On modular approximation property in the Besicovitch-Orlicz space of almost periodic functions

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Abstract. We investigate some convergence questions in the class of Besicovitch-Orlicz spaces of vector valued functions. Next, the existence problem of the projection operator on closed convex subsets is considered in the class of almost periodic functions. This problem was considered in [5], in the case of an Orlicz space. The approximation property obtained in both cases are of the same kind. However, the arguments which are used in the proofs are different.

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I. Introduction and preliminaries

It is well known that in the case of normed uniformly convex spaces, the projection operator on closed convex sets exists.
Uniform convexity is well characterized in the case of Orlicz spaces both for the Luxemburg or the Orlicz norm.

Uniform convexity with respect to a modular was already considered by H. Nakano [9], H. Hudzik [2] in the general case, by H. Hudzik [3], A. Kaminska [4] in the case of an Orlicz space and in [6] in the case of the Besicovitch-Orlicz space of almost periodic functions.

Now, let us agree with some definitions and notations.

I.1. In the sequel, the notation φ will stand for a Young function, i.e. a function $\varphi : R \to R$ which satisfies the conditions: φ is even, convex, satisfies $\varphi(u) = 0$ iff $u = 0$ and $\lim_{u\to\infty} \varphi(u) = \infty$.

This function is called of Δ_2 type when there exist constants $K > 2$ and $u_0 \geq 0$ for which $\varphi(2u) \leq K\varphi(u)$, $\forall\, u \geq u_0$.

If φ is of Δ_2 type with constants $K > 2$ and $u_0 > 0$, we have (cf. [4]): for all u_1, $0 < u_1 \leq u_0$, there exists $K_1 \geq K$ such that $\varphi(2u) \leq K\varphi(u)$ for $u \geq u_1$.

The function φ will be called uniformly convex when, for every $a \in (0, 1)$, there exists $\delta(a) \in (0, 1)$ and $u_0 \geq 0$ such that

$$\varphi\Big(\frac{u + (au)}{2}\Big) \leq (1 - \delta(a))\frac{\varphi(u) + \varphi(au)}{2}\,, \quad \forall\, u \geq u_0.$$

I.2. Let X be a real linear space. A functional $\rho : X \to [0, +\infty]$ will be called (pseudo)modular, if it satisfies:

(i) $\rho(x) = 0$ iff $x = 0$ for a modular, and
(i)$'$ $\rho(0) = 0$ for a pseudomodular,
(ii) $\rho(x) = \rho(-x)\ \forall\, x \in X$,
(iii) $\rho(\alpha x + \beta y) \le \rho(x) + \rho(y);\ \forall\, \alpha, \beta \ge 0,\ \alpha + \beta = 1;\ x, y \in X$.

When, in the case of (iii), we have

(iii)$'$ $\rho(\alpha x + \beta y) \le \alpha\rho(x) + \beta\rho(y);\ \forall\, \alpha, \beta \ge 0,\ \alpha + \beta = 1;\ x, y \in X$,

the (pseudo)modular ρ will be called convex.

The linear space $X_\rho = \{x \in X, \lim_{\alpha\to 0} \rho_\varphi(\alpha x) = 0\}$ associated to the modular ρ_φ is called a modular space.

When ρ is a convex (pseudo)modular, a (pseudo)norm is defined on X by means of the formula (cf. [8])

$$\|x\|_\rho = \inf\{k > 0,\ \rho(x/k) \le 1\}.$$

The modular space X_ρ is called uniformly convex with respect to the (pseudo)modular ρ when:
For every $\varepsilon > 0$ and $r > 0$, there exists a number $p(r, \varepsilon) > 0$ such that, for all $x, y \in X_\rho$ satisfying $\rho(x) \le r$, $\rho(y) \le r$, $\rho(x - y) \ge r\varepsilon$, we have

$$\rho\Big(\frac{x + y}{2}\Big) \le r(1 - p(r, \varepsilon)).$$

The definition of uniform convexity with respect to the norm $\|\ \|_\rho$ can be given in the same way and in this case we can take $r = 1$.

I.3. The Besicovitch-Orlicz space of almost periodic functions.
Let $M(R, E)$ be the set of all real Lebesgue measurable functions with value in a Banach space $(E, \|\ \|)$.

The functional $\rho_\varphi : M(R, E) \to [0, +\infty]$, $\rho_\varphi(f) = \overline{\lim}_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \varphi(\|f(t)\|) d\mu$ is a pseudomodular on $M(R, E)$ (cf. [7], [8]).

The associated modular space $B^\varphi(R, E) = \{f \in M(R, E), \lim_{\alpha\to 0} \rho_\varphi(\alpha f) = 0\} = \{f \in M(R, E),\ \rho_\varphi(\alpha f) < \infty$ for some $\alpha > 0\}$ is called the Besicovitch-Orlicz space.

