On congruences of G-sets

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Abstract. We describe G-sets whose congruences satisfy some natural lattice or multiplicative restrictions. In particular, we determine G-sets with distributive, arguesian, modular, upper or lower semimodular congruence lattice as well as congruence n-permutable G-sets for $n = 2, 2.5, 3$.

Keywords: G-set, congruence lattice, congruence distributivity, congruence modularity, congruence n-permutability

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Introduction

Let A be a non-empty set, G a group, and φ a homomorphism from G into the full transformation group of A. Then A may be considered as a unary algebra with the set G of operations, where an operation $g \in G$ is defined by the rule $g(x) = (\varphi(g))(x)$ for every $x \in A$. These algebras are called *G-sets*. Some basic information about G-sets and, in particular, about their congruences may be found, for example, in [6].

Studying of congruences of unary algebras appears to be sufficiently natural per se, and indeed it became the subject of several papers (see, for example, [1], [3]–[5], [7] where questions close to those of the present paper have been studied for the case of *mono*-unary algebras). Our main motivation, however, comes from a different source. Recent results by M.V. Volkov and the author show that, for some wide classes of semigroup varieties, the structure of subvariety lattices can be described in terms of congruence lattices of certain G-sets (see, in particular, [10], [11], [16], [17]). Such a description effectively reduces questions if the subvariety lattice of a variety $\mathcal{V}$ satisfies a lattice restriction to analogous questions about congruence lattices of some G-sets depending on $\mathcal{V}$. Of course, to make this reduction really useful one must be able to answer the latter questions, this means, to describe G-sets with various restrictions to their congruences. Some partial results of such kind have occasionally arisen in [10], [17] but gradually it became clear that it is worth studying congruences of G-sets more systematically to clarify already existing and to provide a basis for further applications of the general reduction technique.

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In the present paper we describe G-sets whose congruences satisfy certain important lattice or multiplicative restrictions. The paper consists of three sections. Section 1 contains some preliminaries and a general result describing a "good part" of the congruence lattice of an arbitrary G-set. Sections 2 and 3 are respectively devoted to some lattice and multiplicative restrictions to congruences of G-sets. We mention that the results of this paper at first permit to give shorter and simpler proofs of the results of the papers [9], [14], [15] than the original proofs, and at second already found some interesting new applications as, for example, complete descriptions of semigroup varieties with semimodular subvariety lattice [12], with commuting fully invariant congruences on free objects [13], with distributive, modular or semimodular lattice of overcommutative subvarieties or with n-permutable ($n = 2, 2.5, 3$) subcommutative fully invariant congruences on free objects [8].

1. The sublattice of greedy congruences

The congruence lattice of a G-set A is denoted by $Con(A)$. A G-set A is said to be *transitive* if, for all $a, b \in A$, there exists $g \in G$ such that $g(a) = b$. It is well known (see, for example, [6, Lemma 4.20]) that if A is a transitive G-set then the lattice $Con(A)$ is isomorphic to an interval of the subgroup lattice of G (more precisely, $Con(A) \cong [Stab_G(a), G]$, where a is an arbitrary element in A and $Stab_G(a) = \{g \in G \mid g(a) = a\}$).

A transitive G-subset of a G-set A is called an *orbit* of A. Clearly, any G-set is a disjoint union of its orbits. In view of the remark in the previous paragraph, it is natural to investigate the lattice $Con(A)$ modulo the congruence lattices of the orbits of A.

Now let α be a congruence on a G-set A and B and C two different orbits of A. We say that α *connects* B and C if $b\alpha c$ for some $b \in B$ and $c \in C$. We say that α *collapses* B and C and write $B\alpha C$ if $x\alpha y$ for all $x, y \in B \cup C$. We call a congruence α *greedy* if it collapses any pair of orbits it connects. Let $GCon(A)$ denote the set of all greedy congruences of A.

