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Pseudomonotonicity and nonlinear hyperbolic equations

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Abstract. In this paper we consider a nonlinear hyperbolic boundary value problem. We show that this problem admits weak solutions by using a lifting result for pseudomonotone operators and a surjectivity result concerning coercive and monotone operators.

Keywords: pseudomonotone operator, demicontinuous operator, maximal monotone operator, weak solution

Classification: 35A05, 35L20

1. Introduction

Let $T = [0, b]$ and $Z \subseteq R^N$ be a bounded domain with Lipschitz boundary Γ. Also let $D_k = \frac{\partial}{\partial z_k}$, $k \in \{1, 2, \dots, N\}$, and $D = grad$. We consider the following hyperbolic problem:

$$(1)\qquad \left\{ \begin{aligned} &\frac{\partial^2 x}{\partial t^2} + \sum_{k=1}^{N} D_k a_k(t, z, x, Dx) = g(t, z, x) \ \text{ a.e. on } \ T \times Z \\ &x(0, z) = \frac{\partial x}{\partial t}(b, z) = 0 \ \text{ a.e. on } Z,\ x\,|_{T\times\Gamma} = 0. \end{aligned} \right\}$$

This problem can be equivalently rewritten in abstract form as

$$(1^*)\qquad \left\{ \begin{aligned} &x''(t) + A(t, x(t)) = g(t, x(t)) \\ &x(0) = x'(b) = 0 \end{aligned} \right\}$$

where $A(t, .)$ is a pseudomonotone operator (see Section 3). Existence results for second order evolutions in the form of (1^*) were obtained by Barbu [2, Theorem 1.1, p. 268], Papageorgiou [5, Theorem 3.1] and Zeidler [7, Theorem 33A, p. 924]. All three authors examine the equation (inclusion) $x''(t) + B(t, x'(t)) + A(x(t)) \ni f(t, x(t))$ but they assume conditions on $A(.)$ stronger than pseudomonotonicity, namely monotonicity, linearity and boundedness. To the best of the knowledge of the author this is the first time that the notion of pseudomonotonicity is being used in the theory of hyperbolic problems.

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2. Preliminaries

In this section we fix our notation, introduce our assumptions on the data of the problem and recall some basic facts from nonlinear analysis that we will need in the sequel.

Our hypotheses on the functions $a_k(t,z,x,y)$, $k \in \{1,2,\dots,N\}$, are the following:

H(a): $a_k : T \times Z \times R \times R^N \to R$, $k \in \{1,2,\dots,N\}$, are functions such that

(i) $(t,z) \to a_k(t,z,x,y)$ is measurable;

(ii) $(x,y) \to a_k(t,z,x,y)$ is continuous;

(iii) $| a_k(t,z,x,y) | \leq \beta_1(t,z) + c_1(| x | + \|y\|)$ a.e. on $T \times Z$ for all $x \in R$, $y \in R^N$, with $\beta_1 \in L^2(T \times Z)$ and $c_1 > 0$;

(iv) $\sum\limits_{k=1}^{N} (a_k(t,z,x,y) - a_k(t,z,x,y'))(y_k - y_k') > 0$ a.e. on $T \times Z$, for all $x \in R$ and all $y, y' \in R^N$, $y \neq y'$; and

(v) $\sum\limits_{k=1}^{N} a_k(t,z,x,y)y_k \geq c_2\|y\|^2 - \beta_2(t,z)$ a.e. on $T \times Z$, for all $x \in R$, all $y \in R^N$, with $\beta_2 \in L^1(T \times Z)$ and $c_2 > 0$.

Because of hypothesis H(a), we can introduce the semilinear form $a : L^2(T, H_0^1(Z)) \times L^2(T, H_0^1(Z)) \to R$ defined by

$$a(x,y) = \int_0^b \int_Z \sum_{k=1}^{N} a_k(t,z,x,Dx)D_k y(t,z)\,dz\,dt.$$

In what follows by $((.,.))$ we will denote the duality brackets between $L^2(T, H_0^1(Z))$ and $L^2(T, H^{-1}(Z))$.

Definition 2.1. *A function $x \in L^2(T, H_0^1(Z))$ is said to be a (weak) solution of problem (1) if*

$$\Big(\Big(\frac{\partial^2 x}{\partial t^2}, u\Big)\Big) - a(x,u) = \int_0^b \int_Z g(t,z,x)u(t,z)\,dz\,dt$$

for all $u \in L^2(T, H^{-1}(Z))$.

