Choice principles in elementary topology and analysis

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Abstract. Many fundamental mathematical results fail in **ZF**, i.e., in Zermelo-Fraenkel set theory without the Axiom of Choice. This article surveys results — old and new — that specify how much "choice" is needed *precisely* to validate each of certain basic analytical and topological results.

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Introduction

"It is a peculiar fact that all the transfinite axioms are deducible from a single one, the axiom of choice, — the most challenged axiom in the mathematical literature."

David Hilbert (1926)

"A formal system in which $\exists G(x)$ is provable, but which provides no method for finding the x in question, is one in which the existential quantifier fails to fulfill its intended function."

A.H. Goldstein (1968)

"The Axiom of Choice has easily the most tortured history of all the set-theoretic axioms."

Penelope Maddy (1988)

As the above quotes indicate the Axiom of Choice is highly controversial — perhaps the most controversial axiom in all of mathematics. To discuss this issue however is not the purpose of the present note, — see instead the inspiring book by **G.H. Moore (1982)** –, its more modest aim is to demonstrate how much "choice" is needed to make elementary analysis resp. topology click.

The results (all in **ZF**) are ordered by their degree of abstractness.

1. In the realm of the reals

We start by observing that several familiar topological properties of the reals are equivalent to each other and to rather natural choice-principles.

Theorem 1.1 ([15], [29], [30]). *Equivalent are:*

1. *in $\mathbb{R}$, a point x is an accumulation point of a subset A iff there exists a sequence in $A \setminus \{x\}$ that converges to x,*
2. *a function $f\colon \mathbb{R} \to \mathbb{R}$ is continuous at a point x iff it is sequentially continuous at x,*
3. *a real-valued function $f\colon A \to \mathbb{R}$ from a subspace A of $\mathbb{R}$ is continuous iff it is sequentially continuous,*
4. *each subspace of $\mathbb{R}$ is separable,*
5. *$\mathbb{R}$ is a Lindelöf space,*
6. *$\mathbb{Q}$ is a Lindelöf space,*
7. *$\mathbb{N}$ is a Lindelöf space,*
8. *each unbounded subset of $\mathbb{R}$ contains an unbounded sequence,*
9. *the Axiom of Choice for countable collections of subsets of $\mathbb{R}$.*

There exist models of ZF that violate the above conditions ([17], [18]). Observe the fine distinction between conditions 2 and 3 of Theorem 1.1. These may lead one to assume that also the following property is equivalent to the above conditions:

(*) a function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ is continuous iff it is sequentially continuous.

However, this would be a serious mistake: (*) holds in **ZF** (without any choice-assumptions) — see [29]. If, however, we consider functions $f\colon X \longrightarrow \mathbb{R}$ with metric domain we need even more choice than in Theorem 1.1, — see Theorem 2.1.

Proposition 1.2 ([15]). *Equivalent are:*

1. *in $\mathbb{R}$, every bounded infinite set contains a convergent injective sequence,*
2. *every infinite subset of $\mathbb{R}$ is Dedekind-infinite.*

There exist models of **ZF** that violate the above conditions ([18]).

Obviously, the conditions of Theorem 1.1 imply the conditions of Proposition 1.2.

Is the converse true?

Observe that the following slight modifications of condition 1 in Proposition 1.2 hold in **ZF**:

(a) in $\mathbb{R}$, every bounded countable set contains a convergent injective sequence,
(b) in $\mathbb{R}$, for every bounded infinite set there exists an accumulation point.

2. In the realm of pseudometric spaces

In this section we consider (pseudo)metric spaces and various compactness-notions for them.

Theorem 2.1 ([4], [15]). *Equivalent are:*

1. *every separable pseudometric space is a Lindelöf space,*
2. *every pseudometric space with a countable base is a Lindelöf space,*
3. *the Axiom of Choice for countable collections of subsets of $\mathbb{R}$.*

Definition 2.2. A pseudometric space $\mathbf{X}$ is called

1. *Heine-Borel-compact* provided every open cover of $\mathbf{X}$ contains a finite one,
2. *Weierstraß-compact* provided for every infinite subset of $\mathbf{X}$ there exists an accumulation point,
3. *Alexandroff-Urysohn-compact* provided for every infinite subset of $\mathbf{X}$ there exists a complete accumulation point,
4. *sequentially-compact* provided every sequence in $\mathbf{X}$ has a convergent subsequence.

Under the Axiom of Choice the above compactness concepts are equivalent. This is no longer the case in **ZF**.