Lemma 1.1. *For any G-set A, the set $GCon(A)$ forms a sublattice of $Con(A)$.*

Proof: Let $\alpha, \beta \in GCon(A)$ and $\gamma = \alpha \vee \beta$. Suppose that B and C are two different orbits of A and γ connects them, that is, $b\gamma c$ for some $b \in B$ and $c \in C$. Hence there exist elements $a_0, \dots, a_n \in A$ such that

$$b = a_0 \ \alpha \ a_1 \ \beta \ a_2 \ \alpha \ \dots \ \alpha \ a_n = c.$$

Since $a_n \notin B$, there exists the least i such that $a_i \notin B$. Clearly, $i \geq 1$. Thus $a_{i-1} \in B$ and $a_{i-1}\zeta a_i$, where ζ is one of the congruences α and β. Let D be the orbit of A that contains a_i. Then $D \neq B$, and ζ connects B and D. Since ζ is greedy, we have $B\zeta D$. In particular, $x\zeta y$ for all $x, y \in B$ whence $x\gamma y$. Symmetrically, considering the maximal j such that $a_j \notin C$, one can verify that

$x\gamma y$ for all $x, y \in C$. Finally, for every $x \in B$ and $y \in C$, we have $x\ \gamma\ b\ \gamma\ c\ \gamma\ y$ whence $x\gamma y$. We see that $B\gamma C$, thus γ is greedy.

Now consider $\delta = \alpha \wedge \beta$ and suppose that it connects B and C, that is, $b\delta c$ for some $b \in B$ and $c \in C$. Hence $b\alpha c$ and $b\beta c$. Since α and β are greedy, this implies $B\alpha C$ and $B\beta C$. Thus, for every $x, y \in B \cup C$, $x\alpha y$ and $x\beta y$ whence $x\delta y$. We see that $B\delta C$, and therefore, δ is greedy as well. □

We are going to show that the sublattice $GCon(A)$ allows an easy description modulo the congruence lattices of the orbits of A. For a G-set A, we denote by $Orb(A)$ the set of all orbits of A. We denote by $Part(X)$ the partition lattice of a set X.

Proposition 1.2. *Let A be a G-set and $Orb(A) = \{A_i \mid i \in I\}$. Then the lattice $GCon(A)$ is isomorphic to a subdirect product of the lattices $Part(Orb(A))$ and $Con(A_i)$, where i runs over I.*

Proof: We shall write an element of the direct product

$$Part(Orb(A)) \times \prod_{i \in I} Con(A_i)$$

as the vector $(\sigma; \dots, \alpha_i, \dots)$, where σ is a partition of the set $Orb(A)$ and α_i is a congruence on the orbit A_i. Put

$$L = \{(\sigma; \dots, \alpha_i, \dots) \mid \text{ if } \ A_i \sigma A_j \ \text{ for some } \ i \neq j, \ \text{ then } \ \alpha_i = \Delta_i\},$$

where Δ_i is the universal relation on the orbit A_i.

Let $x = (\sigma; \dots, \alpha_i, \dots)$, $y = (\tau; \dots, \beta_i, \dots)$ belong to L. Then $x \vee y = (\sigma \vee \tau; \dots, \alpha_i \vee \beta_i, \dots)$. If $A_i(\sigma \vee \tau)A_j$ for some $i \neq j$, then either $A_i \sigma A_k$ or $A_i \tau A_k$ for some $k \neq i$. Hence either $\alpha_i = \Delta_i$ or $\beta_i = \Delta_i$, and therefore, $\alpha_i \vee \beta_i = \Delta_i$. Now consider $x \wedge y = (\sigma \wedge \tau; \dots, \alpha_i \wedge \beta_i, \dots)$. If $A_i(\sigma \wedge \tau)A_j$ for some $i \neq j$, then $A_i \sigma A_j$ and $A_i \tau A_j$. Hence $\alpha_i = \beta_i = \Delta_i$, and therefore, $\alpha_i \wedge \beta_i = \Delta_i$. We see that both $x \vee y$ and $x \wedge y$ belong to L, that is, L is a sublattice of $Part(Orb(A)) \times \prod_{i \in I} Con(A_i)$. It is obvious that the projections of L on each of the lattices $Part(Orb(A))$ and $Con(A_i)$, $i \in I$, are surjective. Thus the lattice L is a subdirect product of these lattices.

Now we define a mapping $f : GCon(A) \longrightarrow L$ by the following rule: if α is a greedy congruence on A then put $f(\alpha) = (\bar{\alpha}; \dots, \alpha_i, \dots)$, where the congruence α_i is merely the restriction of α to the orbit A_i and $A_i \bar{\alpha} A_j$ if and only if either $i = j$ or α collapses A_i and A_j. It is clear that $f(\alpha) \in L$ for each $\alpha \in GCon(A)$. It can be easily verified that f is an isotone bijection from $GCon(A)$ onto L and the inverse bijection is also isotone. Hence f is a lattice isomorphism. □

Proposition 1.2 implies that the lattice $Con(A)$ has a very simple structure modulo congruence lattices of orbits of A in the case when $Con(A) = GCon(A)$, that is, when every congruence on A is greedy. Our next proposition characterizes G-sets with the latter property.

