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Operational quantities

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Abstract. In this paper we consider maps called operational quantities, which assign a non-negative real number to every operator acting between Banach spaces, and we obtain relations between the kernels of these operational quantities and the classes of operators of the Fredholm theory.

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1. Introduction

Roughly speaking, an operational quantity is a procedure a which determines, for every (continuous linear) operator T from a Banach space X into another Banach space Y, a real number $a(T) \geq 0$. Several authors have considered operational quantities in order to obtain characterizations and perturbation results for the classes of operators of Fredholm theory. In this paper we study general operational quantities and we obtain particular results known and other which are new. We fix our attention in the following aspects:

(1) From an operational quantity a there are derived other operational quantities, taking into account the behavior of a on the restrictions of T to the subspaces M of X, $a(TJ_M)$. For example,

$$ia(T) := \inf\{a(TJ_M) : M \subset X \text{ infinite dimensional subspace of } X\}.$$

From the *norm*

$$n(T) :=\| T \|,$$

from the *injection modulus*

$$j(T) := \inf\{\| Tx \| \ : x \in X, \| x \|= 1\},$$

and from the *surjection modulus*

$$q(T) := \sup\{\varepsilon > 0 : \varepsilon B_Y \subset TB_X\},$$

where B_X denotes the closed unit ball of X, several operational quantities have been derived and certain perturbation results obtained.

(2) For each operational quantity a we consider the *kernel* and the *cokernel* of a:

$$ker(a) := \{T : a(T) = 0\},\ cok(a) := \{T : a(T) > 0\}.$$

If a is an operational quantity verifying $j \leq a \leq n$ (or $q \leq a \leq n$) we obtain several results about the kernel and the cokernel of the operational quantities derived from a and we relate these classes with the operator classes of the Fredholm theory. For example,

$$T \in cok(isa) \Leftrightarrow T \text{ is left semi-Fredholm.}$$

(3) We obtain some results about perturbation of the semi-Fredholm operators by the strictly singular operators and the strictly cosingular operators, in terms of operational quantities. For example, if a and b are operational quantities such that $a(T+S) \leq a(T) + b(S)$ and $b(S) = b(-S)$, for any $T, S \in L(X,Y)$, then

$$S \in cok(a) \text{ and } T \in ker(b) \Rightarrow T + S \in cok(a).$$

Certain operational quantities associated with a space ideal [12] have been used in the definition of some classes of operators which generalize the classes of strictly singular and strictly cosingular operators, and the classes of semi-Fredholm operators (see [5], [6], [8]).

Notation and terminology. Let X, Y be Banach spaces. We shall denote by X^* the dual space of X, $L(X,Y)$ the class of all (continuous linear) operators from X into Y, and B_X the closed unit ball of X. Given $T \in L(X,Y)$, we denote by $T^* \in L(Y^*, X^*)$ its conjugate operator, $R(T)$ its range and by $N(T)$ its kernel. If $M \subset X$ is a (closed) subspace, then J_M stands for the canonical inclusion of M into X, Q_M the quotient map $X \to X/M$ and $\dim(M)$ its dimension.

Finally, given two operational quantities a and b we will write $a \leq \alpha b$, for $\alpha > 0$, if for any operator T we have $a(T) \leq \alpha b(T)$. We will say that a and b are *comparable* if $\alpha a \leq b$ or $\alpha b \leq a$ for some $\alpha > 0$; and we will say that they are *equivalent* if $\alpha a \leq b \leq \beta a$ for some $\beta > \alpha > 0$.

2. Left operational quantities

In this section X is an infinite dimensional Banach space and Y is a Banach space. We consider the following family of (closed) subspaces of X:

$$S(X) := \{M \subset X : M \text{ is an infinite dimensional subspace of } X\},$$
$$S^*(X) := \{M \subset X : M \text{ is a finite codimensional subspace of } X\}.$$

A *left operational quantity* is a procedure a which determines, for every X, Y, a map from $L(X,Y)$ into the non-negative real numbers, such that for every $T \in L(X,Y)$ we have that $\{a(TJ_M) : M \in S(X)\}$ is bounded.