However the following implications remain valid ([4]):

$$\begin{tikzcd}
\boxed{\text{Alexandroff-Urysohn-compact}} \arrow[d] & & \boxed{\text{Heine-Borel-compact}} \arrow[lld] \arrow[d] \\
\boxed{\text{Weierstraß-compact}} \arrow[rd] & & \boxed{\text{complete and totally bounded}} \arrow[ld] \\
& \boxed{\text{sequentially compact}} &
\end{tikzcd}$$

Theorem 2.3 ([4]). *Equivalent are:*

1. *Weierstraß-compact $\Leftrightarrow$ sequentially compact,*
2. *finite $\Leftrightarrow$ Dedekind-finite.*

Theorem 2.4 ([4], [14]). *Equivalent are:*

1. *in a (pseudo)metric space X, a point x is an accumulation point of a subset A iff there exists a sequence in $A \setminus \{x\}$ that converges to x,*
2. *a real-valued function $f\colon \mathbf{X} \to \mathbb{R}$ from a (pseudo)metric space is continuous iff it is sequentially continuous,*
3. *a function between (pseudo)metric spaces is continuous iff it is sequentially continuous,*
4. *subspaces of a separable pseudometric spaces are separable,*

5. *totally bounded pseudometric spaces are separable,*
6. *Heine-Borel-compact pseudometric spaces are separable,*
7. *separable* $\Leftrightarrow$ *countable base,*
8. *separable* $\Leftrightarrow$ *Lindelöf,*
9. *separable* $\Leftrightarrow$ *topologically totally bounded,*
10. *Heine-Borel-compact* $\Leftrightarrow$ *sequentially compact,*
11. *Heine-Borel-compact* $\Leftrightarrow$ *complete and totally bounded,*
12. *sequentially compact* $\Leftrightarrow$ *complete and totally bounded,*
13. *Weierstraß-compact* $\Leftrightarrow$ *complete and totally bounded,*
14. *complete* $\Leftrightarrow$ *each Cauchy-filter converges,*
15. *the Baire Category Theorem for complete, totally bounded pseudometric spaces,*
16. *the Baire Category Theorem for complete pseudometric spaces with countable base,*
17. *the Axiom of Countable Choice.*

Observe that the Baire Category Theorem for complete, separable pseudometric spaces holds in **ZF**. However, the role of the Baire Category theorem for all complete pseudometric spaces is not yet clear:
The Axiom of Dependent Choices implies the Baire Category Theorem for complete pseudometric spaces, and the latter implies the Axiom of Countable Choice.

Is one of these implications an equivalence?

Theorem 2.5 ([4], [10]). *Equivalent are:*

1. *Alexandroff-Urysohn-compact* $\Leftrightarrow$ *Heine-Borel-compact,*
2. *Alexandroff-Urysohn-compact* $\Leftrightarrow$ *Weierstraß-compact,*
3. *Alexandroff-Urysohn-compact* $\Leftrightarrow$ *sequentially compact,*
4. *the Axiom of Choice.*

3. In the realm of topological spaces

Finally, we consider arbitrary topological spaces and various familiar compactness notions there — all equivalent under the Axiom of Choice.

Definition 3.1. A topological space **X** is called

1. *Heine-Borel-compact* provided every open cover of **X** contains a finite one,
2. *Alexandroff-Urysohn-compact* provided for every infinite subset of **X** there exists a complete accumulation-point,
3. *Bourbaki-compact* provided every ultrafilter converges in **X**,
4. *Comfort-compact* provided **X** is homeomorphic to a closed subspace of $[0,1]^I$ for some I.

Theorem 3.2([15]). *Equivalent are:*

1. *every topological space with a countable base is a Lindelöf space,*
2. *the Axiom of Choice for countable collections of subset of the reals.*

Theorem 3.3 ([1], [3], [10], [13], [14], [21], [32]). *Equivalent are:*

1. *the Tychonoff Theorem for Heine-Borel-compact spaces,*
2. *the Tychonoff Theorem for Alexandroff-Urysohn-compact spaces,*
3. *the Čech-Stone Theorem*[1] *for Alexandroff-Urysohn-compactness,*
4. *the Ascoli-Theorem for Alexandroff-Urysohn-compactness,*
5. *products of spaces with finite topologies are Heine-Borel-compact,*
6. *products of spaces with finite topologies are Alexandroff-Urysohn-compact,*
7. *products of finite discrete spaces are Alexandroff-Urysohn-compact,*
8. *spaces with finite topologies are Alexandroff-Urysohn-compact,*
9. *finite products of Alexandroff-Urysohn-compact spaces are Alexandroff-Urysohn-compact,*
10. *finite coproducts of Alexandroff-Urysohn-compact spaces are Alexandroff-Urysohn-compact,*
11. *Heine-Borel-compact ⇔ Alexandroff-Urysohn-compact,*
12. *Bourbaki-compact ⇔ Alexandroff-Urysohn-compact,*
13. *Comfort-compact ⇔ Alexandroff-Urysohn-compact and completely regular,*
14. *the Axiom of Countable Choice and every closed filter is contained in a maximal one,*
15. *the Axiom of Choice.*