Proposition 1.3. *Let A be a G-set. Then each congruence on A is greedy if and only if, for every pair of different orbits $B, C \in Orb(A)$, no non-singleton homomorphic image of B is isomorphic to a homomorphic image of C.*

Proof: **Necessity.** Let B, C be two different orbits of A such that a non-singleton homomorphic image B' of B is isomorphic to a homomorphic image C' of C. Denote by f an isomorphism from B' onto C'. Now we define a partition α of A with only non-singleton classes of the form $X \cup f(X)$, where X runs over B'. It can be straightforwardly verified that α is a congruence. Clearly, α connects B and C, and the restriction of α to B coincides with the kernel of the natural homomorphism from B onto B'. Since the latter set is non-singleton, not all elements of B are α-equivalent. Therefore α does not collapse B and C so it is not greedy.

Sufficiency. Let a congruence α connect two different orbits B and C. Denote by β and γ the restrictions of α to B and C respectively and define a mapping $f : B/\beta \longrightarrow C/\gamma$ by putting

$$f(X) = \{y \in C \mid y\alpha x \ \text{ for all } \ x \in X\}$$

for each β-class X. To prove that the definition is correct, let us check, first of all, that the set $f(X)$ is non-empty for any $X \in B/\beta$. Indeed, there are some $b \in B$ and $c \in C$ such that $b\alpha c$. Fix a β-class X and take an element $x_0 \in X$. Since B is transitive, there exists $g \in G$ such that $x_0 = g(b)$. Hence $x_0 = g(b)\ \alpha\ g(c)$, and therefore, $x\ \alpha\ g(c)$ for all $x \in X$. We have $g(c) \in C$ because C is a G-subset of A. Thus $g(c) \in f(X)$, and the latter set is non-empty. Clearly, $f(X)$ is a γ-class so the correctness of the definition is proved.

Now take any γ-class $Y \in C/\gamma$ and consider the set

$$X = \{x \in B \mid x\alpha y \ \text{ for all } \ y \in Y\}.$$

By symmetry, X is non-empty and is a β-class and obviously $f(X) = Y$. This means that f maps B/β onto C/γ.

Suppose that $f(X) = f(X') = Y$ for some $X, X' \in B/\beta$. Take an element $y \in Y$. Then by definition $x\alpha y\alpha x'$ for all $x \in X$ and $x' \in X'$ whence $x\alpha x'$ and $X = X'$. Thus f one-to-one.

Finally let $X \in B/\beta,\ f(X) = Y$, and $g \in G$. Take an element $x' \in g(X)$. Then $x' = g(x)$ for some $x \in X$. We have $x\alpha y$ for all $y \in Y$ whence $x' = g(x)\ \alpha\ g(y)$. This implies that $g(f(X)) = g(Y) = f(g(X))$. Thus f is an isomorphism.

Now the condition of our Proposition applies yielding that both B/β and C/γ are to be singletons. Hence $x\alpha y$ whenever $x, y \in B$ or $x, y \in C$. Further, for every $x \in B$ and $y \in C$, we have $x\ \beta\ b\ \alpha\ c\ \gamma\ y$, and therefore, $x\alpha y$. Thus α collapses B and C so α is greedy. □

The characterization given by Proposition 1.3 says roughly speaking that all congruences of a G-set are greedy if and only if its orbits are extremely different. By this reason we shall call G-sets all whose congruences are greedy *segregated*.

2. Lattice restrictions

Recall that an element z of a lattice L is said to *cover* an element $x \in L$ if $x < z$ and there is no $y \in L$ such that $x < y < z$. A lattice $\langle L;\ \vee, \wedge\rangle$ is called (*weakly*) *upper semimodular* if, for all $x, y \in L$, their join $x \vee y$ covers y whenever x covers their meet $x \wedge y$ (respectively, whenever x and y cover $x \wedge y$). (*Weakly*) *lower semimodular* lattices are defined dually.

