $\mathbf{R}_d^+$, and for all $f \in L_1(\Omega; X)$ and $t = (t_1, \dots, t_d) \in \mathbf{R}_d^+$ we have

$$|\tilde{T}(t)f(\omega)| \le \tilde{P}_1(t_1)\dots\tilde{P}_d(t_d)|f|(\omega) \quad \text{a.e. on } \Omega.$$

Lemma. *Suppose the semigroups $T_1, \dots, T_d$ commute, and let $\tilde{T} = \{\tilde{T}(t) : t \in \mathbf{R}_d^+\}$ be the d-parameter semigroup defined by (3). Then to any $u = (u_1, \dots, u_d) \in \mathbf{R}_d^+$ there corresponds a positive linear contraction $\tau(u)$ defined in $L_1(\Omega; \mathbf{R})$, called the linear modulus of $\tilde{T}(u)$, such that*

- (i) $|\tilde{T}(u)f| \le \tau(u)|f| \le \tilde{P}_1(u_1)\dots\tilde{P}_d(u_d)|f|$ *a.e. on Ω for all $f \in L_1(\Omega; X)$,*
- (ii) $\tau(u)g = \sup\{\sum_{i=1}^k |\tilde{T}(u)f_i| : f_i \in L_1(\Omega; X),\ \sum_{i=1}^k |f_i| \le g,\ 1 \le k < \infty\}$ *for all $g \in L_1^+(\Omega; X)$,*
- (iii) $\tau(s+t) \le \tau(s)\tau(t)$ *for all $s, t \in \mathbf{R}_d^+$,*

(iv) *if* $u \in \mathbf{P}_d$ *then*

$$\tau(u) = \text{ \textit{strong}-} \lim_{\substack{t \to u \\ t \geq u}} \tau(t).$$

PROOF: See the proof of Lemma 1 in [7]. □

3. Proof of the theorem

We first consider the case $d = 1$. For $u > 0$ let $\varphi_u(x) = u^{-2}\varphi(xu^{-2})$, where

$$\varphi(x) = \begin{cases} 2^{-1}\pi^{-\frac{1}{2}}x^{-\frac{3}{2}}e^{-\frac{1}{4x}} & (x > 0), \\ 0 & (x \leq 0). \end{cases}$$

Define

$$Q_1(u)f = \int_0^\infty \varphi_u(x)P_1(x)f\,dx \quad \text{for} \quad f \in L_1(\Omega;\mathbf{R}).$$

It follows (cf. [3], [1]) that $Q_1 = \{Q_1(u) : u > 0\}$ becomes a strongly continuous semigroup of positive linear contractions in $L_1(\Omega;\mathbf{R})$ such that for all $f \in L_1^+(\Omega;\mathbf{R})$ and $\alpha > 0$

$$\frac{1}{\alpha}\int_0^\alpha P_1(t)f\,dt \leq C_1 \cdot \frac{1}{\sqrt{\alpha}}\int_0^{\sqrt{\alpha}} Q_1(u)f\,du \quad \text{a.e. on } \Omega, \tag{4}$$

where C_1 is an absolute constant, and also such that

$$\|Q_1(u)\|_\infty \leq M'K \quad \text{for all} \quad u > 0, \tag{5}$$

where

$$M' = \int_0^\infty \left|\frac{\partial \varphi_u(x)}{\partial x}\right| x\,dx < \infty$$

(M' does not depend on $u > 0$). Thus we have

$$q\text{-}\sup_{\alpha>0} \frac{1}{\alpha}\int_0^\alpha P_1(t)f\,dt \leq C_1 \cdot q\text{-}\sup_{\alpha>0} \frac{1}{\alpha}\int_0^\alpha Q_1(u)f\,du,$$

where $q\text{-}\sup_{\alpha>0}$ means that the supremum is taken as α ranges along a countable dense subset of the positive real numbers.