A left operational quantity a is said to be *s-increasing* if for every X, Y and every $T \in L(X, Y)$ the following implication holds:

$$N, M \in S(X) \text{ and } N \subset M \Rightarrow a(TJ_N) \le a(TJ_M);$$

analogously, a is said to be *s-decreasing* if

$$N, M \in S(X) \text{ and } N \subset M \Rightarrow a(TJ_N) \ge a(TJ_M).$$

The norm n and the surjection modulus q are s-increasing and the injection modulus j is s-decreasing.

From a left operational quantity a we can derive several operational quantities:

Definition 2.1. *Let a be a left operational quantity. We define, for $T \in L(X, Y)$,*

$$\begin{aligned} ia(T) &:= \inf\{a(TJ_M) : M \in S(X)\}, \\ sa(T) &:= \sup\{a(TJ_M) : M \in S(X)\}, \\ i^*a(T) &:= \inf\{a(TJ_M) : M \in S^*(X)\}, \\ s^*a(T) &:= \sup\{a(TJ_M) : M \in S^*(X)\}. \end{aligned}$$

Note that the functions ia, sa, i^*a and s^*a also are left operational quantities and we can iterate the procedure of Definition 2.1 and obtain many derivated operational quantities from a: sia, $s^*s^*ii^*a$, $i^*isa, \dots$. It is surprising that we obtain only three different new quantities if a is s-monotone [11]: if a is s-increasing, then ia, sia and i^*a are the only different derivated left operational quantities which are obtained from a by means of Definition 2.1; moreover,

$$ia \le sia \le i^*a \le a.$$

Analogously, if a is s-decreasing, only are obtained sa, isa, s^*a and moreover

$$a \le s^*a \le isa \le sa.$$

Remark 2.2. (1) For any left operational quantity a, the quantity ia is s-decreasing and the quantity sa is s-increasing.

(2) If a is s-increasing, then $sa = s^*a = a$, and sia and i^*a are s-increasing.

(3) If a is s-decreasing, then $ia = i^*a = a$, and isa and s^*a are s-decreasing.

(4) Given $T \in L(X, Y)$ and $M \in S(X)$, if a is s-increasing, then we have that, for $b = ia, sia, i^*a$,

$$N \in S^*(M) \Rightarrow b(TJ_M) = b(TJ_N);$$

analogously, if a is s-decreasing, then we obtain the same implication for $b = s^*a, isa, sa$. $\square$

From the norm n and from the injection modulus j the following left operational quantities have been derived: in by Gramsch (see [14]); sin, sj and s^*j by Schechter [14]; $i^*n(T)$ by Sedaev [16] and Lebow and Schechter [10]; isj by González and Martinón [6].

Now we obtain the kernel or the cokernel of certain left operational quantities. Recall some definitions, for $T \in L(X,Y)$: T is an *injection* ($T \in Inj$) if $R(T)$ is closed and $N(T) = \{0\}$, or equivalently if $j(T) > 0$; T is *left* (or *upper*) *semi-Fredholm* ($T \in SF_l$) if $R(T)$ is closed and $dim(N(T)) < \infty$; T is *compact* ($T \in Co$) if TB_X is a relatively compact subset; T is *strictly singular* ($T \in SS$) if $TJ_M \in Inj \Rightarrow dim(M) < \infty$.

Note that if $ker(a) = Co$ or $ker(a) = SS$, then a is non s-decreasing. Also, if $Inj \subset cok(a) \subset SF_l$, then a is non s-increasing.

Theorem 2.3. *Let a be a left operational quantity such that $j \leq a \leq n$. Then*

(1) *a s-increasing $\Rightarrow cok(ia) = SF_l$;*
(2) *$cok(isa) = SF_l$;*
(3) *a s-decreasing $\Rightarrow cok(s^*a) = SF_l$.*

PROOF: Let $T \in L(X,Y)$.

(1) Assume $\varepsilon := ia(T) > 0$. For every $M \in S(X)$, we have that $a(TJ_M) \geq \varepsilon > 0$. Because

$$a(TJ_{N(T)}) = a(0) \leq n(0) = 0,$$

we have that $N(T)$ is a finite dimensional subspace of X. Moreover, if $R(T)$ is not closed, then there exists a subspace $M \in S(X)$ such that $n(TJ_M) < \varepsilon \leq a(TJ_M)$ [3, Corollary III.1.10]: a contradiction. Consequently $T \in SF_l$.

