How restrictive is topological dynamics?

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Abstract. Let T be a permutation of an abstract set X. In ZFC, we find a necessary and sufficient condition it terms of cardinalities of the T-orbits that allows us to topologize (X,T) as a topological dynamical system on a compact Hausdorff space. This extends an early result of H. de Vries concerning compact metric dynamical systems. An analogous result is obtained for $\mathbf{Z}^2$-actions without periodic points.

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Introduction

By topological dynamics one usually means the study of a transformation group of a topological phase space. In the most common setup the infinite cyclic group $\mathbf{Z}$ acts on a compact Hausdorff space X, that is to say we study iterates of a single homeomorphism T of X. From an abstract point of view — forgetting the topology — we are left with a permutation T of a set X, i.e. an abstract dynamical system (X,T). A natural question arises whether it is possible to make such an abstract system into a compact Hausdorff one with T a homeomorphism. Some results in this direction have been known for a long time. De Vries proved in [V] that, assuming the continuum hypothesis, if $|X| = c$, the cardinality of continuum, then (X,T) can always be endowed with a compact metric in which T is a homeomorphism. We also mention other papers where under suitable assumptions T is realized as an isometry ([I-J-K], [K], [I]), or it is embedded into a linear system on a Hilbert space ([E],[J], where noninvertible transformation are also studied).

In the present note we characterize completely the abstract systems possessing compact Hausdorff models. As we will see, this requirement is only restrictive if $|X| < c$, in which case the argument relies on two results from [I-J-K]. We also illustrate how the problem complicates in the case of $\mathbf{Z}^2$-actions.

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1. Z-actions

Let T be a permutation of a set X.

1. First we are assuming that there are no periodic points, i.e., all the orbits under the T-action are infinite.

If $|X| = c$ then an obvious topological model for (X, T) is provided by any irrational rotation $x \to x + \alpha$ of the circle $\mathbf{T} = \mathbf{R}/\mathbf{Z}$. There is also no problem in extending this to $|X| > c$ as now we may just take $X = \mathbf{T} \times Y$ with $T(x, y) = (x + \alpha, y)$, where Y is any compact Hausdorff space of a suitable cardinality, e.g. a closed initial interval of the ordinals with its order topology.

On the other hand, if $|X| < c$, no such model exists. Indeed, suppose (X, T) is a compact Hausdorff dynamical system with no periodic points and $|X| < c$. Without loss of generality we may assume that the system is minimal. Observe that X is not dense-in-itself because otherwise we would have $|X| \geq c$ by the familiar Cantor set construction. There must exist an isolated point in X, whence by minimality all the points are isolated, so X is discrete. By compactness, X is finite, which contradicts the absence of periodic points.

The case of infinite orbits is now completely settled.

2. Now assume there are no infinite orbits, in other words the system is pointwise periodic. Denote by ν_k the number of orbits of length k.

Suppose first that $|X| \geq c$. We have

$$\nu_1 + 2\nu_2 + 3\nu_3 + \ldots = |X|$$

so by König's theorem, $\nu_k \geq c$ for at least one of the k's. We fix one such k. Now for any natural number n choose a relatively prime $p_n \in \{1, \ldots, n-1\}$ such that $p_n/n \to 1/k$ as $n \to \infty$. Such a sequence p_n exists, e.g. as a simple consequence of the Prime Number Theorem: for any $\epsilon > 0$ and $0 < \lambda < 1$ there are approximately $2\epsilon\lambda n/\log(\lambda n)$ primes situated between $(1-\epsilon)\lambda n$ and $(1+\epsilon)\lambda n$, while the number of all prime divisors of n is bounded by $\log_2 n$. Without loss of generality we may assume that there are exactly c orbits of length k (if there are more, we can make the others into a disjoint closed-and-open compact dynamical system of the type $Z \times Y$ as in 1, where Z is a single k-cycle). We also assume for simplicity that there is only one orbit of each length $n \neq k$.

Now the argument runs as in de Vries [V] where, however, the author ignored the necessity of p_n being relatively prime with respect to n. First take the circle model for the k-orbits with $\mathbf{T}$ rotated by $Tx = x + 1/k$. Next choose a sequence of concentric n-cycles of distinct radii $r_n \to 1$ with the n points situated corresponding to the angles $2\pi j/n$, $j = 0, \ldots, n-1$ (we construct only those n-cycles for which $\nu_n \neq 0$). Now define the transformation T on the n-cycle by $j \to j + p_n$ modulo n. Since $(n, p_n) = 1$, we still have a single n-cycle. Since the n-cycles converge to $\mathbf{T}$ with $p_n/n \to 1/k$, we obtain a compact dynamical system. If there are more orbits of each length, we pass to suitable products as before and make the whole "packs" of n-cycles converge to the circle — one can easily do this by a quotient space construction.