Proposition 2.1. *Let A be a G-set. If the lattice $Con(A)$ is either weakly upper semimodular or weakly lower semimodular then A is segregated.*

Proof: Suppose that A is not segregated. By Proposition 1.3 there exist two different orbits $B, C \in Orb(A)$ and congruences $\beta \in Con(B)$, $\gamma \in Con(C)$ such that $B/\beta \cong C/\gamma$ and $|B/\beta| > 1$. For any $\mu \in Con(B)$ and $\nu \in Con(C)$, we denote by $\mu \oplus \nu$ a congruence on A given by the following rule: $(x, y) \in \mu \oplus \nu$ if and only if either $x = y$ or $x, y \in B$ and $x\mu y$ or $x, y \in C$ and $x\nu y$. Now put $\alpha = \beta \oplus \gamma$. Clearly, the coideal $[\alpha)$ of the lattice $Con(A)$ is isomorphic to the lattice $Con(A/\alpha)$. Since any interval of a weakly upper (lower) semimodular lattice is a weakly upper (lower) semimodular lattice by itself, it is sufficient to verify that the lattice $Con(A/\alpha)$ is neither weakly upper semimodular nor weakly lower semimodular. In other words, without any loss we may (and will) consider A/α instead of A and so assume that A simply has two non-singleton isomorphic orbits B and C. Further, it is easy to see that, for any G-subset D of A, the lattice $Con(D)$ is isomorphic to the ideal $(\rho_D]$ of $Con(A)$, where ρ_D is the congruence on A given by the rule: $x\rho_D y$ if and only if either $x = y$ or $x, y \in D$. Hence we may (and will) consider $B \cup C$ instead of A. In other words, we will assume that $A = B \cup C$.

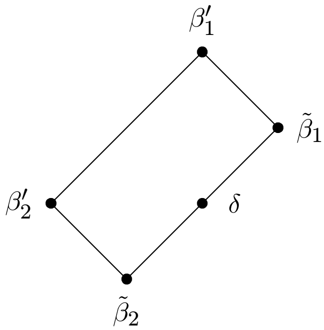
We shall make use of two constructions which extend congruences of B to congruences of A. Fix an isomorphism f from B onto C. For every $x \in A$, put $x' = x$ whenever $x \in B$, and $x' = f^{-1}(x)$ whenever $x \in C$. Clearly, $x' \in B$ for each $x \in A$. Now, for every congruence μ on B, we define a binary relation μ' on A as follows: $x\mu' y$ if and only if $x'\mu y'$. It is easy to see that μ' is a congruence on A. Clearly, μ'-classes are exactly $X \cup f(X)$ where X runs over the set of all μ-classes and $f(X) = \{f(x) \mid x \in X\}$. We note that $x\mu' x'$ for every $x \in A$. Further, for every congruence μ on B, we denote by $f(\mu)$ the set of all pairs of the form $(f(x), f(y))$, where (x, y) runs over μ. Clearly, $f(\mu)$ is a congruence on C. Put $\tilde{\mu} = \mu \oplus f(\mu)$. It is clear that $\tilde{\mu} < \mu'$.

Let us verify that μ' covers $\tilde{\mu}$ for every $\mu \in Con(B)$. Suppose that $\tilde{\mu} < \zeta \leq \mu'$ for a congruence $\zeta \in Con(A)$, and fix a pair $(x, y) \in \zeta \setminus \tilde{\mu}$. Clearly, $\tilde{\mu}|_B = \mu'|_B = \mu$ whence $\zeta|_B = \mu$. By symmetry, $\tilde{\mu}|_C = \mu'|_C = \zeta|_C = f(\mu)$. Hence x and y should belong to different orbits, and therefore, we may assume that $x \in B$ an $y \in C$. In this case we have $f(x)\ \tilde{\mu}\ y$. Let now $z\mu' t$. If $z, t \in B$ or $z, t \in C$ then $z\zeta t$ because $\mu'|_B = \zeta|_B$ and $\mu'|_C = \zeta|_C$. Suppose that $z \in B$ and $t \in C$. Then $f(z)\ \tilde{\mu}\ t$. Since B is transitive, there exists $g \in G$ with $g(x) = z$. Hence

$$z = g(x)\ \zeta\ g(y)\ \tilde{\mu}\ g(f(x)) = f(g(x)) = f(z)\ \tilde{\mu}\ t$$

whence $z\zeta t$. Thus $\mu' = \zeta$, and μ' covers $\tilde{\mu}$.

The lattice $Con(B)$ is non-trivial because $|B| > 1$. Since this lattice is algebraic (as the congruence lattice of a universal algebra), it is weakly atomic [2, Theorem 2.2], this means that every non-singleton interval of $Con(B)$ contains a two-element subinterval. In particular, there exist congruences $\beta_1, \beta_2 \in Con(B)$ such that β_1 covers β_2. Now we consider the following five congruences on A: $\tilde{\beta}_2, \beta_2', \tilde{\beta}_1, \beta_1'$, and $\delta = \beta_1 \oplus f(\beta_2)$ (see the diagram).