Let $T \in SF_l$. There exists $P \in S^*(X)$ such that TJ_P is an injection; that is, $j(TJ_P) > 0$. For any $M \in S(X)$ we obtain $P \cap M \in S(X)$ and

$$0 < j(TJ_P) \leq j(TJ_{P\cap M}) \leq a(TJ_{P\cap M}) \leq a(TJ_M);$$

hence $ia(T) \geq j(TJ_P) > 0$. Consequently, $T \in cok(ia)$.

(2) Because the quantity sa is s-increasing (Remark 2.2(1)) and $j \leq sa \leq n$, from (1) we obtain $cok(isa) = SF_l$.

(3) Let $T \in SF_l$. There exists $P \in S^*(X)$ such that TJ_P is an injection, hence $a(TJ_P) \geq j(TJ_P) > 0$. Hence $s^*a(T) > 0$; that is, $T \in cok(s^*a)$.

Assume $s^*a(T) > 0$. From $s^*a \leq isa$ and (2) we obtain $T \in SF_l$. □

In the above theorem there appear the following results, which are well known:

$$SF_l = cok(in)\ [14]\ = cok(isj)\ [6]\ = cok(s^*j)\ [14].$$

In [9] is proved that

$$s^*j \leq isj \leq in,$$

but they are not equivalent. For Hilbert spaces the situation is simpler. In [15] it is proved that $s^*j(T) = in(T)$, for $T \in L(H,K)$, where H is an infinite dimensional Hilbert space and K is any Hilbert space.

Theorem 2.4. *Let a be a left operational quantity such that $j \leq a \leq n$. Then*
(1) *a s-decreasing $\Rightarrow ker(sa) = SS$;*
(2) *$ker(sia) = SS$;*
(3) *a s-increasing $\Rightarrow Co \subset ker(i^*a) \subset SS$.*

PROOF: Let $T \in L(X, Y)$.

(1) Assume $sa(T) = 0$. For any $M \in S(X)$ we obtain $a(TJ_M) = 0$, hence $j(TJ_M) = 0$. Then TJ_M is a not an injection for all $M \in S(X)$; that is, $T \in SS$.
Let $T \in SS$. If $M \in S(X)$ and $\varepsilon > 0$, then there exists $N \in S(M)$ such that $n(TJ_N) < \varepsilon$ [3, Theorem III.2.1]. Because $a(TJ_M) \leq a(TJ_N)$, we obtain $a(TJ_M) = 0$. Hence $sa(T) = 0$.

(2) Because ia is s-decreasing, from (1) we obtain $ker(sia) = SS$.

(3) Let $T \in Co(X, Y)$. For any $\varepsilon > 0$, there exists $P \in S^*(X)$ such that $n(TJ_P) < \varepsilon$ [3, Theorem III.2.3], hence $a(TJ_P) < \varepsilon$. Consequently $i^*a(T) = 0$.
Assume $i^*a(T) = 0$. If $\varepsilon > 0$, then there is $P \in S^*(X)$ such that $a(TJ_P) < \varepsilon$, hence $j(TJ_P) < \varepsilon$. For any $M \in S(X)$ we have that

$$j(TJ_M) \leq j(TJ_{P \cap M}) \leq a(TJ_{P \cap M}) \leq a(TJ_P) < \varepsilon.$$

Consequently $j(TJ_M) = 0$; that is, TJ_M is not an injection, for all $M \in S(X)$. Hence $T \in SS$. $\square$

The Theorem 2.3(1) and Theorem 2.4(2) generalize [2, Theorem 7] and [2, Proposition 9], respectively. Moreover, from Theorem 2.4 we obtain [14]:

$$SS = ker(sin) = ker(sj).$$

In [7] it is showed that $sj \leq sin$, but they are not equivalent.