Finally our hypotheses on the function $g(t,z,x)$ are the following:

H(g): $g : T \times Z \times R \to R$ is a function such that

(i) $(t,z) \to g(t,z,x)$ is measurable;

(ii) $x \to g(t,z,x)$ is continuous; and

(iii) $| g(t,z,x) | \leq \beta_3(t,z)$ for almost all $(t,z) \in T \times Z$ and $x \in R$, with $\beta_3 \in L^2(T \times Z)$.

Suppose now that X is a separable Hilbert space with inner product $(.,.)$. We recall the following generalization of the notion of a maximal monotone operator (see Zeidler [7, p. 585]).

Definition 2.2. *An operator $A : X \to X$ is said to be "pseudomonotone" if $x_n \xrightarrow{w} x$ in X as $n \to \infty$ and $\limsup_{n\to\infty}(A(x_n), x_n - x) \le 0$, imply that $(A(x), x - y) \le \liminf_{n\to\infty}(A(x_n), x_n - y)$ for all $y \in X$.*

Remark 2.1. *A monotone hemicontinuous operator or a completely continuous operator $A : X \to X$ is pseudomonotone. Pseudomonotonicity is preserved by addition and it is easy to see that it implies property (M) (i.e. if $x_n \xrightarrow{w} x$ in X, $A(x_n) \xrightarrow{w} u$ in X as $n \to \infty$ and $\limsup_{n\to\infty}(A(x_n), x_n - x) \le 0$, then $A(x) = u$). For details we refer to Zeidler* [7, pp. 583–588].

We will also need the following notion:

Definition 2.3. *Suppose that $L : D(L) \subseteq X \to X$ is a linear maximal monotone operator and $V : X \to X$ is a bounded nonlinear operator. We say that $V(.)$ is "pseudomonotone with respect to $D(L)$" if for $\{x_n\}_{n\ge1} \subseteq D(L)$ such that $x_n \xrightarrow{w} x$ in X and $L(x_n) \xrightarrow{w} L(x)$ in X as $n \to \infty$ and also $\limsup_{n\to\infty}(V(x_n), x_n - x) \le 0$, then we have $V(x_n) \xrightarrow{w} V(x)$ in X and $(V(x_n), x_n) \to (V(x), x)$ as $n \to \infty$.*

The following proposition will be used in order to prove the maximal monotonicity of the differential operator (see Zeidler [7, Theorem 32L, p. 897]).

Proposition 2.1. *A linear operator $L : D(L) \subseteq X \to X$ is maximal monotone if and only if L is densely defined, closed and both L and L^* are monotone.*

Now let $L : D(L) \subseteq L^2(T, X) \to L^2(T, X)$ be defined by $Lx = -x''$ for all $x \in D(L) = \{y \in L^2(T, X) : y'' \in L^2(T, X),\ y(0) = y'(T) = 0\}$ (here the time derivatives of $x(.)$ and $y(.)$ are understood in the sense of vector-valued distributions). With the use of the above proposition we easily see that $L(.)$ is maximal monotone.

Consider now an operator $A : T \times X \to X$ satisfying the following hypotheses:

H(A): $A : T \times X \to X$ is an operator such that

(i) $t \to A(t, x)$ is measurable;

(ii) $x \to A(t, x)$ is demicontinuous and pseudomonotone (recall that demicontinuity means that if $x_n \to x$ in X as $n \to \infty$, then $A(t, x_n) \xrightarrow{w} A(t, x)$ in X as $n \to \infty$);

(iii) $\|A(t, x)\| \le \beta_4(t) + c_3\|x\|$ a.e. on T with $c_3 > 0$, $\beta_4 \in L^2(T)_+ = \{f \in L^2(T) : f(t) \ge 0 \text{ a.e.}\}$; and

(iv) $(A(t, x), x) \ge c_4\|x\|^2 - \beta_5(t)\|x\| - \beta_6(t)$ for almost all $t \in T$, all $x \in X$, with $c_4 > 0$, $\beta_6 \in L^1(T)$, $\beta_5 \in L^2(T)_+$.

Let $\widehat{A} : L^2(T, X) \to L^2(T, X)$ be the Nemitsky (superposition) operator corresponding to $A(t, x)$ i.e. $\widehat{A}(x)(.) = A(., x(.))$. We will show that, in some sense, the pseudomonotonicity of $A(t, .)$ can be lifted to $\widehat{A}(.)$.