All lines on this diagram represent in actual fact the cover relation. Indeed, as shown above, β_1' covers $\tilde{\beta}_1$, and β_2' covers $\tilde{\beta}_2$. The definition of δ easily implies that δ covers $\tilde{\beta}_2$ and is covered by $\tilde{\beta}_1$. It remains to check that β_1' covers β_2'. Suppose that $\beta_2' \leq \zeta \leq \beta_1'$ for some congruence $\zeta \in Con(A)$. Since $\beta_1'|_B = \beta_1$, $\beta_2'|_B = \beta_2$, and β_1 covers β_2, we have either $\zeta|_B = \beta_1$ or $\zeta|_B = \beta_2$.

First suppose that $\zeta|_B = \beta_1$. Let $x\beta_1'y$. Then $x'\beta_1 y'$ whence $x'\zeta y'$. We then have $x \ \beta_2' \ x' \ \zeta \ y' \ \beta_2' \ y$ whence $x\zeta y$. Thus $\zeta = \beta_1'$. Suppose now that $\zeta|_B = \beta_2$. Let $x\zeta y$. Then $x' \ \beta_2' \ x \ \zeta \ y \ \beta_2' \ y'$ whence $x'\zeta y'$. Therefore $x'\beta_2 y'$ and $x\beta_2' y$. Thus $\zeta = \beta_2'$ in this case. We have proved that β_1' covers β_2'.

We see that β_2' and δ cover $\tilde{\beta}_2 = \beta_2' \wedge \delta$ while $\beta_2' \vee \delta = \beta_1'$ does not cover δ. On the other hand, $\beta_1' = \beta_2' \vee \tilde{\beta}_1$ covers β_2' and $\tilde{\beta}_1$ while $\tilde{\beta}_1$ does not cover $\tilde{\beta}_2 = \beta_2' \wedge \tilde{\beta}_1$. Hence the lattice $Con(A)$ is neither weakly upper semimodular nor weakly lower semimodular. $\square$

We are now well prepared to prove the main result of this section.

Theorem 2.2. *Let A be a G-set and $\mathcal{L}$ a class of lattices closed under taking intervals and subdirect products. Suppose that every lattice in $\mathcal{L}$ is either weakly upper semimodular or weakly lower semimodular. Then the lattice $Con(A)$ belongs to $\mathcal{L}$ if and only if:*

(a) *the lattice $Con(B)$ belongs to $\mathcal{L}$ for every orbit B of A;*
(b) *the lattice $Part(Orb(A))$ belongs to $\mathcal{L}$;*
(c) *A is segregated.*

Proof: **Necessity.** The fact that A is segregated immediately follows from Proposition 2.1. To get (a) and (b), we note that the lattices $Con(B)$ and

$Part(Orb(A))$ are isomorphic to some intervals of the lattice $Con(A)$. Indeed, $Con(B) \cong (\rho_B]$ (see the proof of Proposition 2.1), and the partition lattice $Part(Orb(A))$ is isomorphic to the coideal $[\pi)$ of $Con(A)$, where π is the partition of A into orbits.

Sufficiency immediately follows from Proposition 1.2. □

It is well known that, for any set X, the lattice $Part(X)$ is upper semimodular; if $|X| > 3$ then $Part(X)$ is not weakly lower semimodular; if $|X| = 3$ then $Part(X) \cong M_3$ is arguesian; finally $Part(X)$ is distributive if and only if $|X| \leq 2$. It is known also that the class of all (weakly) upper semimodular lattices as well as the class of all (weakly) lower semimodular lattices are closed under taking intervals and subdirect products. Combining these observations with Theorem 2.2, we immediately obtain a variety of corollaries dealing with some concrete lattice restrictions to the congruence lattice of a G-set. We explicitly formulate only three of them here.

Corollary 2.3. *Let A be a G-set. The lattice $Con(A)$ is (weakly) upper semimodular if and only if, for every orbit B of A, the lattice $Con(B)$ is (weakly) upper semimodular and A is segregated.* □

Corollary 2.4. *Let A be a G-set. The lattice $Con(A)$ is modular (arguesian, lower semimodular, weakly lower semimodular) if and only if, for every orbit B of A, the lattice $Con(B)$ is modular (respectively, arguesian, lower semimodular, weakly lower semimodular), A has ≤ 3 orbits and is segregated.* □

In fact, it is clear that every lattice quasiidentity (in particular, identity) which is strictly weaker than distributivity and stronger than modularity might be added to the list of the conditions in Corollary 2.4.