Define for $f \in L_p^+(\Omega;\mathbf{R})$ with $1 \leq p < \infty$,

$$Q_1^* f = q\text{-}\sup_{\alpha>0} \frac{1}{\alpha}\int_0^\alpha Q_1(u)f\,du.$$

By (5) together with Theorem 3 in [5], we see that

(i) if $1 < p < \infty$ then there exists a constant $K(p)$ with

$$\|Q_1^* f\|_p \le K(p)\|f\|_p \quad \text{for all} \quad f \in L_p^+(\Omega; \mathbf{R}), \tag{6}$$

(ii) if $p = 1$ then there exits a constant $K(1)$ with

$$\mu(\{\omega : Q_1^* f(\omega) > \alpha\}) \le \frac{1}{\alpha} K(1)\|f\|_1 \tag{7}$$

for all $f \in L_1^+(\Omega; \mathbf{R})$ and $\alpha > 0$; hence $Q_1^* f < \infty$ a.e. on Ω for all $f \in L_1^+(\Omega; \mathbf{R})$.

We now prove that if $f \in L_1(\Omega; X)$ then

$$q\text{–}\lim_{\alpha \to \infty} \frac{1}{\alpha} \int_\alpha^{\alpha+1} T_1(t) f \, dt = 0 \quad \text{on } \Omega. \tag{8}$$

For this purpose, by (1) it is enough to show that

$$q\text{–}\lim_{\alpha \to \infty} \frac{1}{\alpha} \int_\alpha^{\alpha+1} P_1(t) g \, dt = 0 \quad \text{a.e. on } \Omega \tag{9}$$

for any $g \in L_1^+(\Omega; \mathbf{R})$. To do so, let $0 < h \in L_1(\Omega; \mathbf{R}) \cap L_\infty(\Omega; \mathbf{R})$ be any function. Then we have

$$\begin{aligned} \frac{1}{\alpha} \int_\alpha^{\alpha+1} P_1(t) g \, dt &= A_\alpha(P_1) h \cdot \frac{\int_\alpha^{\alpha+1} P_1(t) g \, dt}{\int_0^\alpha P_1(t) h \, dt} \\ &\le K\|h\|_\infty \cdot \frac{\int_\alpha^{\alpha+1} P_1(t) g \, dt}{\int_0^\alpha P_1(t) h \, dt}, \end{aligned}$$

and

$$q\text{–}\lim_{\alpha \to \infty} \frac{\int_\alpha^{\alpha+1} P_1(t) g \, dt}{\int_0^\alpha P_1(t) h \, dt} = 0$$

a.e. on $\{\omega : q\text{–}\sup_{\alpha > 0}(\int_0^\alpha P_1(t) h \, dt)(\omega) > 0\}$ by virtue of the Chacon-Ornstein lemma (cf. Lemma 3.2.3 in [6]). Hence (9) follows.

Next let $1 < p < \infty$ be fixed. We observe that the net $\{A_\alpha(T_1) : \alpha > 0\}$ is ergodic with respect to the one-parameter semigroup $T_1 = \{T_1(t) : t > 0\}$ of bounded linear operators in $L_p(\Omega; X)$ in the sense of Chapter 2 of [6]. Indeed, for any $t > 0$ we have

$$\begin{aligned} \|T_1(t) A_\alpha(T_1) - A_\alpha(T_1)\|_p &= \Big\| \frac{1}{\alpha} \int_\alpha^{\alpha+t} T_1(u) \, du - \frac{1}{\alpha} \int_0^t T_1(u) \, du \Big\|_p \\ \le \Big\| \frac{1}{\alpha} \int_\alpha^{\alpha+t} T_1(u) \, du \Big\|_p &+ \frac{1}{\alpha} \Big\| \int_0^t T_1(u) \, du \Big\|_p \\ \le \Big\| \frac{1}{\alpha} \int_\alpha^{\alpha+t} P_1(u) \, du \Big\|_p &+ \frac{1}{\alpha} \Big\| \int_0^t P_1(u) \, du \Big\|_p \to 0 \quad \text{as} \quad \alpha \to \infty, \end{aligned}$$