This space is endowed with the norm (cf. [8])

$$\|f\|_\rho = \inf\{k > 0,\ \rho_\varphi(f/k) \le 1\},\ f \in B^\varphi(R, E).$$

Let now A be the set of generalized trigonometric polynomials, i.e.

$$A = \{P(t) = \sum_{j=1}^{n} a_j e^{i\lambda_j t},\ a_j \in E,\ \lambda_j \in R\}.$$

The Besicovitch-Orlicz space of almost periodic functions, denoted B^φp.p. (R,E), is the closure of the set A in $B^\varphi(R,E)$ with respect to the pseudonorm $\| \|_\varphi$:

$$B^\varphi\text{p.p.}\,(R,E) = \{f \in B^\varphi(R,E),\ \exists\, P_n \in A,\ n = 1,2,\ldots,\ \text{s.t.} \\ \lim_{n\to\infty} \|f - P_n\|_\varphi = 0\}.$$

We shall also be concerned with the space

$$\tilde{B}^\varphi\text{p.p.}\,(R,E) = \{f \in B^\varphi(R,E),\ \exists\, \{P_n\} \subset A \text{ s.t.} \\ \lim_{n\to\infty} \rho_\varphi(\lambda(f - P_n)) = 0, \quad \text{for some} \quad \lambda > 0\}.$$

Some structural and topological properties of all these spaces are considered in [1], [6] and [7].

To every $f \in B^\varphi$p.p. (R,E) we can associate a formal Fourier series. Questions concerning the convergence of the Fourier series are not considered. However, we have the following approximation result (cf. [1], [7]):
For all $f \in B^\varphi$p.p. (R,E), there exists a sequence $\{P_n\}$ of trigonometric polynomials called the Bochner-Fejer polynomials satisfying

(i) $\lim_{n\to\infty} \rho_\varphi(f - P_n) = 0$,
(ii) $\rho_\varphi(P_n) \le \rho_\varphi(f)$.

The most important geometrical properties of the spaces $\tilde{B}^\varphi$p.p. (R,E) and B^φp.p. (R,E) are characterized in [6] and [7].

Concerning the uniform convexity, we have the following result ([6]).

Theorem. *The space $\tilde{B}^\varphi$p.p. (R,E) is uniformly convex with respect to the pseudomodular ρ_φ if and only if*

(i) *φ is strictly convex on R,*
(ii) *φ is uniformly convex on $[d,\infty[$, with some $d \ge 0$,*
(iii) *E is uniformly convex.*

Remarks. Supposing φ satisfies the Δ_2-condition, the theorem may be stated for the norm and then we get the following approximation property:
Let C be a normed closed convex subset in B^φp.p. (R,E). Then there exists a projection operator $P : X \to C$ such that

$$\forall\, x \in X,\ \|x - Px\|_\varphi = \inf\{\|x - z\|_\varphi,\ z \in C\}.$$

II. Results

We shall first investigate some natural questions on convergence of sequences in the space $B^\varphi(R,E)$. The usual convergence results of the Lebesgue measure theory are not valid in the space $B^\varphi(R,E)$ as it can be seen from the following example:
The sequence $\{f_k\}$, where $f_k(x) = 0$ if $x > k$ and $f_k(x) = 1$ if $x \le k$, satisfies the

hypothesis of the classical convergence results of the Lebesgue theory, however, we have
$\lim_{k\to\infty}\rho_\varphi(f_k)\neq\rho_\varphi(f)$ (in fact $\rho_\varphi(f)>\lim_{k\to\infty}(f_k)$).
This shows that the Lebesgue and Fatou's convergence results cannot be directly used. In the sequel, we shall prove some results of this nature in the space $B^\varphi(R,E)$.

II.1. Let $P(R)$ be the family of subsets of R and $\Sigma(R)$ the Σ-algebra of Lebesgue measurable sets.

For $A\in\Sigma$, we define the set function

$$\overline{\mu}(A)=\overline{\lim_{T\to\infty}}\frac{1}{2T}\int_{-T}^{T}\chi_A(t)\,d\mu=\overline{\lim_{T\to\infty}}\frac{1}{2T}\mu(A\cap[-T,T]).$$

Clearly, $\overline{\mu}$ is null on sets with μ-finite measure. Moreover, $\overline{\mu}$ is not σ-additive. As usual, a sequence of Σ-measurable functions $\{f_k\}$ will be called $\overline{\mu}$-convergent to a function f, when, for all $\varepsilon>0$,

$$\lim_{k\to\infty}\overline{\mu}\{x\in R,\ \|f_k(x)-f(x)\|>\varepsilon\}=0.$$

Lemma 1. *Let $\{f_n\}$ be a sequence of functions in $B^\varphi(R,E)$, modular convergent to a function $f\in B^\varphi(R,E)$, i.e. $\lim_{n\to\infty}\rho_\varphi(f_n-f)=0$. Then $\{f_n\}$ is $\overline{\mu}$-convergent to f.*