Remark 2.5. The result (3) in the above theorem is the better possible. In fact, the quantity $a = n$ verifies $j \leq a \leq n$, a is s-increasing and

$$ker(i^*a) = Co \quad \text{[3, Theorem III.2.3]};$$

also the quantity $a = sj$ verifies $j \leq a \leq n$, a is s-increasing and $i^*a = a$ (Remark 2.2(4)), hence $ker(i^*a) = SS$. $\square$

Remark 2.6. Let H be an infinite dimensional Hilbert space and K any Hilbert space. For $T \in L(H, K)$, we have that

$$sj(T) = sin(T) = i^*n(T).$$

(The last equality implies that in Hilbert spaces agree the concepts of strictly singular operator and compact operator, a well known result.) We give a proof inspired by [15]. It is enough to show that $i^*n(T) \leq sj(T)$. Assume $0 < i^*n(T)$.

Given $\varepsilon > 0$, there exists an infinite-dimensional subspace M of H such that $j(TJ_M) > i^*n(T) - \varepsilon$.

Let P_1 be a finite-codimensional subspace of H. Then there exists $x_1 \in P_1$, $\|x_1\| = 1$, such that

$$\|Tx_1\| > n(TJ_{P_1}) - \varepsilon \geq i^*n(T) - \varepsilon.$$

We denote $M_1 :=< x_1 >^{\perp}$, the orthogonal of the subspace generated by x_1, and $N_1 :=< Tx_1 >^{\perp}$.

Let $P_2 := M_1 \cap T^{-1}N_1 \cap P_1$, which is a finite-codimensional subspace of H. There exists $x_2 \in P_2$, $\|x_2\| = 1$, such that

$$\|Tx_2\| > n(TJ_{P_2}) - \varepsilon \geq i^*n(T) - \varepsilon.$$

Moreover

$$x_1 \perp x_2 \text{ and } Tx_1 \perp Tx_2.$$

In this way, we construct orthogonal sequences (x_n) and (Tx_n) such that

$$\|x_n\| = 1 \text{ and } \|Tx_n\| > i^*n(T) - \varepsilon.$$

Let M be the closed subspace generated by the sequence (x_n). Take $z \in M$; that is, $z = \sum \alpha_n x_n$, for a suitable sequence of scalars $(\alpha_n) \in \ell_2$. Then

$$\begin{aligned}\|Tz\|^2 &= \|T \sum \alpha_n x_n\|^2 = \sum \|\alpha_n T x_n\|\|^2 \\ &= \sum \mid \alpha_n \mid^2 \|Tx_n\|^2 > \sum \mid \alpha_n \mid^2 (i^*n(T) - \varepsilon)^2 \\ &= (i^*n(T) - \varepsilon)^2 \sum \|\alpha_n x_n\|^2 = (i^*n(T) - \varepsilon)^2 \|z\|^2.\end{aligned}$$

Hence $j(TJ_M) \geq i^*n(T) - \varepsilon$. □

Theorem 2.7. *Let a be a left operational quantity. Then*

$$Inj \subset cok(a) \subset SF_l \Rightarrow ker(sa) = SS \text{ and } cok(s^*a) = SF_l.$$

Proof: Let $T \in L(X, Y)$. Assume $T \in \ker(sa)$. For all $M \in S(X)$, we have that $a(TJ_M) = 0$, hence $TJ_M \notin Inj$, that is, $T \in SS$.

If $T \in SS$ and $M \in S(X)$, then $TJ_M \notin SF_l$, hence $a(TJ_M) = 0$; consequently $T \in ker(sa)$.

If $T \in cok(s^*a)$, then there exists $P \in S^*(X)$ such that $a(TJ_P) > 0$, hence $TJ_P \in SF_l$ and consequently $T \in SF_l$.

For $T \in SF_l$, we have that there exists $P \in S^*(X)$ such that $TJ_P \in Inj$, hence $a(TJ_P) > 0$; that is, $T \in cok(s^*a)$. □

3. Right operational quantities

In this section X is a Banach space and Y is an infinite dimensional Banach space. We study the right operational quantities in a similar way as we have done before for the left operational quantities. We omit the proofs, because they are similar to the proofs of Section 2. We consider the following families of (closed) subspaces of Y:

$$Q(Y) := \{U \subset Y : Y/U \text{ is an infinite dimensional}\},$$
$$Q_*(Y) := \{U \subset Y : U \text{ is a finite dimensional subspace of } Y\}.$$

A *right operational quantity* is a procedure a which determines, for every X, Y, a map from $L(X,Y)$ into the non-negative real numbers, such that for every $T \in L(X,Y)$ we have that $\{a(Q_U T) : U \in Q(Y)\}$ is bounded.