by the Riesz convexity theorem together with (1) and (2). Since X is reflexive by hypothesis, $L_p(\Omega; X)$ is also reflexive. Thus by a mean ergodic theorem (cf. Theorem 2.1.5 in [6]) for any $f \in L_p(\Omega; X)$ the limit

$$\lim_{\alpha\to\infty} A_\alpha(T_1)f$$

exists in the L_p-norm, and we have $L_p(\Omega; X) = F \oplus N$, where

$$\begin{aligned} F &= \{f \in L_p(\Omega; X) : T_1(t)f = f \ \text{ for all } \ t > 0\}, \\ N &= \text{ the closed linear span of } \ \{f - T_1(t)f : f \in L_p(\Omega; X), \ t > 0\}. \end{aligned}$$

Since (9) holds for all $g \in L_1^+(\Omega; \mathbf{R})$, (6) together with an approximation argument proves that (9) holds for all $g \in L_p^+(\Omega; \mathbf{R})$. By this and (1), for all $f \in L_p(\Omega; X)$ we have

$$q\text{–}\lim_{\alpha\to\infty} \frac{1}{\alpha} \int_\alpha^{\alpha+1} T_1(t)f\,dt = 0 \ \text{ a.e. on } \Omega. \tag{10}$$

Here clearly $\alpha + 1$ can be replaced by any $\alpha + u$ with $u > 0$. So for $u > 0$ we have

$$\begin{aligned} &q\text{–}\lim_{\alpha\to\infty} A_\alpha(T_1)(f - T_1(u)f) \\ &\quad = q\text{–}\lim_{\alpha\to\infty} \Big(\frac{1}{\alpha} \int_0^u T_1(t)f\,dt - \frac{1}{\alpha} \int_\alpha^{\alpha+u} T_1(t)f\,dt \Big) \\ &\quad = 0 \ \text{ a.e. on } \Omega, \end{aligned}$$

whence (1), (4), (6) and Banach's convergence principle (cf. Theorem 1.7.2 in [6]) prove that for any $f \in L_p(\Omega; X)$ the limit

$$q\text{–}\lim_{\alpha\to\infty} \frac{1}{\alpha} \int_0^\alpha T_1(t)f\,dt$$

exists a.e. on Ω. Since $L_p(\Omega; X) \cap L_1(\Omega; X)$ is dense in $L_1(\Omega; X)$, (7) and Banach's convergence principle prove that the theorem holds for $d = 1$.

Since the case $d = 1$ has been done, we now proceed by an induction argument. First suppose that the semigroups $T_1, \dots, T_d$ commute and the semigroups $P_1, \dots, P_d$ are both L_1 and L_∞ contraction semigroups. Let $\tilde{T} = \{\tilde{T}(t) : t \in \mathbf{R}_d^+\}$ and $\{\tau(t); t \in \mathbf{R}_d^+\}$ be as in the lemma. We notice that $\|\tau(t)\|_p \le 1$ for all $1 \le p \le \infty$ and $t \in \mathbf{R}_d^+$, and that if $u \in \mathbf{P}_d$ then $\tau(u) = \text{strong-lim}_{t\to u, t\ge u}\, \tau(t)$ in $L_p(\Omega; \mathbf{R})$ for each $1 \le p < \infty$. For $u = (u_1, \dots, u_d) \in \mathbf{P}_d$ and $g \in L_p(\Omega; \mathbf{R})$ with $1 \le p < \infty$, define

$$\begin{aligned} S(u)g &= S(u_1, \dots, u_d)g \\ &= \int_0^\infty \cdots \int_0^\infty \varphi_{u_1}(x_1) \dots \varphi_{u_d}(x_d) \tau(x_1, \dots, x_d) g\,dx_1 \dots dx_d. \end{aligned} \tag{11}$$