Proof: Putting $E_n(\varepsilon)=\{x\in R,\|f_n(x)-f(x)\|>\varepsilon\}$, we have

$$\begin{aligned}\overline{\mu}(E_n(\varepsilon))&=\overline{\lim_{T\to\infty}}\frac{1}{2T}\int_{-T}^{T}\chi_{E_n(\varepsilon)}(x)\,d\mu\\ &\le\overline{\lim_{T\to\infty}}\frac{1}{2T}\int_{-T}^{T}\frac{\|f_n(x)-f(x)\|}{\varepsilon}\chi_{E_n(\varepsilon)}(x)\,d\mu\\ &\le\frac{1}{\varepsilon}\overline{\lim_{T\to\infty}}\frac{1}{2T}\int_{-T}^{T}\|f_n(x)-f(x)\|\,d\mu.\end{aligned}\tag{+}$$

Moreover, from the continuity of φ and Jensen's inequality

$$\begin{aligned}&\varphi\Big(\overline{\lim_{T\to\infty}}\frac{1}{2T}\int_{-T}^{T}\|f_n(x)-f(x)\|\,d\mu\Big)\\ &\le\overline{\lim_{T\to\infty}}\varphi\Big(\frac{1}{2T}\int_{-T}^{T}\|f_n(x)-f(x)\|\,d\mu\Big)\\ &\le\overline{\lim_{T\to\infty}}\frac{1}{2T}\int_{-T}^{T}\varphi(\|f_n(x)-f(x)\|)\,d\mu.\end{aligned}\tag{++}$$

Finally, if $\delta > 0$ is given, from the modular convergence of f_n to f, there exists $n_0 \in \mathbb{N}$ such that $\overline{\lim}_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \varphi(\|f_n(x) - f(x)\|)\, d\mu \leq \varphi(\delta\varepsilon)$ and then, in virtue of (++)

$$\overline{\lim_{T\to\infty}} \frac{1}{2T} \int_{-T}^{T} \|f_n(x) - f(x)\|\, d\mu \leq \delta\varepsilon.$$

Now, using (+), we get

$$\overline{\mu}(E_n(\varepsilon)) \leq \delta,\ \forall\, n \geq n_0.$$

This means exactly that the sequence $\{f_n\}$ is $\overline{\mu}$-convergent to f. □

Lemma 2. *Let $h \in \tilde{B}^{\varphi}$p.p.$(R, E)$ be such that $\rho_\varphi(h) = a$, $a > 0$. Then for every $\overline{\theta} \in (0, 1)$ there exists $\beta > 0$ and $T_0 > 0$ such that*

$$(*) \qquad \mu\{\overline{G} \cap [-T, T]\} \geq \overline{\theta}\ 2T,\ \forall\, T \geq T_0,\ \textit{where } \overline{G} = \{t \in R,\ \|h(t)\| \leq \beta\}.$$

Proof: Let $\overline{\theta} \in (0, 1)$ and take $\beta > 0$ such that $\varphi(\beta)\ (1 - \overline{\theta}) > 2\rho_\varphi(h) = 2a$. Now, supposing that $(*)$ does not hold, we have:
There exists a sequence $\{T_n\}$ increasing to infinity and satisfying

$$\mu\{\overline{G} \cap [-T_n, T_n]\} < \overline{\theta}\ 2T_n$$

and then we get

$$\begin{aligned}\frac{1}{2T_n} \int_{-T_n}^{T_n} \varphi(\|h(t)\|)\, d\mu &\geq \frac{1}{2T_n} \int_{(\overline{G}\cap[-T_n,T_n])'} \varphi(\|h(t)\|)\, d\mu \\ &\geq \frac{1}{2T_n} \varphi(\beta)(1 - \overline{\theta})\ 2T_n \geq 2a.\end{aligned}$$

Finally, letting n tend to infinity we get

$$\rho_\varphi(h) \geq \overline{\lim_{n\to\infty}} \frac{1}{2T_n} \int_{-T_n}^{T_n} \varphi(\|h(t)\|)\, d\mu = 2\rho_\varphi(h),$$

a contradiction. □

Lemma 3. *Let $g \in B^{\varphi}$p.p.(R, E). Then, for every $\varepsilon > 0$, there exists $\delta > 0$ and $T_0 > 0$ such that $\rho_\varphi(g\chi_Q) \leq \varepsilon$ for all sets Q for which $\mu(Q \cap [-T, T]) \leq \delta\ 2T$ for $T \geq T_0$.*

Proof: Let $\varepsilon > 0$ be given. If $\rho_\varphi(g) = 0$, the result is trivial. We then assume $\rho_\varphi(g) > 0$. From the definition, there exists a trigonometric polynomial P_ε such that $\rho_\varphi(2(g - P_\varepsilon)) < \varepsilon/2$. Moreover, the trigonometric polynomial P_ε is uniformly bounded on R. So, put $M(\varepsilon) = \sup_R \varphi(2\|P_\varepsilon(t)\|)$.