Corollary 2.5. *Let A be a G-set. The lattice $Con(A)$ is distributive if and only if, for every orbit B of A, the lattice $Con(B)$ is distributive, A has ≤ 2 orbits and is segregated.* □

3. Multiplicative restrictions

As we have mentioned at the beginning of Section 1, the congruence lattice of a transitive G-set A is isomorphic to the interval $[Stab_G(a), G]$ of the subgroup lattice of G where a is an arbitrary element of A. It is easy to check (see [8]) that this result has a natural multiplicative analogue. Namely, let $\mathcal{B}(A)$ be the semigroup of all binary relations on the set A and let $\mathcal{P}(G)$ denote the semigroup of all subsets of the group G (where the product of two subsets $C, D \subseteq G$ is defined as the subset $\{cd \mid c \in C,\ d \in D\}$). Then the isomorphism between $Con(A)$ and $[Stab_G(a), G]$ extends to an isomorphism between the subsemigroup of $\mathcal{B}(A)$ generated by congruences of A and the subsemigroup of $\mathcal{P}(G)$ generated by subgroups of G containing the subgroup $Stab_G(a)$. Taking this into account, we may, as by studying the lattice restrictions, investigate the multiplicative behavior of congruences of G-sets modulo orbits.

Let α and β be congruences on a G-set A. Put $\alpha \circ_2 \beta = \alpha\beta$ and, for any positive integer $n > 2$,

$$\alpha \circ_n \beta = \begin{cases} (\alpha \circ_{n-1} \beta)\beta & \text{if } n \text{ is even,} \\ (\alpha \circ_{n-1} \beta)\alpha & \text{if } n \text{ is odd.} \end{cases}$$

A G-set A is called *congruence n-permutable* if $\alpha \circ_n \beta = \beta \circ_n \alpha$ for all congruences α, β on A.

Lemma 3.1. *Let $n > 1$ be an integer. If a G-set A is congruence n-permutable then A contains $\leq n$ orbits.*

Proof: Clearly, any partition of the set $Orb(A)$ induces a partition of A, and the latter partition is a congruence on A. Therefore our lemma immediately follows from the fact that, for any set X with $> n$ elements, there exist partitions α and β of X such that $\alpha \circ_n \beta \neq \beta \circ_n \alpha$. This fact is certainly known but, for completeness' sake, we will prove it here.

Thus, let $X = \{x_1, \dots, x_{n+1}, \dots\}$. We define the partitions α and β of X as follows:

if n is even then non-singleton α-classes are exactly $\{x_1, x_2\}$, $\{x_3, x_4\}$, $\dots$, $\{x_{n-1}, x_n\}$, and non-singleton β-classes are exactly $\{x_2, x_3\}$, $\{x_4, x_5\}$, $\dots$, $\{x_n, x_{n+1}\}$;

if n is odd then non-singleton α-classes are exactly $\{x_1, x_2\}$, $\{x_3, x_4\}$, $\dots$, $\{x_n, x_{n+1}\}$, and non-singleton β-classes are exactly $\{x_2, x_3\}$, $\{x_4, x_5\}$, $\dots$, $\{x_{n-1}, x_n\}$.

Then we have

$$x_1 \ \alpha \ x_2 \ \beta \ x_3 \ \alpha \ \dots \ \alpha \ x_n \ \beta \ x_{n+1}$$

for n being even and

$$x_1 \ \alpha \ x_2 \ \beta \ x_3 \ \alpha \ \dots \ \beta \ x_n \ \alpha \ x_{n+1}$$

for n being odd. In both cases $(x_1, x_{n+1}) \in \alpha \circ_n \beta$. Suppose that $(x_1, x_{n+1}) \in \beta \circ_n \alpha$. If n is even there exist $y_2, y_3, \dots, y_n \in X$ such that

$$x_1 \ \beta \ y_2 \ \alpha \ y_3 \ \beta \ \dots \ \beta \ y_n \ \alpha \ x_{n+1}.$$

From the definition of the partitions α and β, we have in succession $y_2 = x_1$, $y_3 \in \{x_1, x_2\}$, $y_4 \in \{x_1, x_2, x_3\}$, $\dots$, $y_n \in \{x_1, x_2, \dots, x_{n-1}\}$, and $x_{n+1} \in \{x_1, x_2, \dots, x_n\}$, a contradiction. The case when n is odd is completely analogous.