We have shown that any pointwise periodic system with $|X| \geq c$ has a compact model. In the case $|X| < c$ we need a finer analysis.

According to [I-J-K], we say that the set of orbits of (X, T) is *finitely based* if there exists a finite family of orbits of lengths $k_1, \ldots, k_r$ such that the length of any orbit in X is a multiple of some k_i.

It is not hard to see that if the orbits are finitely based then a compact model exists. One way of seeing this is a symbolic construction of X as a closed subshift in $[0,1]^{\mathbf{Z}}$. Clearly there is no loss of generality in assuming that all the periods are multiples dk of a single number k. Also, by using suitable products to obtain "packs" of cycles of the same length and passing to a quotient space as before, we may restrict ourselves to the case where there is only one orbit of each length. Now define the word $W = W_1 = 10\ldots0$ (a 1 followed by $k-1$ symbols 0) and for each $d = 2, 3, \ldots$ set $W_d = W'W\ldots W$, where W' the word W with the first symbol replaced by $1_d = 1-1/d$. Now form periodic sequences $x_d = \ldots W_d W_d \ldots$ in $[0,1]^{\mathbf{Z}}$, where the first symbol of some W_d always occurs at the zero coordinate (use only those d's that actually occur in the given abstract system). It is clear that x_d is dk-periodic and since $1_d \to 1$,the least closed subshift containing the elements x_d consists only of the x_d's and their translates, and has exactly the orbits we need.

It remains to see what happens if $|X| < c$ and the orbits are not finitely based. We show that in this case no compact model exists. To this end first note that $|X| < c$ implies that X is 0-dimensional (if not, there would exist a nontrivial connected component which, by a Cantor set construction would be of cardinality at least c). We now quote two results from [I-J-K]:

(a) *If (X, T) is a 0-dimensional pointwise periodic compact Hausdorff dynamical system then (X, T) is equicontinuous;*

(b) *If the equicontinuous system (X, T) is pointwise periodic then the orbits are finitely based.*

Putting this together we can see that in the pointwise periodic case a compact model does not exists if and only if $|X| < c$ and the orbits are not finitely based.

3. Finally we consider the mixed case and assume that there are both finite and infinite orbits in our abstract system (X, T). We will show that now a compact model always exists regardless of the cardinality of X.

It is clear how to "wind around" an infinite orbit (or a "pack" of them) about a finite cycle — so there is no difficulty in finding a compact model if the set of finite orbits is finite or finitely based. As before, it is also not difficult to find a compact model with a preassigned number of orbits of given lengths, once a model with single orbit of each kind has been constructed. Therefore we may assume that (X, T) consists of a single infinite orbit and an infinite collection of finite orbits which are not finitely based and have only one representative of each length. An example of this would be an infinite orbit along with finite orbits of all prime lengths, one of each kind. Our task is to show that such orbits can coexist in a compact model.

Let $n_1 = N < n_2 < n_3 < \ldots$ be the lengths of finite orbits that occur in

(X, T). Of course we may assume

$$n_2 \geq N^2,$$

for any finite collection of orbits has a compact model. It will also be convenient to assume that no $n_k,\ k \geq 2,$ is a multiple of N (we take care of the multiples as in the finitely based case by winding them around N).

As a model for (X, T) we will construct a closed subshift of $\{0,1\}^{\mathbf{Z}}$. First let $B = 01\ldots1$ (N symbols). Now for every $n = n_k,\ k \geq 2,$ write $n = qN + r,$ where $0 < r < N,\ q = rp + s$. Note that such a representation exists with $r > 0$, $p > 0$, and $1 \leq s \leq r$ thanks to our assumptions that N does not divide n_k and $n_k \geq N^2$. We define

$$C_n = B\ldots B0B\ldots B0\ldots B\ldots B0B\ldots B,$$

where there are r concatenations of p words B each, separated by single zeros and followed by a terminal concatenation of s words B. Since $n = rpN + r + sN$, the length of C_n is exactly n and the sequence $y_n = \ldots C_n C_n \ldots$, with the first symbol of C_n at the zero coordinate, is n-periodic with no shorter period (indeed, the block $0B\ldots B$ with more than p consecutive words B occurs exactly once every n positions). The least closed subshift containing the points $y_{n_k},\ k = 2, 3, \ldots$ and the N-periodic sequence $y_N = \ldots BBB\ldots$ consists exactly of these sequences and their translates plus one infinite orbit generated by the sequence $y = \ldots BBB0BBB\ldots$. The latter is obtained as the limit of a subsequence of some translates of the y_{n_k}'s whenever the zero coordinate remains at a bounded distance from the nearest separating 0 in C_{n_k}; otherwise we will end up with a translate of y_N. It is now clear that the orbits of the compact subshift obtained by our construction have the same structure as in the abstract model (X, T).