A right operational quantity a is said to be *q-increasing* if for every X, Y and every $T \in L(X,Y)$ the following implication holds:

$$V, U \in Q(Y) \text{ and } V \supset U \Rightarrow a(Q_V T) \leq a(Q_U T);$$

analogously, a is said to be *q-decreasing* if

$$V, U \in Q(Y) \text{ and } V \supset U \Rightarrow a(Q_V T) \geq a(Q_U T).$$

The norm n and the injection modulus j are q-increasing and the surjection modulus q is q-decreasing.

We derive several right operational quantities from one given.

Definition 3.1. Let a be a right operational quantity. We define, for $T \in L(X,Y)$,

$$ia'(T) := \inf\{a(Q_U T) : U \in Q(Y)\},$$
$$sa'(T) := \sup\{a(Q_U T) : U \in Q(Y)\},$$
$$i_*a'(T) := \inf\{a(Q_U T) : U \in Q_*(Y)\},$$
$$s_*a'(T) := \sup\{a(Q_U T) : U \in Q_*(Y)\}.$$

Now also we obtain only three different right operational quantities derived from one given ([11]): if a is q-increasing, then ia', sia' and i_*a' are the only different derivated right operational quantities which are obtained from a by means of Definition 3.1. Moreover,

$$ia' \leq sia' \leq i_*a' \leq a;$$

analogously, if a is q-decreasing there are obtained only sa', isa' and s_*a' and moreover

$$a \leq s_*a' \leq isa' \leq sa'.$$

Remark 3.2. (1) For any right operational quantity a, the quantity ia' is q-decreasing and sa' is q-increasing.

(2) If a is q-increasing, then $sa' = s_*a' = a$ and sia' and i_*a' are q-increasing.

(3) If a is q-decreasing, then $ia' = i_*a' = a$, and isa' and s_*a' are q-decreasing.

(4) Given $T \in L(X,Y)$ and $U \in Q(Y)$, if a is q-increasing, for $b = ia', sia', i_*a'$, we have that

$$V \in Q(Y), U \subset V, V/U \text{ is finite dimensional } \Rightarrow b(Q_U T) = b(Q_V T);$$

analogously, if a is q-decreasing, for $b = s_*a', isa', sa'$, also the above implication is obtained. □

From the norm n and from the surjection modulus q the following right operational quantities have been derived: in' and sin' by Weis [18]; i_*n' by Fajnshtejn and Shulman (see [1]) and Zemánek [19]; sq' and s_*q' by Zemánek [19]; isq' by González and Martinón [6].

Now we give some results about the kernel or the cokernel of certain right operational quantities. Recall the definitions of some classes of operators of the Fredholm theory. Let $T \in L(X,Y)$: T is a *surjection* ($T \in Sur$) if $R(T) = Y$, or equivalently $q(T) > 0$; T is *right* (or *lower*) *semi-Fredholm* ($T \in SF_r$) if $R(T)$ is closed and $dim(Y/R(T)) < \infty$; T is *strictly cosingular* ($T \in SC$) if $Q_M T \in Sur \Rightarrow dim(Y/M) < \infty$.

Theorem 3.3. *Let a be a right operational quantity. Then*

$$Sur \subset cok(a) \subset SF_r \Rightarrow ker(sa') = SC \text{ and } cok(s_*a') = SF_r.$$

If moreover $q \le a \le n$, then

(1) a *q-increasing* $\Rightarrow cok(ia') = SF_r$ *and* $Co \subset ker(i_*a') \subset SC$;
(2) $cok(isa') = SF_r$ *and* $ker(sia') = SC$;
(3) a *q-decreasing* $\Rightarrow cok(s_*a') = SF_r$ *and* $ker(sa') = SC$.