Let $\overline{\theta} \in (0,1)$ satisfy $(1-\overline{\theta})M(\varepsilon) < \varepsilon/4$. By Lemma 1 there exists $\beta > 0$ and $T_0 > 0$ such that

$$\mu(\overline{G} \cap [-T,T\,]) \geq \overline{\theta}\ 2T,\ \forall T \geq T_0,\ \text{where } \overline{G} = \{t \in R,\ \|g(t)\| \leq \beta\}.$$

We then get for $T \geq T_0$

$$\begin{aligned}
&\frac{1}{2T}\int_{(\overline{G}\cap[-T,T\,])'} \varphi(\|g(t)\|)\,d\mu \\
(*) \qquad &\leq \frac{1}{2T}\int_{(\overline{G}\cap[-T,T\,])'} \frac{1}{2}\{\varphi(2\|g(t)-P_\varepsilon(t)\|)+\varphi(2\|P_\varepsilon(t)\|)\}\,d\mu \\
&\leq \varepsilon/4 + \frac{1}{2}M(\varepsilon)(1-\overline{\theta})2T\frac{1}{2T} \leq \varepsilon/2.
\end{aligned}$$

Let now $\delta = \varepsilon/2\varphi(\beta)$ and $Q \subset R$ be such that $\mu(Q \cap [-T,T\,]) \leq \delta\ 2T$, $\forall\, t \geq T_0$. Putting $Q_1 = Q \cap (\overline{G} \cap [-T,T\,])$ and $Q_2 = Q \cap (\overline{G} \cap [-T,T\,])'$, we have $Q = Q_1 \cup Q_2$. Moreover,

$$\frac{1}{2T}\int_{Q_1} \varphi(\|g(t)\|)\,d\mu \leq \varphi(\beta)\frac{1}{2T}\mu(Q_1) \leq \varphi(\beta) \quad \delta \leq \varepsilon/2$$

and, from $(*)$

$$\frac{1}{2T}\int_{Q_2} \varphi(\|g(t)\|)\,d\mu \leq \frac{1}{2T}\int_{(\overline{G}\cap[-T,T\,])'} \varphi(\|g(t)\|)\,d\mu \leq \varepsilon/2.$$

Finally, we have

$$\begin{aligned}
\frac{1}{2T}\int_{Q\cap[-T,T\,]} \varphi(\|g(t)\|)\,d\mu &= \frac{1}{2T}\int_{(Q_1\cup Q_2)\cap[-T,T\,]} \varphi(\|g(t)\|)\,d\mu \\
&= \frac{1}{2T}\int_{Q_1} \varphi(\|g(t)\|)\,d\mu + \frac{1}{2T}\int_{Q_2} \varphi(\|g(t)\|)\,d\mu \leq \varepsilon,
\end{aligned}$$

i.e. $\rho_\varphi(g\chi_Q) \leq \varepsilon$. □

Lemma 4. *Let $\{f_k\}$, $f_k \in B^\varphi(R,E)$ be a sequence of functions $\overline{\mu}$-convergent to some $f \in B^\varphi(R,E)$. Suppose that there exists $g \in B^\varphi p.p.\,(R,E)$ such that* $\max(\|f_k(x)\|, \|f(x)\|) \leq g(x)$. *Then we have*

$$\lim_{k\to\infty} \rho_\varphi(f_k) = \rho_\varphi(f).$$

Proof: Applying Lemma 2 to the function g, we get that for every $\overline{\theta} \in (0,1)$ there exits $M > 0$ and $T_0 > 0$ such that

$$\mu(\overline{G} \cap [-T,T\,]) \leq \overline{\theta}\ 2T,\ \forall T \geq T_0,\ \text{where } \overline{G} = \{x \in R,\ \|g(x)\| \geq M\}.$$

Hence it follows that $\overline{\mu}(\overline{G}) \leq \overline{\theta}$.