Proposition 2.2. *Let $A : T \times X \to X$ be an operator satisfying hypothesis* H(A) *and $L : D(L) \subseteq L^2(T,X) \to L^2(T,X)$ be the linear maximal monotone operator defined by $L(x) = -x''$ for all $x \in D(L) = \{y \in L^2(T,X) : y'' \in L^2(T,X), y(0) = y'(T) = 0\}$. Then the Nemitsky operator $\widehat{A} : L^2(T,X) \to L^2(T,X)$ is demicontinuous and pseudomonotone with respect to $D(L)$.*

Proof: We will first prove the demicontinuity of $\widehat{A}(.)$. So let $x_n \to x$ in $L^2(T,X)$ as $n \to \infty$. By passing to a subsequence if necessary, we may assume that $x_n(t) \to x(t)$ in X a.e. on T as $n \to \infty$. Then hypothesis H(A)(ii) implies that for every $y \in L^2(T,X)$ we have that $(A(t,x_n(t)),y(t)) \to (A(t,x(t)),y(t))$ a.e. on T. Because of hypothesis H(A)(iii) we can apply the extended dominated convergence theorem (see Ash [1, Theorem 7.52, p. 295]) and get that

$$((\widehat{A}(x_n),y)) = \int_0^b (A(t,x_n(t)),y(t))\,dt \to \int_0^b (A(t,x(t)),y(t))\,dt = ((\widehat{A}(x),y))$$

(here $((.,.))$ denotes the inner product of $L^2(T,X)$). Since $y \in L^2(T,X)$ was arbitrary, we conclude that $\widehat{A}(x_n) \xrightarrow{w} \widehat{A}(x)$ in $L^2(T,X)$ as $n \to \infty$, which shows that $\widehat{A}(.)$ is demicontinuous.

Next we will show that $\widehat{A}(.)$ is pseudomonotone with respect to $D(L)$. So assume that $x, x_n \in D(L)$, $n \geq 1$, $x_n \xrightarrow{w} x$ in $L^2(T,X)$, $x_n'' \xrightarrow{w} x''$ in $L^2(T,X)$ as $n \to \infty$ and

$$\limsup_{n\to\infty}((\widehat{A}(x_n), x_n - x)) = \limsup_{n\to\infty} \int_0^b (A(t,x_n(t)), x_n(t) - x(t))\,dt \leq 0.$$

Since $x_n'' \xrightarrow{w} x''$ in $L^2(T,X)$ as $n \to \infty$ we have

$$x_n'(t) = -\int_t^T x_n''(s)\,ds \xrightarrow{w} -\int_t^T x''(s)\,ds = x'(t)$$

for every $t \in T$, i.e. $x_n'(t) \xrightarrow{w} x'(t)$ in X for every $t \in T$. Similarly we can show that $x_n(t) \xrightarrow{w} x(t)$ in X for every $t \in T$.

Now let $\xi_n(t) = (A(t,x_n(t)), x_n(t) - x(t))$. Also let $B \subseteq T$ be the Lebesgue-null set outside of which hypotheses H(A)(iii) and (iv) hold. Then we have:

$$\begin{aligned} \xi_n(t) \geq \vartheta_n(t) = c_4\|x_n(t)\|^p - \beta_5(t)\|x_n(t)\| - \beta_6(t) \\ -(\beta_4(t) + c_3\|x_n(t)\|)\|x(t)\| \quad \text{for all} \quad t \in T\backslash B. \end{aligned} \tag{2}$$

Let also $\vartheta_n(t) = 0$ for $t \in B$. Set $C = \{t \in T : \liminf_{n\to\infty} \xi_n(t) < 0\}$. This is a Lebesgue measurable subset of T. We will show that $C \subseteq B$ and therefore $\lambda(C) = 0$ (here $\lambda(.)$ is the Lebesgue measure on T). Indeed, if $t \in C \cap (T \setminus B)$, then consider a subsequence $\{n_m\}_{m\geq 1}$ of N such that $\liminf_{n\to\infty} \xi_n(t) = \lim_{n\to\infty} \xi_{n_m}(t) < 0$. Exploiting the fact that $A(t,.)$ is pseudomonotone we obtain that $(A(t, x_{n_m}(t)), x_{n_m}(t) - x(t)) \to 0$, contradicting the fact that $t \in C$. So $\lambda(C) = 0$, which means that $0 \leq \liminf_{n\to\infty} \xi_n(t)$ a.e. on T. Then from the extended Fatou's lemma (see Ash [1, Theorem 7.5.2, p. 295]), we obtain

$$0 \leq \int_0^b \liminf_{n\to\infty} \xi_n(t)\, dt \leq \liminf_{n\to\infty} \int_0^b \xi_n(t)\, dt \leq \limsup_{n\to\infty} \int_0^b \xi_n(t)\, dt \leq 0.$$