$S(u)$ becomes a positive linear contraction in $L_p(\Omega;\mathbf{R})$ for each $1 \le p < \infty$. Further, by putting

$$\tilde{\tau}(x_1, \tilde{x}_1, \ldots, x_x, \tilde{x}_d) = \tau(x_1, \ldots, x_d),$$

we get for all $g \in L_p(\Omega;\mathbf{R})$ with $1 \le p < \infty$

$$S(u)g =$$
$$= \int_0^\infty \cdots \int_0^\infty \varphi_{u_1}(x_1)\varphi_{u_1}(\tilde{x}_1)\ldots\varphi_{u_d}(x_d)\varphi_{u_d}(\tilde{x}_d)\tilde{\tau}(x_1,,\tilde{x}_1,\ldots,\tilde{x}_d)g\,dx_1\ldots d\tilde{x}_d.$$

Thus it follows from the lemma (iii) and a standard calculation (cf. p. 700 in [3]) that if $u, t \in \mathbf{P}_d$ and $g \in L_p^+(\Omega;\mathbf{R})$ with $1 \le p < \infty$ then

$$S(u)S(t)g \ge S(u+t)g \quad \text{a.e. on } \Omega, \tag{12}$$

that is, $S = \{S(t) : t \in \mathbf{P}_d\}$ becomes a d-parameter *sub-semigroup* of positive linear contractions in $L_p(\Omega;\mathbf{R})$ for each $1 \le p < \infty$. Since S is strongly continuous on $\mathbf{P}_d$, the proof of Lemma VIII.7.13 in [3] shows that there exists a constant $C_d > 0$, dependent only on d, and a strongly continuous one-parameter sub-semigroup $S^1 = \{S^1(t) : t > 0\}$ of positive linear contractions in $L_1(\Omega;\mathbf{R})$ such that

$$\|S^1(t)\|_\infty \le 1 \quad \text{for all} \quad t > 0, \tag{13}$$

and also such that for all $g \in L_p^+(\Omega;\mathbf{R})$ with $1 \le p < \infty$

$$q\text{-}\sup_{\alpha>0} \frac{1}{\alpha^d} \int_{[0,\alpha]^d} \tau(u)g\,du \le C_d \cdot q\text{-}\sup_{\alpha>0} \frac{1}{\alpha} \int_0^\alpha S^1(t)g\,dt \quad \text{a.e. on } \Omega. \tag{14}$$

Let us fix a $g \in L_p^+(\Omega;\mathbf{R})$ with $1 \le p < \infty$, and let $\mathbf{Q}^+$denote the set of all positive rational numbers. Since

$$\lim_{n\to\infty} \left\| \frac{1}{r}\int_0^r S^1(t)g\,dt - \frac{1}{r(n!)} \sum_{i=0}^{r(n!)-1} S^1\Big(\frac{i}{n!}\Big)g \right\|_p = 0$$

for all $r \in \mathbf{Q}^+$, where $S^1(0) = I$, the Cantor diagonal method can be applied to choose a subsequence (n') of (n) such that

$$\frac{1}{r}\int_0^r S^1(t)g\,dt = \lim_{n'\to\infty} \frac{1}{r(n'!)} \sum_{i=0}^{r(n'!)} S^1\Big(\frac{i}{n'!}\Big)g$$
$$\le \liminf_{n'\to\infty} \frac{1}{r(n'!)} \sum_{i=0}^{r(n'!)-1} \Big(S^1\Big(\frac{1}{n'!}\Big)\Big)^i g \quad \text{a.e. on } \Omega$$

for all $r \in \mathbf{Q}^+$. Thus putting

$$g^*(n') = \sup_{k \geq 1} \frac{1}{k} \sum_{i=0}^{k-1} \Big(S^1\Big(\frac{1}{n'!}\Big)\Big)^i g, \tag{15}$$

and for each $a > 0$

$$E(n', a) = \{\omega \in \Omega : g^*(n')(\omega) > a\},$$

we see that the function

$$g^* = \sup_{r \in \mathbf{Q}^+} \frac{1}{r} \int_0^r S^1(t) g \, dt \tag{16}$$

satisfies

$$g^* \leq \liminf_{n' \to \infty} g^*(n') \quad \text{a.e. on } \Omega, \tag{17}$$