Using the uniform continuity of φ on $[0, M]$ and the fact that if $x \in (\overline{G})'$, $\|f_k(x)\|$ and $\|f(x)\|$ are in $[0, M]$, we can write:
For every $\varepsilon > 0$, there exits $\eta > 0$ (η depends on M) such that

$$\forall\, x \in (\overline{G})',\ |\varphi(\|f_k(x)\|) - \varphi(\|f(x)\|)| > \varepsilon \Rightarrow \|f_k(x) - f(x)\| > \eta.$$

Now, since $\{f_n\}$ is $\overline{\mu}$-convergent to f, we have

$$\lim_{k\to\infty} \overline{\mu}\{x \in R,\ \|f_k(x) - f(x)\| > \eta\} = 0$$

and, in particular,

$$\lim_{k\to\infty} \overline{\mu}\{x \in G',\ |\varphi(\|f_k(x)\|) - \varphi(\|f(x)\|)| > \varepsilon\} = 0.$$

Moreover,

$$\begin{aligned}
&\overline{\mu}\{x \in R,\ |\varphi(\|f_k(x)\|) - \varphi(\|f(x)\|)| > \varepsilon\} \\
&\ \leq \overline{\mu}\{x \in (\overline{G})',\ |\varphi(\|f_k(x)\|) - \varphi(\|f(x)\|)| > \varepsilon\} \\
&\ + \overline{\mu}\{x \in G,\ |\varphi(\|f_k(x)\|) - \varphi(\|f(x)\|)| > \varepsilon\} \\
&\ \leq \overline{\mu}\{x \in (\overline{G})',\ |\varphi(\|f_k(x)\|) - \varphi(\|f(x)\|)| > \varepsilon\} + \overline{\mu}(\overline{G})
\end{aligned}$$

and then

$$\begin{aligned}
&\overline{\lim_{k\to\infty}}\, \overline{\mu}\{x \in R,\ |\varphi(\|f_k(x)\|) - \varphi(\|f(x)\|)| > \varepsilon\} \\
&\leq \overline{\lim_{k\to\infty}}\, \overline{\mu}\{x \in (\overline{G})',\ |\varphi(\|f_k(x)\|) - \varphi(\|f(x)\|)| > \varepsilon\} + \overline{\mu}(\overline{G}) \leq 0 + \overline{\theta}.
\end{aligned}$$

Finally, since $\overline{\theta}$ is arbitrary, it follows that

$$\overline{\lim_{k\to\infty}}\, \overline{\mu}\{x \in R,\ |\varphi(\|f_k(x)\|) - \varphi(\|f(x)\|)| > \varepsilon\} = 0.$$

This means that the sequence $\varphi(\|f_k\|)$ is $\overline{\mu}$-convergent to $\varphi(\|f\|)$. We can then write:
For every $\varepsilon > 0$ and $\delta > 0$, there exists $k_0 \in \mathbb{N}$ such that, for all $k \geq k_0$, there exists T_k for which we have

$$\mu\{x \in [-T, T],\ |\varphi(\|f_k(x)\|) - \varphi(\|f(x)\|)| > \varepsilon\} \leq \delta\, 2T, \quad \forall\, T \geq T_k.$$

We take δ as in Lemma 3, then we define for $k \geq k_0$ and $T \geq T_k$ the set $E_2 = \{x \in R,\ |\varphi(\|f_k(x)\|) - \varphi(\|f(x)\|)| > \varepsilon\}$ and denote by E_2' its complement. From the hypothesis $\mu(E_2 \cap [-T, T]) \leq \delta\, 2T$, $\forall\, T \geq T_k$, and by Lemma 3

$$\frac{1}{2T} \int_{E_2 \cap [-T,T]} |\varphi(\|f_k(x)\|) - \varphi(\|f(x)\|)|\, d\mu \leq \frac{1}{2T} \int_{E_2 \cap [-T,T]} \varphi(\|g(t)\|)\, d\mu \leq 2\varepsilon.$$

In the same way

$$\frac{1}{2T}\int_{E_2'\cap[-T,T]} |\varphi(\|f_k(x)\|)-\varphi(\|f(x)\|)|\,d\mu \\ \le \frac{1}{2T}\int_{-T}^{T}\sup_{E_2'}|\varphi(\|f_k(x)\|)-\varphi(\|f(x)\|)|\,d\mu \le \varepsilon$$

for every $k \ge k_0$ and $T \ge T_k$, whence we have

$$\frac{1}{2T}\int_{-T}^{T}[\varphi(\|f_k(x)\|)-\varphi(\|f(x)\|)]\,d\mu \le 3\varepsilon$$

and then, for all $k \ge k_0$,

$$\overline{\lim_{T\to\infty}}\,\Big|\frac{1}{2T}\int_{-T}^{T}[\varphi(\|f_k(x)\|)-\varphi(\|f(x)\|)]\,d\mu\Big| \le 3\varepsilon.$$

This means exactly that

$$\lim_{k\to\infty}\Big(\overline{\lim_{T\to\infty}}\,\Big|\frac{1}{2T}\int_{-T}^{T}[\varphi(\|f_k(x)\|)-\varphi(\|f(x)\|)]\,d\mu\Big|\Big) = 0,$$

from which it follows immediately that

$$\lim_{k\to\infty}\rho_\varphi(f_k) = \rho_\varphi(f).$$

□

Corollary 5. *Let $\{f_k\}$ be a sequence of functions from $B^\varphi(R,E)$. Suppose that*

(i) *there exists $f \in B^\varphi(R,E)$ such that $\lim_{k\to\infty}\rho_\varphi(f-f_k)=0$,*

(ii) *there exits $g \in B^\varphi$p.p.(R,E) such that $\|f_k(x)\| \le g(x)$, $\forall\, x \in R$.*

Then $\lim_{k\to\infty}\rho_\varphi(f_k)=\rho_\varphi(f)$.