Hence $\int_0^b \xi_n(t)\, dt \to 0$ as $n \to \infty$. Since $0 \leq \liminf_{n\to\infty} \xi_n(t)$ a.e. on T, we deduce that $\xi_n^-(t) \to 0$ a.e. on T as $n \to \infty$. Moreover from (2) we see that $\{\vartheta_n\}_{n\geq 1}$ is uniformly integrable in $L^1(T)$. Then $0 \leq \xi_n^-(t) \leq \vartheta_n^-(t)$ a.e. on T and of course $\{\vartheta_n^-\}_{n\geq 1}$ is also uniformly integrable. By applying the extended dominated convergence theorem we see that $\int_0^b \xi_n^-(t)\, dt \to 0$ as $n \to \infty$. Therefore we have

$$\lim_{n\to\infty} \int_0^b | \, \xi_n(t) \, | \, dt = \lim_{n\to\infty} \int_0^b (\xi_n(t) + 2\xi_n^-(t))\, dt = 0,$$

so by passing to a subsequence if necessary, we may assume that $\xi_n(t) \to 0$ a.e. on T as $n \to \infty$. Since $A(t,.)$ is pseudomonotone, we obtain that $(A(t, x_n(t)), x_n(t)) \to (A(t, x(t)), x(t))$ a.e. on T and $A(t, x_n(t)) \xrightarrow{w} A(t, x(t))$ a.e. on T as $n \to \infty$. A final application of the extended dominated convergence theorem gives that $\widehat{A}(x_n) \xrightarrow{w} \widehat{A}(x)$ in $L^2(T, X)$ and $((\widehat{A}(x_n), x_n)) \to ((\widehat{A}(x), x))$ as $n \to \infty$. Therefore $\widehat{A}(.)$ is pseudomonotone with respect to $D(L)$. $\square$

In the proof of our main theorem in the next section we will need the following well known surjectivity result (see B.-A. Ton [6] or Lions [4, Theorem 1.2, p. 319]):

Proposition 2.3. *If Y is a reflexive Banach space, $L : D(L) \subseteq Y \to Y^*$ is a linear maximal monotone operator and $G : Y \to Y^*$ is a bounded, demicontinuous operator which is pseudomonotone with respect to $D(L)$ and satisfies $\frac{(G(y),y)_{Y^*,Y}}{\|y\|_Y} \to +\infty$ as $\|y\|_Y \to \infty$ (i.e. $G(.)$ is coercive) then $L + G$ is surjective; i.e. $R(L + G) = Y^*$.*

3. Main theorem

Theorem 3.1. *If hypotheses* H(a) *and* H(g) *hold, then problem* (1) *has a solution.*

Proof: In what follows, for notational simplicity, we write $X = H_0^1(Z)$ and $X^* = H^{-1}(Z)$. Note that X is a separable Hilbert space. Let $\widehat{a} : T \times X \times X \to R$ be the following time dependent semilinear form:

$$\widehat{a}(t,x,y) = \int_Z \sum_{k=1}^N a_k(t,z,x(t,z),Dx(t,z))D_k y(t,z)\,dz.$$

Because of hypothesis H(a)(iii), $y \to \widehat{a}(t,x,y)$ is a continuous linear functional defined on X. Therefore the equation

$$(A_1(t,x),y) = \widehat{a}(t,x,y)$$

defines uniquely a function $A_1 : T \times X \to X^*$. Using hypothesis H(a) we can verify that $t \to A_1(t,x)$ is measurable, $x \to A_1(t,x)$ is demicontinuous, $\|A_1(t,x)\|_* \leq \beta_1'(t) + c_1'\|x\|$ a.e. on T for all $x \in X$, with $\beta_1' \in L^2(T)$, $c_1' > 0$ and $(A_1(t,x),x) \geq c_2'\|x\|^2 - \beta_2'(t)$ a.e. on T for all $x \in X$, with $\beta_2' \in L^1(T)$ and $c_2' > 0$. Now let $f : T \times X \to L^2(T \times Z)$ be defined by $f(t,x)(z) = -g(t,z,x(z))$. From hypotheses H(g)(i) and (ii) we see that the function $t \to f(t,x)$ is measurable, the function $x \to f(t,x)$ is continuous and

$$\begin{aligned}|(f(t,x),x)| &\leq \int_Z |g(t,z,x(z))x(z)|\,dz\\ &\leq c(t)\|x\|_2 \ \text{ a.e. on } T \text{ with } \ c(.) \in L^2(T).\end{aligned}$$