and

$$\{\omega : g^*(\omega) > a\} \subset \liminf_{n' \to \infty} E(n', a).$$

Therefore by Fatou's lemma, if $g \in L_1^+(\Omega; \mathbf{R})$ then

$$\begin{aligned} \int_{\{g^* > a\}} (a - \min\{a, g\}) \, d\mu &\leq \liminf_{n' \to \infty} \int_{E(n', a)} (a - \min\{a, g\}) \, d\mu \\ &\leq \int_\Omega (g - \min\{a, g\}) \, d\mu \qquad \text{(by Theorem 1 in [5])}, \end{aligned}$$

so that

$$\mu(\{g^* > a\}) \leq \frac{1}{a} \|g\|_1,$$

whence $g^* < \infty$ a.e. on Ω. By this together with (14) and the lemma, we have for all $f \in L_1(\Omega; X)$

$$q\text{–}\sup_{\alpha > 0} \alpha^{-d} \Big| \int_0^\alpha \cdots \int_0^\alpha \tilde{T}(t_1, \ldots, t_d) f \, dt_1 \ldots dt_d \Big| < \infty \quad \text{a.e. on } \Omega. \tag{18}$$

Let $1 < p < \infty$. If $g \in L_p^+(\Omega; \mathbf{R})$ then the function g^* in (16) satisfies, by (17) and Fatou's lemma,

$$\|g^*\|_p \leq \liminf_{n' \to \infty} \|g^*(n')\|_p.$$

From (13) it follows (cf. [5]) that there exists a constant $\tilde{K}(p) > 0$ such that

$$\|g^*(n')\|_p \leq \tilde{K}(p) \|g\|_p;$$

thus

$$\|g^*\|_p \le \tilde{K}(p)\|g\|_p \qquad (g \in L_p^+(\Omega;\mathbf{R})). \tag{19}$$

Let $f \in L_p(\Omega; X)$ and $t > 0$ be fixed. Since

$$q\text{–}\lim_{\alpha\to\infty} \frac{1}{\alpha}\int_0^\alpha T_d(u)[f - T_d(t)f]\,du = 0 \ \text{ a.e. on } \Omega,$$

the functions

$$M(\alpha)[f - T_d(t)f] = \sup_{\substack{b>\alpha\\ b\in\mathbf{Q}^+}} \frac{1}{b}\Big|\int_0^b T_d(u)[f - T_d(t)f]\,du\Big| \qquad (\alpha > 0)$$

satisfy

$$\lim_{\alpha\to\infty} M(\alpha)[f - T_d(t)f] = 0 \ \text{ a.e. on } \Omega.$$

Further, since

$$M(\alpha)[f - T_d(t)f] \le \sup_{r\in\mathbf{Q}^+} \frac{1}{r}\Big|\int_0^r T_d(u)[f - T_d(t)f]\,du\Big| \in L_p(\Omega;\mathbf{R})$$

by the preceding argument for $d = 1$, it follows from Lebesgue's convergence theorem that

$$\lim_{\alpha\to\infty} \|M(\alpha)[f - T_d(t)f]\|_p = 0.$$

This together with the inequalities (14) for the case $d-1$ and (19) yield

$$q\text{–}\lim_{\alpha\to\infty} \alpha^{-(d-1)}\int_0^\alpha\cdots\int_0^\alpha \tilde{T}(u_1,\dots,u_{d-1}) \Big(\frac{1}{\alpha}\int_0^\alpha T_d(s)[f - T_d(t)f]\,ds\Big)\,du_1\dots du_{d-1} = 0 \ \text{ a.e. on } \Omega.$$