Proof: It follows directly from Lemma 1 and Lemma 4. □

Proposition 6. *Let $\{f_k\}$, $k \ge 1$, be a sequence of functions in $B^\varphi(R,E)$ and $f \in B^\varphi$p.p.(R,E) such that $\lim_{k\to\infty}\rho_\varphi(f-f_k)=0$. Then we have*

$$\underline{\lim}_{k\to\infty}\,\rho_\varphi(f_k) \ge \rho_\varphi(f).$$

Proof: We define a sequence $\{g_k\}$ as follows

$$g_k(x) = \begin{cases} f_k(x) & \text{if } \|f(x)\| \ge \|f_k(x)\| \\ f(x) & \text{if } \|f(x)\| < \|f_k(x)\|. \end{cases}$$

It is easily seen that $\|g_k(x)\| \leq \|f(x)\|$, $\forall\, x \in R$. Now, since $\|g_k(x) - f(x)\| \leq \|f_k(x) - f(x)\|$, we get $\lim_{k\to\infty} \rho_\varphi(g_k - f) = 0$. Moreover, f being in B^φp.p. (R, E), so is the real function $\|f(x)\|$. It follows from Corollary 5 that

$$\lim_{k\to\infty} \rho_\varphi(g_k) = \rho_\varphi(f).$$

Now, since $\|g_k(x)\| \leq \|f_k(x)\|$, we also have $\rho_\varphi(g_k) \leq \rho_\varphi(f_k)$, and then

$$\varliminf_{k\to\infty} \rho_\varphi(f_k) \geq \varliminf_{k\to\infty} \rho_\varphi(g_k) = \rho_\varphi(f).$$

□

Corollary 7. *Let $f \in B^\varphi$p.p.$\,(R, E)$ and $\{f_k\}$ be a sequence of Bochner-Fejer approximation polynomials. Then we have*

$$\lim_{k\to\infty} \rho_\varphi(f_k) = \rho_\varphi(f).$$

PROOF: From the result in I.3 (see also [1]) we have

$$\lim_{k\to\infty} \rho_\varphi(f - f_k) = 0 \quad \text{and} \quad \rho_\varphi(f_k) \leq \rho_\varphi(f).$$

By the last properties and the result in Proposition 6, we get:
$\varliminf_{k\to\infty} \rho_\varphi(f_k) \geq \rho_\varphi(f) \geq \varlimsup_{k\to\infty} \rho_\varphi(f_k)$ and then $\lim_{k\to\infty} \rho_\varphi(f_k) = \rho_\varphi(f)$.

□

Now, sketching with some modifications the proof in [8], we state:

Proposition 8. *Let $\{x_n\}$ be a sequence in B^φp.p.$\,(R, E)$ satisfying the modular Cauchy condition, i.e. such that $\lim_{m,n\to\infty} \rho_\varphi(x_n - x_m) = 0$. Then there exists a subsequence $\{x_{n_k}\}$ and $x \in \tilde{B}^\varphi$p.p.$\,(R, E)$ such that*

$$\lim_{k\to\infty} \rho_\varphi(x_{n_k} - x) = 0.$$

PROOF: Let ε_k be a sequence of real numbers decreasing to zero and let $\{n_k\}$ be the corresponding sequence of integers such that

$$\rho_\varphi(x_{n_k} - x_n) < \varepsilon_k, \quad \forall\, n \geq n_k.$$

We define by induction a sequence $T_1, T_2, \dots, T_{i-1}$ as follows: $T_0 = 0$ and, if $T_1, T_2, \dots, T_{i-1}$ are defined, we take T_i such that

(i) if $T > T_i$, $\frac{1}{2T} \int_{-T}^{T} \varphi(\|x_{n_i}(t) - x_{n_k}(t)\|)\, d\mu < \varepsilon_k$ for $k = 1, 2, \dots, i-1$ and $\frac{1}{2T} \int_{-T}^{T} \varphi(\|x_{n_i}(t) - x_{n_{i+1}}(t)\|)\, d\mu < \varepsilon_i$,

(ii) $T_i > 2T_{i-1}$.