Also from hypothesis H(g)(iii) we have

$$\|f(t,x)\|_2 \leq \|\beta_3(t,.)\|_2 \ \text{ a.e. on } T.$$

Now let $A : T \times X \to X^*$ be defined by $A(t,x) = A_1(t,x) + f(t,x)$. It is easy to see that $t \to A(t,x)$ is measurable, $x \to A(t,x)$ is demicontinuous, $\|A(t,x)\|_* \leq \overline{\beta}(t) + \overline{c}\|x\|$ a.e. on T with $\overline{\beta} \in L^2(T)$, $\overline{c} > 0$, and $|\ (A(t,x),x)\ | \geq c_2'\|x\|^2 - \beta_2'(t)\|x\| - \beta_3'(t)$ a.e. on T with $c_2' > 0$, $\beta_2' \in L^2(T)$ and $\beta_3' \in L^1(T)$. Moreover from Theorem 3.1 of Gossez-Mustonen [3] we know that $A(t,.)$ is pseudomonotone. Thus if $\widehat{A} : L^2(T,X) \to L^2(T,X^*)$ is the Nemitsky operator corresponding to $A(t,x)$, then by virtue of Proposition 2.2 we see that $\widehat{A}$ is pseudomonotone with respect to $D(L)$.

Let $\widehat{f} : L^2(T,X) \to L^2(T \times Z)$ be the Nemitsky operator corresponding to $f(t,x)$. We claim that the operator $x \to \widehat{A}_1(x) + \widehat{f}(x) = \widehat{A}(x)$ is coercive. Indeed, from hypothesis H(a)(iv) we know that

$$((\widehat{A}_1(x),x)) \geq c_2\|x\|^2_{L^2(T,X)} - \|\beta_2\|_{L^1(T\times Z)}. \tag{3}$$

Also using Young's inequality with $\varepsilon > 0$ we have

$$((\widehat{f}(x), x)) = (\widehat{f}(x), x)_{L^2(T;L^2(Z))} \leq \|\widehat{f}(x)\|_{L^2(T\times Z)} \|x\|_{L^2(T\times Z)}$$
$$\leq \frac{\varepsilon^2}{2} \|\widehat{f}(x)\|^2_{L^2(T\times Z)} + \frac{1}{2\varepsilon^2} \|x\|^2_{L^2(T\times Z)}.$$

But by virtue of hypothesis H(g)(iii) and Minkowski's inequality we have

$$\|\widehat{f}(x)\|^2_{L^2(T\times Z)} \leq \gamma_1 + \gamma_2 \|x\|^2_{L^2(T,X)} \quad \text{for some} \quad \gamma_1, \gamma_2 > 0.$$

Hence it follows that

$$((\widehat{f}(x), x)) \leq \frac{\varepsilon^2}{2}\gamma_1 + \frac{\varepsilon^2}{2}\gamma_2 \|x\|^2_{L^2(T,X)} + \frac{1}{2\varepsilon^2} \|x\|^2_{L^2(T\times Z)}. \tag{4}$$

From (3) and (4) we get

$$\begin{aligned} ((\widehat{A}_1(x) + \widehat{f}(x), x)) &\geq (c - \frac{1}{2\varepsilon^2}\gamma_2) \|x\|^2_{L^2(T,X)}, \\ &- \frac{1}{2\varepsilon^2} \|x\|^2_{L^2(T\times Z)} - \frac{\varepsilon^2}{2}\gamma_1. \end{aligned} \tag{5}$$

Choose $\varepsilon > 0$ so that $c_2 > \frac{1}{2\varepsilon^2}\gamma_2$. Then from (5) we deduce that $x \to \widehat{A}_1(x) + \widehat{f}(x)$ is coercive.

Proposition 2.3 implies that the operator $L + \widehat{A}$ is surjective. Therefore there exists $x \in D(L)$ such that

$$Lx + \widehat{A}(x) = 0. \tag{6}$$

But (6) is equivalent to (1), therefore x solves (1). □

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