Since $L_p(\Omega; X) = F_d \oplus N_d$, where

$$F_d = \{h \in L_p(\Omega; X) : T_d(t)h = h \ \text{ for all } \ t > 0\},$$
$$N_d = \text{ the closed linear span of } \{h - T_d(t)h : h \in L_p(\Omega; X),\ t > 0\},$$

we then apply the induction hypothesis together with Banach's convergence principle (cf. (14), (16) and (19)) to show for any $f \in L_p(\Omega; X)$ the limit

$$q\text{–}\lim_{\alpha\to\infty} \alpha^{-d}\int_0^\alpha\cdots\int_0^\alpha \tilde{T}_1(t_1,\dots,t_d)f\,dt_1\dots dt_d$$

exists a.e. on Ω. This and (18) for $f \in L_1(\Omega; X)$ prove that the conclusion of the theorem holds, when $T_1, \dots, T_d$ commute and $P_1, \dots, P_d$ are both L_1 and L_∞ contraction semigroups.

Finally suppose that the semigroups $P_1, \dots, P_d$ commute. For $u = (u_1, \dots, u_d) \in \mathbf{P}_d$ and $g \in L_1(\Omega; \mathbf{R})$, define

$$\begin{aligned} Q(u)g &= Q(u_1, \dots, u_d)g \\ &= \int_0^\infty \cdots \int_0^\infty \varphi_{u_1}(x_1) \dots \varphi_{u_d}(x_d) P_1(x_1) \dots P_d(x_d) g \, dx_1 \dots dx_d. \end{aligned} \tag{20}$$

It follows from (5) (cf. [8]) that $Q = \{Q(u) : u \in \mathbf{P}_d\}$ becomes a d-parameter *semigroup* of positive linear contraction in $L_1(\Omega; \mathbf{R})$ such that

$$\|Q(u)\|_\infty \le (M'K)^d \quad \text{for all} \quad u \in \mathbf{P}_d. \tag{21}$$

Thus there exists a strongly continuous one-parameter *semigroup* $Q^1 = \{Q^1(t) : t > 0\}$ of positive linear contractions in $L_1(\Omega; \mathbf{R})$ such that $\|Q^1(t)\|_\infty \le (M'K)^d$ for all $t > 0$, and if $g \in L_p^+(\Omega; \mathbf{R})$ with $1 \le p < \infty$ then

$$\begin{aligned} q\text{-}\sup_{\alpha > 0} \frac{1}{\alpha^d} \int_0^\alpha \cdots \int_0^\alpha P_1(u_1) \dots P_d(u_d) g \, du_1 \dots du_d \\ \le C_d \cdot q\text{-}\sup_{\alpha > 0} \frac{1}{\alpha} \int_0^\alpha Q^1(t) g \, dt < \infty \quad \text{a.e. on } \Omega. \end{aligned}$$

Since (1) implies that if $f \in L_p(\Omega; X)$ with $1 \le p < \infty$ then the function

$$Mf = q\text{-}\sup_{\alpha > 0} \frac{1}{\alpha^d} \left| \int_0^\alpha \cdots \int_0^\alpha T_1(t_1) \dots T_d(t_d) f \, dt_1 \dots dt_d \right|$$

satisfies

$$Mf \le q\text{-}\sup_{\alpha > 0} \frac{1}{\alpha^d} \int_0^\alpha \cdots \int_0^\alpha P_1(u_1) \dots P_d(u_d) |f| \, du_1 \dots du_d,$$

it follows that

$$Mf < \infty \quad \text{a.e. on } \Omega$$

for all $f \in L_p(\Omega; X)$ with $1 \le p < \infty$. Using this and the fact that the one-parameter semigroup $T_d = \{T_d(t) : t > 0\}$ of bounded linear operators in $L_p(\Omega; X)$ with $1 < p < \infty$ satisfies the mean ergodic theorem, we can prove that the conclusion of the theorem holds in this case, too. We may omit the details.

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Department of Mathematics, Shibaura Institute of Technology, Omiya, 330 Japan

Department of Mathematics, Faculty of Science, Okayama University, Okayama, 700 Japan

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