Now, we define a function $x(t)$ as follows: $x(t) = 0$ if $|t| \in [0, T_1$, and $x(t) = x_{n_i}(t)$ if $|t| \in [T_i, T_{i+1}[$, $i = 1, 2, \ldots$. Let $k \in \mathbb{N}$ be fixed. For a given T, take $m \in \mathbb{N}$ such that $T_m \le T < T_{m+1}$. We can thus write

$$\begin{aligned}
\frac{1}{2T}\int_{-T}^{T} \varphi(\|x_{n_k}(t) - x(t)\|)\,d\mu &= \frac{1}{2T}\int_{-T_1}^{T_1} \varphi(\|x_{n_k}(t)\|)\,d\mu \\
&+ \frac{1}{2T}\sum_{i=1}^{k-1}\int_{T_i \le |t| < T_{i+1}} \varphi(\|x_{n_k}(t) - x_{n_i}(t)\|)\,d\mu \\
&+ \frac{1}{2T}\sum_{i=k}^{m-1}\int_{T_i \le |t| < T_{i+1}} \varphi(\|x_{n_k}(t) - x_{n_i}(t)\|)\,d\mu \\
&+ \frac{1}{2T}\int_{T_m \le |t| \le T} \varphi(\|x_{n_k}(t) - x_m(t)\|)\,d\mu \\
&= (1) + (2) + (3) + (4).
\end{aligned}$$

We shall estimate all these terms. From Lemma 3, there exists $\delta_k > 0$ such that $\frac{1}{2T}\mu([0, T_1[\,) < \delta_k$ and then

$$\frac{1}{2T}\int_{-T_1}^{T_1} \varphi(\|x_{n_k}(t)\|)\,d\mu < \varepsilon_k.$$

For the term (2), we have

$$\begin{aligned}
&\frac{1}{2T}\sum_{i=1}^{k-1}\int_{T_i \le |t| < T_{i+1}} \varphi(\|x_{n_k}(t) - x_{n_i}(t)\|)\,d\mu \\
&\le \frac{1}{2T}\sum_{i=1}^{k-1}\int_{-T_{i+1}}^{T_{i+1}} \varphi(\|x_{n_k}(t) - x_{n_i}(t)\|)\,d\mu \\
&\le \frac{1}{2T}\sum_{i=1}^{k-1} 2\varepsilon_i T_{i+1} \le \frac{2\varepsilon_1}{2T}\sum_{i=1}^{k-1} T_{i+1}
\end{aligned}$$

and this later tends to zero when T tends to infinity.

For the term (3), we have

$$\begin{aligned}
&\frac{1}{2T}\sum_{i=k}^{m-1}\int_{T_i \le |t| < T_{i+1}} \varphi(\|x_{n_k}(t) - x_{n_i}(t)\|)\,d\mu \\
&\le \frac{1}{2T_m}\sum_{i=k}^{m-1}\frac{2T_{i+1}}{2T_{i+1}}\int_{-T_{i+1}}^{T_{i+1}} \varphi(\|x_{n_k}(t) - x_{n_i}(t)\|)\,d\mu \\
&\le \frac{1}{2T_m}\sum_{i=k}^{m-1} 2T_{i+1}\varepsilon_k \le \frac{\varepsilon_k}{2T_m}\sum_{i=1}^{m} T_i \\
&\le \frac{\varepsilon_k}{2T_m}\sum_{i=1}^{m} T_m \frac{1}{2^{m-i}} \le 2\varepsilon_k.
\end{aligned}$$

For the term (4), we have

$$\begin{aligned}
&\frac{1}{2T}\int_{T_m\le|t|<|T|}\varphi(\|x_{n_k}(t)-x(t)\|)\,d\mu\\
&\quad=\frac{1}{2T}\int_{T_m\le|t|<|T|}\varphi(\|x_{n_k}(t)-x_{n_m}(t)\|)\,d\mu\\
&\quad\le\frac{1}{2T}\int_{-T}^{T}\varphi(\|x_{n_k}(t)-x_{n_m}(t)\|)\,d\mu\le\varepsilon_k.
\end{aligned}$$

Finally, letting T tend to infinity, we get

$$\rho_\varphi(x_{n_k}-x)\le 4\varepsilon_k$$

and then $\lim_{k\to\infty}\rho_\varphi(x_{n_k}-x)=0$.