$\square$

Lemma 3.2. *If a G-set A is congruence 3-permutable then it is segregated.*

Proof: It is well known that every lattice of 3-permutable equivalences is modular. Now Proposition 2.1 applies.

□

Lemma 3.3. *Let A be a segregated G-set, $Orb(A) = \{A_1, A_2, A_3\}$, $\alpha, \beta \in Con(A)$, and $x, y \in A$. If $(x, y) \in \alpha \circ_n \beta$ for some $n \geq 2$ then one of the following conditions holds:*

(a) *there exists $i \in \{1, 2, 3\}$ such that $x, y \in A_i$ and $(x, y) \in \alpha_i \circ_n \beta_i$ where α_i (respectively, β_i) is the restriction of α (respectively, of β) to A_i;*
(b) *$(x, y) \in \alpha\beta \cup \beta\alpha$.*[1]

Proof: Let Δ be the universal relation on A. Clearly, if $\alpha = \Delta$ or $\beta = \Delta$ then $(x, y) \in \alpha \cup \beta$, and therefore, the condition (b) holds. Hence we may assume that $\alpha, \beta \neq \Delta$.

For any congruence μ on A, we say that an orbit is *μ-saturated* if it is the union of μ-classes. Since A is segregated, for any congruence $\mu \neq \Delta$ on A, either all the orbits A_1, A_2, A_3 are μ-saturated or μ collapses two orbits A_i and A_j while the third orbit A_k is μ-saturated ($\{i, j, k\} = \{1, 2, 3\}$).

Let now $x, y \in A$ and $(x, y) \in \alpha \circ_n \beta$ for some $n \geq 2$. By the trivial induction it is sufficient to verify that either the condition (a) holds or $(x, y) \in \alpha \circ_{n-1} \beta \cup \beta \circ_{n-1} \alpha$. There exist elements $z_1, \dots, z_{n-1} \in A$ such that either $x \ \alpha \ z_1 \ \beta \ z_2 \ \alpha \ \dots \ \alpha \ z_{n-1} \ \beta \ y$ or $x \ \alpha \ z_1 \ \beta \ z_2 \ \alpha \ \dots \ \beta \ z_{n-1} \ \alpha \ y$.

Take the orbit A_i containing the element x and consider three cases.

Case 1. *A_i is α-saturated and β-saturated.*

In this case, $z_1, z_2, \dots, z_{n-1}, y \in A_i$. Thus $x, y \in A_i$ and $(x, y) \in \alpha_i \circ_n \beta_i$. We have the condition (a) fulfilled.

Case 2. *A_i is α-saturated but not β-saturated.*

In this case β collapses A_i and some other orbit A_j while the third orbit A_k is α-saturated ($\{i, j, k\} = \{1, 2, 3\}$). Therefore, $z_1 \in A_i$ and $z_2 \in A_i \cup A_j$. Hence $x\beta z_2$ and $(x, y) \in \beta \circ_{n-1} \alpha$.

Case 3. *A_i is not α-saturated.*

Here α collapses A_i and some other orbit A_j while the third orbit A_k is α-saturated ($\{i, j, k\} = \{1, 2, 3\}$). Hence $z_1 \in A_i \cup A_j$. If $z_2 \in A_i \cup A_j$ then $x\alpha z_2$ whence $(x, y) \in \alpha \circ_{n-1} \beta$. Thus we may assume that $z_2 \in A_k$ whence $z_3 \in A_k$. Suppose that $z_1 \in A_i$. Then β connects A_i and A_k, and therefore, β collapses A_i and A_k. Thus $x\beta z_3$ and $(x, y) \in \beta \circ_{n-1} \alpha$. Finally let $z_1 \in A_j$. Then β connects A_j and A_k and so it collapses these two orbits. Thus $z_1 \beta z_3$, that is, $(x, y) \in \alpha \circ_{n-1} \beta$. □

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[1] Here and below $\cup$ denotes the set-theoretic join of binary relations.

We say that an algebra A is *congruence* 2.5-*permutable* if the join $\alpha \vee \beta$ in the congruence lattice of A is equal to the union $\alpha\beta \cup \beta\alpha$ for all congruences α and β. Clearly, each 2-permutable algebra is 2.5-permutable and each 2.5-permutable algebra is 3-permutable — this observation explains the term we use. The property of being 2.5-permutable is less exotic than it might seem, it frequently arises, for example, by studying fully invariant congruences on free semigroups, see [15].