We have obtained the following result.

Theorem 1. *Let X be a set and $T : X \to X$ be a bijection. Then (X, T) can be made into a compact Hausdorff dynamical system iff none of the following holds*

(1) *$|X| < c$ and all the orbits are infinite,*
(2) *$|X| < c$, all the orbits are finite and are not finitely based.*

2. Remarks on $\mathbf{Z}^2$-actions

In this section we consider an abstract dynamical system (X, S, T), where S, T are two commuting permutations of the set X. In other words, we study the $\mathbf{Z}^2$-action $((m, n), x) \to S^m T^n x$ on X. We are seeking a realization of our abstract system as a compact Hausdorff space with two commuting homeomorphisms.

In general a solution seems much more complicated than in the case of a $\mathbf{Z}$-action. We only present the solution in the case where all the $\mathbf{Z}^2$-orbits are infinite.

First note that any $\mathbf{Z}^2$-orbit can be viewed as a homomorphic image of $\mathbf{Z}^2$ (actually the orbit is identified with the quotient group of $\mathbf{Z}^2$ divided by the stabilizer of any point in the orbit). Consequently, there are three types of infinite orbits: $\mathbf{Z}, \mathbf{Z}^2$, and $\mathbf{Z} \times n$, the latter meaning the product of $\mathbf{Z}$ with the finite cyclic group of order n, i.e. the collection of points $S^m T^k x_0$, where $m \in \mathbf{Z}$, $k = 0, 1, \dots, n-1$.

We will say that the orbit Y of the type $\mathbf{Z} \times n$ is *minimal* if there are no other orbits $\mathbf{Z} \times k$, where $k \neq n$ and k divides n. We use the same notation for the $n \times \mathbf{Z}$ orbits.

In any orbit of the type $\mathbf{Z}$ we can distinguish two integers m, n such that S, T restricted to the orbit correspond to the translations by m, n in $\mathbf{Z}$, respectively. We will then say that the orbit is of the type $\mathbf{Z}(q)$, where $q = m/n$ and $q = \infty$ if $n = 0$. Clearly m, n are always relatively prime and we may assume $n \geq 0$.

Now we are in a position to state our result.

Theorem 2. *Let X be a set on which on which $\mathbf{Z}^2$ acts by permutations and assume that all the orbits are infinite. Then X can be made into a compact Hausdorff space with continuous $\mathbf{Z}^2$ action iff none of the following holds*

(1) *for some n the number of minimal orbits of at least one of the types $\mathbf{Z} \times n$, $n \times \mathbf{Z}$ is positive and less than c,*
(2) *for some q the number of $\mathbf{Z}(q)$ orbits is positive and less than c,*
(3) *there exist only $\mathbf{Z} \times \mathbf{Z}$ orbits and $0 < |X| < c$.*

PROOF: We first handle the case where for some n there are at least c minimal orbits $\mathbf{Z} \times k$ and the only other orbits are of the type $\mathbf{Z} \times dk$, $d = 2, 3, \dots$. As in Section 1, we reduce the task of finding a compact model to the case where there are exactly c orbits $\mathbf{Z} \times k$ and only one orbit of each kind $\mathbf{Z} \times dk$. We arrange the $\mathbf{Z} \times k$ orbits into the 2-torus $\mathbf{T}^2$ with the rotations $S(x, y) = (x + \alpha, y)$, $T(x, y) = (x, y + 1/k)$, where α is an irrational number. Now, as in the finitely based case of Section 1 we wind up the other orbits around the torus. Instead of the symbolic description of Section 1 we now represent any other orbit geometrically as an infinite sequence of dk-loops wound around the torus with the x-coordinate of the j-th loop equal to $j\alpha$, $j \in \mathbf{Z}$. As $|j| \to \infty$, the loops converge to the surface of the torus and the T-rotation along the loop approaches the $1/n$-rotation of the y-coordinate on the torus. The S-rotation of any point coincides with the α-rotation of the x-coordinate. It is clear that such a system is a compact model for (X, S, T).

If, apart from the specified orbits, there is one (or more) orbit of the type $\mathbf{Z} \times \mathbf{Z}$, we can easily wind it up around the same torus as a base, using infinite loops at the same x-coordinates $j\alpha$.