It remains to show that $x\in\tilde{B}^\varphi\text{p.p.}\,(R,E)$. Since $x_{n_k}\in B^\varphi\text{p.p.}\,(R,E)$, there exists a trigonometric polynomials $P_k^{(n_k)}$ for which $\rho_\varphi(x_{n_k}-P_k^{(n_k)})<2\varepsilon_k$. We then deduce

$$\rho_\varphi\Big(\frac{x-P_k^{(n_k)}}{2}\Big)\le\frac{1}{2}\rho_\varphi(x-x_{n_k})+\frac{1}{2}\rho_\varphi(x_{n_k}-P_k^{(n_k)})\le 3\varepsilon_k.$$

This means that $x\in\tilde{B}^\varphi\text{p.p.}\,(R,E)$. □

Theorem. *Let $C\subset B^\varphi\text{p.p.}\,(R,E)$ be convex and closed with respect to the modular convergence, i.e. if $\rho_\varphi(x_n-x)\to 0$, $x_n\in C$, then $x\in C$. We suppose that the following properties hold:*

(i) *φ is strictly convex on R and uniformly convex on $[d,+\infty[$, with $d\ge 0$.*

For $f\in B^\varphi\text{p.p.}\,(R,E)$ we define

$$d_{\rho_\varphi}(f,C)=\inf\{\rho_\varphi(f-g),\ g\in C\}.$$

Then there exists a unique g_0 such that $\rho_\varphi(f-g_0)=d\rho_\varphi(f,C)$. The correspondence $P:B^\varphi\text{p.p.}\,(R,E)\to C$, where $P(f)=g_0$ is called the projection operator on C.

Proof: We first notice that $d_{\rho_\varphi}(f,C)$ is always finite. Indeed, if f,g are in $B^\varphi\text{p.p.}\,(R,E)$, so is $f-g$ and then $\rho_\varphi(f-g)$ is finite (cf. [6]). We use the notation $d=d\rho_\varphi(f,C)$ and we may suppose $d>0$. From the definition of d, there exists a sequence $f_n\in C$ for which

$$\rho_\varphi(f-f_n)\le\Big(1+\frac{1}{n}\Big)d.$$

Using arguments similar to those in [5], we show that the sequence $\{\frac{1}{2}f_n\}$ is a modular Cauchy sequence. Now, from Proposition 7, we can extract a subsequence convergent to some $g \in \tilde{B}^{\varphi}$p.p. (R,E). For simplicity, we use the same notation for this subsequence. We shall show that $2g \in C$. Since $\lim_{m\to\infty} \rho_\varphi(\frac{1}{2}(f_n + f_m) - (\frac{1}{2}f_n + g)) = 0$, it follows from ρ_φ-closedness of C and $\frac{1}{2}(f_n + f_m) \in C$ that $\frac{1}{2}f_n + g \in C$. Moreover, since $\lim_{n\to\infty} \rho_\varphi((\frac{1}{2}f_n + g) - 2g) = 0$, by similar arguments, we get $2g \in C$. It is also clear that $\lim_{n\to\infty} \rho_\varphi[(f - \frac{1}{2}(f_n + f_m)) - (f - (g + \frac{1}{2}f_m))] = 0$. Since $f - (g + \frac{1}{2}f_m) \in B^\varphi$p.p. (R,E), we can apply Proposition 6 to get

$$\varliminf_{n\to\infty} \rho_\varphi[f - \frac{1}{2}(f_n + f_m)] \geq \rho_\varphi[f - (g + \frac{1}{2}f_m)].$$

Using the inequality $\rho_\varphi[f - \frac{1}{2}(f_n + f_m)] \leq \frac{1}{2}[\rho_\varphi(f - f_n) + \rho_\varphi(f - f_m)]$ we get

$$(*) \qquad \rho_\varphi[f - (g + \frac{1}{2}f_m)] \leq \frac{1}{2}(d + \rho_\varphi(f - f_m)).$$

In the same manner, we also have $\lim_{m\to\infty} \rho_\varphi[f - (g + \frac{1}{2}f_m) - (f - 2g)] = 0$ and, since $f - 2g \in B^\varphi$p.p. (R,E), by Proposition 6,

$$\varliminf_{m\to\infty} \rho_\varphi[f - (g + \frac{1}{2}f_m)] \geq \rho_\varphi(f - 2g).$$

Then, using $(*)$, we get $\rho_\varphi(f - 2g) \leq d$.

Finally, since $2g \in C$, we also have $\rho_\varphi(f - 2g) \geq d$ and then $\rho_\varphi(f - 2g) = d$. So, $g_0 = 2g$ is the element of the best approximation. To see that this element is unique, suppose there exists h_0 with the same properties. Then we have: $\rho_\varphi(f - g_0) = \rho_\varphi(f - h_0) = d$, whence $\rho_\varphi[f - \frac{1}{2}(g_0 + h_0)] \leq d$ and, since $\frac{1}{2}(g_0 + h_0) \in C$, we also have $\rho_\varphi[f - \frac{1}{2}(g_0 + h_0)] \geq d$. Finally, since ρ_φ is uniformly convex (and so also strictly convex), we get $g_0 = h_0$. □

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