From Theorem 3.3 we obtain

$$SF_r = cok(in') \text{ [18] } = cok(s_*q') \text{ [19] } = cok(isq') \text{ [6]}$$

and

$$SC = ker(sin') \text{ [18] } = ker(sq') \text{ [19]}.$$

In [9] it is proved that $s_*q' \le isq' \le in'$, but they are not equivalent and $sq' \le sin'$, but they are not equivalent.

Remark 3.4. The inclusions in the Theorem 3.3(1) are the better possible result. In fact, the quantity $a = n$ verifies $q \le a \le n$, a is q-increasing and $ker(i_*a') = Co$ [17]; also the quantity $a = sq'$ verifies $q \le a \le n$, a is q-increasing and $i_*a' = a$, hence $ker(i_*a') = SC$. □

Remark 3.5. If H is any Hilbert space and K is an infinite dimensional Hilbert space, for $T \in L(H,K)$, we obtain

$$s_*q'(T) = in'(T) \text{ and } sq'(T) = i_*n'(T).$$

In fact, based on $q(T) = j(T^*)$, where T^* denotes the dual operator of T, it is easy to prove that

$$s_*q'(T) = s^*j(T^*) = in(T^*) = in'(T).$$

Analogously is proved the another equality.

4. Perturbation

In this section X is an infinitedimensional Banach space and Y is a Banach space. We obtain several perturbation theorems in terms of operational quantities. We refer to [15] for some interesting results about perturbation of classes of operators.

First we prove that $cok(a)$ is open if a verifies a simple condition, hence $cok(a)$ can be perturbed by operators with a small norm.

Proposition 4.1. *Let $T, S \in L(X,Y)$. If the left operational quantity a verifies*

$$a(T+S) \leq a(T) + n(S),$$

then $cok(a) \cap L(X,Y)$ is open in $L(X,Y)$, and consequently $ker(a) \cap L(X,Y)$ is closed.

Proof: Note that $a(T) \leq a(T+S) + n(S)$. Assume $T \in cok(a) \cap L(X,Y)$. If $n(S) < a(T)$, then $S + T \in cok(a)$. Hence $cok(a) \cap L(X,Y)$ is open. □

In the following results we give some properties about the behaviour of the operational quantities on the sum of two operators.

Proposition 4.2. *If a and b are s-increasing left operational quantities, and, for every $T, S \in L(X,Y)$,*

$$a(T+S) \leq a(T) + b(S),$$

then

(1) $ia(T+S) \leq ia(T) + sib(S)$;
(2) $ia(T+S) \leq sia(T) + ib(S)$;
(3) $sia(T+S) \leq sia(T) + sib(S)$;
(4) $i^*a(T+S) \leq i^*a(T) + i^*b(S)$.

Proof: (1) Let $M \in S(X)$. We have that

$$a((T+S)J_M) \leq a(TJ_M) + b(SJ_M) \leq a(T) + b(SJ_M).$$

Taking infimum, $ia(T+S) \le a(T) + ib(S)$. Hence

$$ia(T+S) \le ia((T+S)J_M) \le a(TJ_M) + ib(SJ_M) \le a(TJ_M) + sib(S).$$

Taking infimum, we obtain the result: $ia(T+S) \le ia(T) + sib(S)$.

(2) Analogous to (1).

(3) For $M \in S(X)$, we obtain from (1)

$$ia((T+S)J_M) \le ia(TJ_M) + sib(SJ_M) \le ia(TJ_M) + sib(S).$$

Taking supremum we obtain: $sia(T+S) \le sia(T) + sib(S)$.

(4) Let $\varepsilon > 0$. There exist $P, N \in S^*(X)$ such that

$$a(TJ_P) < i^*a(T) + \varepsilon \text{ and } b(TJ_N) < i^*b(S) + \varepsilon.$$

Then

$$\begin{aligned} i^*a(T+S) &\le a((T+S)J_{P\cap N}) \le a(TJ_{P\cap N}) + b(SJ_{P\cap N}) \\ &\le a(TJ_P) + b(SJ_N) < i^*a(T) + i^*b(S) + 2\varepsilon. \end{aligned}$$

The conclusion is clear. $\square$

It is possible to obtain analogous results to the above proposition for a s-decreasing and b s-increasing, a s-increasing and b s-decreasing, and also for a and b s-decreasing.