The Axiom of Dependent Choices implies the Tychonoff Theorem for countable collections of Heine-Borel-compact spaces, and the latter implies the Axiom of Countable Choice ([7]). Is one of these implications an equivalence? It is known however that the Axiom of Countable Choice does not imply the Axiom of Dependent Choices ([20]).

Theorem 3.4 ([2], [3], [10], [12], [14], [22], [25], [28]). *Equivalent are:*

1. *the Tychonoff Theorem for Heine-Borel-compact Hausdorff spaces,*
2. *the Čech-Stone Theorem*[1] *for Heine-Borel-compactness,*
3. *the Ascoli-Theorem for Heine-Borel-compactness,*
4. *the Ascoli-Theorem for Bourbaki-compactness,*
5. *the Ascoli-Theorem for Comfort-compactness,*
6. *products of finite spaces are Heine-Borel-compact,*
7. *Heine-Borel-compact ⇔ Bourbaki-compact,*
8. *Heine-Borel-compact and completely regular ⇔ Comfort-compact,*
9. *every z-filter is fixed ⇔ every z-ultrafilter is fixed,*
10. *in the ring $C(X)$, every ideal is fixed iff every maximal ideal is fixed,*

[1]The *Čech-Stone Theorem* states that compact Hausdorff spaces form an epireflective subcategory of the category of Hausdorff spaces and continuous maps (i.e., that for every Hausdorff space X there exists a compact Hausdorff space βX and a continuous dense map $r\colon X \longrightarrow \beta X$ such that for every compact Hausdorff space Y and every continuous map $f\colon X \longrightarrow Y$ there exists a continuous map $\bar{f}\colon \beta X \longrightarrow Y$ with $f = \bar{f} \circ r$).

11. *in the ring $C^*(X)$, every ideal is fixed iff every maximal ideal is fixed,*
12. *every filter is contained in an ultrafilter,*
13, *every zero-filter is contained in a maximal one,*
14. *the Boolean Prime Ideal Theorem.*

The Boolean Prime Ideal Theorem is known to be properly weaker than the Axiom of choice [9]. This is a deep result, but the gap — surprisingly — is even wider. The Boolean Prime Ideal Theorem together with the Axiom of Countable Choice (or even with the Axiom of Dependent Choices) is properly weaker than the Axiom of Choice ([26]).

Theorem 3.5([10])**.** *Equivalent are:*

1. *the Tychonoff Theorem for Bourbaki-compact spaces,*
2. *either the Axiom of Choice or all ultrafilters are fixed.*

This result is particularly surprising and rather strange indeed, since the two parts of condition 2 are rather far apart. The second part is not void, as one might suspect, since there exist models of **ZF** with no free ultrafilters ([5]).

Proposition 3.6 ([10])**.** *Equivalent are:*

1. *Alexandroff-compactifications of discrete spaces are Alexandroff-Urysohn-compact,*
2. *finite $\Leftrightarrow$ Dedekind-finite.*

The condition *"finite $\Leftrightarrow$ Dedekind-finite"* of the above proposition is properly weaker than the Axiom of Countable Choice ([19]). This is perhaps not surprising, but rather hard to prove.

Theorem 3.7 ([7], [31])**.** *Equivalent are:*

1. *Urysohn's Lemma for orderable*[2] *spaces,*
2. *Tietze-Urysohn's Extension Theorem for orderable*[2] *spaces,*
3. *orderable*[2] *spaces are normal,*
4. *orderable*[2] *spaces are collectionwise normal,*
5. *orderable*[2] *spaces are collectionwise Hausdorff,*
6. *orderable*[2] *spaces are monotonically normal,*
7. *completely orderable*[2] *spaces are hereditarily normal,*
8. *the Axiom of Choice for collections of pairwise disjoint convex open subsets of some completely ordered*[2] *set,*
9. *the Axiom of Choice for the collection of all non-empty convex subsets of some completely ordered*[2] *set.*

Let us sum things up: Topology with "choice" may be as unreal as a soap-bubble dream, but topology without "choice" is as horrible as nightmare.

[2]Here *order* means *linear order.*

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