The main result of this section describes (modulo orbits) G-sets with n-permutable congruences for $n = 2, 2.5, 3$. By $\lceil n \rceil$ we denote the least integer such that $n \leq \lceil n \rceil$.

Theorem 3.4. *Let $n \in \{2, 2.5, 3\}$. A G-set A is congruence n-permutable if and only if every orbit of A is congruence n-permutable, A has $\leq \lceil n \rceil$ orbits and is segregated.*

Proof: **Necessity** follows from Lemmas 3.1 and 3.2 combined with the evident fact that congruence n-permutability is inherited by orbits.

Sufficiency. We may assume without loss of generality that A has exactly $\lceil n \rceil$ orbits $A_1, \dots, A_{\lceil n \rceil}$ (if necessary, we can always add a one-element orbit). For any congruence μ on A, denote by μ_i the restriction of μ to A_i, $i = 1, \dots, n$.

We first consider the case $n = 3$. Let $\alpha, \beta \in Con(A)$, $x, y \in A$, and $(x, y) \in \alpha\beta\alpha$. By Lemma 3.3 either $x, y \in A_i$ and $(x, y) \in \alpha_i\beta_i\alpha_i$ for some $i \in \{1, 2, 3\}$ or $(x, y) \in \alpha\beta \cup \beta\alpha$. Since A_i is congruence 3-permutable, we see that, in the first case, $(x, y) \in \beta_i\alpha_i\beta_i \subseteq \beta\alpha\beta$. Clearly, in the second case, $(x, y) \in \beta\alpha\beta$ too. We have proved that $\alpha\beta\alpha \subseteq \beta\alpha\beta$, and, by symmetry, $\alpha\beta\alpha = \beta\alpha\beta$. Thus A is congruence 3-permutable.

Now consider the case $n = 2.5$. Let $\alpha, \beta \in Con(A)$, $x, y \in A$, and $(x, y) \in \alpha \vee \beta$. It is well known that $\alpha \vee \beta = \bigcup_{m=2}^{\infty} \alpha \circ_m \beta$. We may therefore assume that $(x, y) \in \alpha \circ_m \beta$ for some $m \geq 2$. By Lemma 3.3 either $x, y \in A_i$ and $(x, y) \in \alpha_i \circ_m \beta_i$ for some $i \in \{1, 2, 3\}$ or $(x, y) \in \alpha\beta \cup \beta\alpha$. In the first case, $(x, y) \in \alpha_i \vee \beta_i$. Since A_i is congruence 2.5-permutable, this implies that $(x, y) \in \alpha_i\beta_i \cup \beta_i\alpha_i \subseteq \alpha\beta \cup \beta\alpha$. Thus $(x, y) \in \alpha\beta \cup \beta\alpha$ in both cases whence $\alpha \vee \beta \subseteq \alpha\beta \cup \beta\alpha$. The opposite inclusion is evident. We have proved that A is congruence 2.5-permutable.

Finally consider the case $n = 2$. Let Δ be the universal relation on A. Since A is segregated, for any congruence μ on A, either $\mu = \Delta$ or both the orbits A_1 and A_2 are μ-saturated. Take two arbitrary congruences α and β on A. Clearly, $\alpha\beta = \beta\alpha$ if $\alpha = \Delta$ or $\beta = \Delta$. Suppose that $\alpha, \beta \neq \Delta$, $x, y \in A$ and $(x, y) \in \alpha\beta$, that is, $x \ \alpha \ z \ \beta \ y$ for some $z \in A$. Take the orbit A_i containing the element z. Since A_i is both α-saturated and β-saturated, we have $x, y \in A_i$. Hence $x \ \alpha_i \ z \ \beta_i \ y$, that is, $(x, y) \in \alpha_i\beta_i$. Since A_i is congruence 2-permutable, we then have $(x, y) \in \beta_i\alpha_i \subseteq \beta\alpha$. Thus $\alpha\beta \subseteq \beta\alpha$, and by symmetry $\alpha\beta = \beta\alpha$. We have proved that A is congruence 2-permutable. □

We conclude the paper with showing that there exists no analogue of Theorem 3.4 for congruence n-permutability with $n > 3$. Indeed, let G be the two-element group, A_1 and A_2 two disjoint copies of G which are considered as G-sets under natural (regular) action of G, and $A = A_1 \cup A_2$. It is easy to check that A is congruence 4-permutable (and even 3.5-permutable in the evident sense), although A fails to be segregated in view of Proposition 1.3.

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