Taking in account that $n(T+S) \le n(T) + n(S)$ and $j(T+S) \le j(T) + n(S)$, Proposition 4.2 and other similar propositions can be applied to obtain several relations between the left operational quantities derived from the norm and from the injection modulus.

Corollary 4.3. *If a and b are left operational quantities such that*

$$a(T+S) \le a(T) + b(S) \text{ and } b(S) = b(-S),$$

for any $T, S \in L(X,Y)$, then

(1) $b(S) < a(T) \Rightarrow T, T+S \in cok(a)$;
(2) $T \in cok(a)$ *and* $S \in ker(b) \Rightarrow T+S \in cok(a)$;
(3) $T \in ker(a)$ *and* $S \in ker(b) \Rightarrow T+S \in ker(a)$;
(4) *(4)* $S \in ker(b) \Rightarrow \forall T,\ a(T+S) = a(T)$.

If $a(T+S) \le a(T) + b(S)$ and $b(S) = b(-S)$, then the above theorem can be applied to every pair of left operational quantities derived from a and b:

For a and b s-increasing, from Proposition 4.2: (ia, sib), (sia, sib), (i^*a, i^*b).

For a s-decreasing and b s-increasing, from Proposition 4.2: (sa, sib), (isa, sib), (s^*a, i^*b).

From the inequalities $n(T+S) \le n(T) + n(S)$ and $j(T+S) \le j(T) + n(S)$, we obtain the following corollary, which contains some quantitative versions of the well-known theorems $SF_l + SS \subset SF_l$ and $SF_l + Co \subset SF_l$.

Corollary 4.4. *Let $T, S \in L(X,Y)$.*
(1) $sin(S) < in(T) \Rightarrow T, T+S \in SF_l$ [S];
(2) $sin(S) < isj(T) \Rightarrow T, T+S \in SF_l$;
(3) $i^*n(S) < s^*j(T) \Rightarrow T, T+S \in SF_l$ [1];
(4) $S \in SS \Rightarrow \forall T, in(T+S) = in(T)$;
(5) $S \in SS \Rightarrow \forall T, sin(T+S) = sin(T)$;
(6) $S \in SS \Rightarrow \forall T, sj(T+S) = sj(T)$ [14];
(7) $S \in Co \Rightarrow \forall T, i^*n(T+S) = i^*n(T)$.

Analogous results can be obtained for the right operational quantities.

5. Examples

In this section we give some examples of operational quantities and we apply the results of the above sections.

Example 5.1. The *minimum modulus* of $T \in L(X,Y)$, $T \neq 0$, is defined in the following way:

$$\gamma(T) := \inf\{\| Tx \| : x \in X,\ dist(x, N(T)) = 1\};$$

moreover, we put $\gamma(0) = 0$. The following equivalence is well-known ([3, Theorem IV.1.6])

$$R(T) \text{ is closed and } R(T) \neq \{0\} \Leftrightarrow \gamma(T) > 0.$$

The operational quantity γ is not monotone, but verifies the inequality

$$j \leq \gamma \leq n.$$

In fact, if T is not injective, then it is clear $0 = j(T) \leq \gamma(T)$; if T is injective, then $j(T) = \gamma(T)$. Moreover, $\gamma(T) = j(\hat{T}) \leq n(\hat{T}) = n(T)$, where $\hat{T} : X/N(T) \to Y$ is the operator defined by $\hat{T}(x + N(T)) := Tx$. From Theorems 2.3(2) and 2.4(2), we obtain

$$cok(is\gamma) = SF_l \text{ and } ker(si\gamma) = SS.$$

Since $q(T) = j(T^*)$ ([12, Proposition B.3.8]), $\gamma(T) = \gamma(T^*)$ ([3, Corollary IV.1.9]) and $n(T) = n(T^*)$, we have

$$q \leq \gamma \leq n.$$

We deduce from Theorem 3.3(2) the following equalities:

$$cok(is\gamma') = SF_r \text{ and } ker(si\gamma') = SC.$$

□

Example 5.2. If A is an operator ideal in the sense of Pietsch [12], then we consider the quantity n_A defined in the following way, for $T \in L(X,Y)$:

$$n_A(T) := \inf\{n(T-K) : K \in A \cap L(X,Y)\}.$$

Then, n_A is s-increasing, q-increasing and $n_A \leq n$. From $n_A \leq n$, taking into account the proofs of Theorems 2.3 and 2.4, we can write

$$cok(in_A) \subset SF_l, \;\; SS \subset ker(sin_A), \;\; Co \subset ker(i^* n_A).$$