Now consider the more general situation where all the minimal orbits $\mathbf{Z} \times k$ and $n \times \mathbf{Z}$ occur at least c times each and there are no orbits of the type $\mathbf{Z}$. Again, we may assume that there are exactly c orbits of each minimal type. To further simplify the wording, assume that there are no orbits of the type $n \times \mathbf{Z}$ and, for each minimal $\mathbf{Z} \times k$, denote by Y_k the 2-torus along with the accompanying orbits

$\mathbf{Z} \times dk$ and $\mathbf{Z} \times \mathbf{Z}$ as above (in general only some positive integers k will occur). We fix one Y_{k_0} and then make the others approach it as in 2 of Section 1. There is no problem with the x-coordinate, as we may use the same α for all the tori. For the y-coordinates, we choose the numbers $p_n/n \to 1/k_0$ as before and change the definition of T on each Y_n to let it rotate by p_n/n rather than $1/n$ (this does not change the structure of the orbits nor does it affect the continuity of S on Y_n). For the $\mathbf{Z} \times \mathbf{Z}$-orbits, S can also be changed accordingly. The resulting system will be compact and will realize the abstract dynamics of X.

If, on the other hand, there are less than c orbits of some minimal type, say, $\mathbf{Z} \times n$ then there is no compact model. Indeed, the set of these orbits would then be closed and invariant, so in particular it would constitute an S-dynamical system (Y, S) with $|Y| < c$ and no periodic points, contradicting Theorem 1.

Of course the same argument applies if there are only $\mathbf{Z} \times \mathbf{Z}$-orbits and $0 < |X| < c$, which excludes the possibility of a compact model in the cases (1) and (3) of the theorem. Since there is an easy model (two irrational rotations) if there are at least c orbits of the type $\mathbf{Z} \times \mathbf{Z}$, we have settled the problem if there are no $\mathbf{Z}$-orbits.

Now we study the $\mathbf{Z}$-orbits. As we will see they can be treated independently from the other types. First observe that if there are c orbits of the type $\mathbf{Z}(q)$, where $q = m/n$, $n \neq 0$, then we can model them on $\mathbf{T}$ letting T be any irrational rotation α and S be the rotation $\beta = q\alpha$ modulo 1.

If there are c orbits $\mathbf{Z}(q)$ for each q that occurs in the system, we may choose one type, say, $\mathbf{Z}(q_0)$, realize it as above as a 1-torus Y_0 with rotations α, β and make the other types $\mathbf{Z}(q_k)$ — more precisely, their associated tori Y_k — converge concentrically to Y_0. Now, however, each Y_k is wound several times around Y_0 in such a way that the angles corresponding to its rotations α_k and $\beta_k = q_k\alpha_k$ of Y_k converge to the angles of α and β, respectively as $k \to \infty$. This is possible as there is no a priori bound for the number of times Y_k encircles Y_0. More specifically, it is an exercise to see that given $\epsilon > 0$ and $q_k \in \mathbf{Q}$, there exists a positive integer K such that if $|m| > K$ or $|n| > K$ (here $\beta/\alpha = m/n$) then we can find irrational numbers α', β' such that $|\alpha - \alpha'| < \epsilon$, $|\beta - \beta'| < \epsilon$, and

$$\frac{P + \beta'}{Q + \alpha'} = q_k$$

for some integers P, Q. The numbers P, Q will determine how many times Y_k encircles Y_0.

Clearly we may change the roles of S and T if $q_0 = \infty$. If some of the cardinalities exceed c, we use the same approach as in Section 1 (passing to products and a quotient space). □

References

[E] Edelstein M., *On the representation of mappings of compact metrizable spaces as restrictions of linear transformations*, Canad. J. Math **22** (1970), 372–375.

[I] Iwanik A., *Period structure for pointwise periodic isometries of continua*, Acta Univ. Carolin. – Math. Phys. **29** (1988), no. 2, 19–21.

[I-J-K] Iwanik A., Janos L., Kowalski Z., *Periods in equicontinuous topological dynamical systems*, in: Nonlinear Analysis, Th. M. Rassias Ed., World Scientific Publ. Co. Singapore, 1987, pp. 355–365.

[J] Janos L., *Compactification and linearization of abstract dynamical systems*, preprint.

[K] Kowalski Z.S., *A characterization of periods in equicontinuous topological dynamical systems*, Bull. Polish Ac. Sc. **38** (1990), 121–124.

[V] de Vries H., *Compactification of a set which is mapped onto itself*, Bull. Acad. Polon. Sci. **5** (1957), 943–945.

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