Analogously,

$$cok(in'_A) \subset SF_r, \;\; SC \subset ker(sin'_A), \;\; Co \subset ker(i_* n'_A).$$

If $A = Co$, the operator ideal of all compact operators, then we have $i^* n \leq n_{Co} \leq n$, and consequently

$$i^* n_{Co} = i^* n, \;\; sin_{Co} = sin, \;\; in_{Co} = in.$$

Also $i_* n' \leq n_{Co} \leq n$, hence

$$i_* n'_{Co} = i_* n', \;\; sin'_{Co} = sin', \;\; in'_{Co} = in'.$$

If $A = SS$, the operator ideal of all strictly singular operators, then we have ([13]) $sin \leq n_{SS} \leq n_{Co}$. Hence

$$sin \leq i_* n_{SS} \leq i_* n, \;\; in_{SS} = in, \;\; sin_{SS} = sin.$$

If $A = SC$, the operator ideal of all strictly cosingular operators, then $sin' \leq n_{SC} \leq n_{Co}$. Hence

$$sin' \leq i_* n'_{SC} \leq i_* n', \;\; in'_{SC} = in', \;\; sin'_{SC} = sin'.$$

□

Example 5.3. The *Hausdorff measure of noncompactness* of $T \in L(X,Y)$ is defined in the following way:

$$h(T) := \inf\{\varepsilon > 0 : \exists\, C \subset Y \text{ relatively compact}, \;\; TB_X \subset C + \varepsilon B_Y\}.$$

Fajnstejhn [1] proved that

$$h = i_* n',$$

hence $ker(h) = Co$. Moreover, h is s-increasing and q-increasing. Consequently, the right operational quantities derived from h agree with the right operational quantities derived from the norm:

$$i_* h' = i_* n' = h, \;\; sih' = sin' \text{ and } ih' = in'.$$

On the other hand, the left operational quantities derived from h are related with the quantity ν, a "hybrid" operational quantity derived from the norm (see Example 5.4); the quantity ih is equivalent to in ($ih \leq in \leq 2ih$ [9]), and, consequently, sih is equivalent to sin ($sih \leq sin \leq 2sih$). Rakocevic [13] considered the quantity ih and Tylli [17] the quantities ih and ih'. □

Example 5.4. In [4] a "hybrid" quantity is studied: for $T \in L(X,Y)$ is defined

$$\nu(T) := i^*(i_*n')(T) = i_*(i^*n)'(T),$$

related with the Hausdorff measure of noncompactness (see Example 5.3) by means of $\nu(T) = i^*h(T)$; that is,

$$\begin{aligned}\nu(T) &= \inf\{\inf\{n(Q_U T J_M) : U \in Q_*(Y)\} : M \in S^*(X)\}\\ &= \inf\{\inf\{n(Q_U T J_M) : M \in S^*(X)\} : U \in Q_*(Y)\}.\end{aligned}$$

The quantity ν is s-increasing and q-increasing. It verifies $i^*\nu = i_*\nu' = \nu$ and $ker(\nu) = Co$. Moreover, $i^*\nu = \nu = i^*h$, $si\nu = sih$, $i\nu = ih$. □

Remark 5.5. It is easy prove that if an operational quantity a is s-decreasing and q-decreasing and moreover $a(0) = 0$, then $a = 0$; hence all the operational quantities derived from a are null. From this result, we obtain the following corollary: if a is an operational quantity such that $a(0) = 0$, and a is s-decreasing and q-increasing, then $ia' = sia' = 0$; analogously, if $a(0) = 0$ and a is s-increasing and q-decreasing, then $ia = sia = 0$.

The injection modulus j is s-decreasing and q-increasing, and it satisfies $ij' = sij' = i_*j' = 0$. The surjection modulus q is s-increasing and q-decreasing, and it verifies $iq = siq = i^*q = 0$. □

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