Separation of $(n+1)$-families of sets in general position in $\mathbf{R}^n$

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Abstract. In this paper the main result in [1], concerning $(n+1)$-families of sets in general position in $\mathbf{R}^n$, is generalized. Finally we prove the following theorem: If $\{A_1, A_2, \ldots, A_{n+1}\}$ is a family of compact convexly connected sets in general position in $\mathbf{R}^n$, then for each proper subset I of $\{1, 2, \ldots, n+1\}$ the set of hyperplanes separating $\cup\{A_i : i \in I\}$ and $\cup\{A_j : j \in \overline{I}\}$ is homeomorphic to S_n^+.

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1. Introduction

In this paper we continue the investigation of a previous article [1], regarding the separability of the members of an $(n+1)$-family of sets in general position in $\mathbf{R}^n$. In the beginning we recall some definitions and notations.

A family $\mathcal{A}$ of sets in $\mathbf{R}^n$ is said to be in general position if any m-flat, $0 \le m \le n-1$, intersects at most $m+1$ members of $\mathcal{A}$. Let $m = \min\{n+1, card\,\mathcal{A}\}$. It is easy to see that the family $\mathcal{A}$ is in general position if and only if for every choice of sets $A_1, A_2, \ldots, A_m \in \mathcal{A}$ and every choice of points $x_1 \in A_1,\ x_2 \in A_2, \ldots, x_m \in A_m$, the set $\{x_1, x_2, \ldots, x_m\}$ is affinely independent.

A set $A \subset \mathbf{R}^n$ is called (cf.[5] and [8, p. 174]) convexly connected if there is no hyperplane H such that $H \cap A = \emptyset$ and A contains points in both open halfspaces determined by H.

If A is a compact set and H a hyperplane in $\mathbf{R}^n$, then the distance between A and H is defined to be $d(A, H) = \min\{\|x - y\| : x \in A,\ y \in H\}$. If $H = \{x \in \mathbf{R}^n : \langle x, b\rangle = \lambda\}$ is a hyperplane, the corresponding closed halfspace $\{x \in \mathbf{R}^n : \langle x, b\rangle \le \lambda\}$, $\{x \in \mathbf{R}^n : \langle x, b\rangle \ge \lambda\}$ are denoted respectively by $H^{\le}$, $H^{\ge}$. A set A is said to be separated from a set B by the hyperplane H provided that A lies in one of the closed halfspaces $H^{\le}$, $H^{\ge}$ and B lies in the other. The set A is strictly separated from B by H provided that the separating hyperplane H is disjoint from both A and B. If $\mathcal{A}$ is a family of sets containing at least two members, we say that a hyperplane H separates the members of $\mathcal{A}$ if there exists a nontrivial partition $(\mathcal{B}, \mathcal{C})$ of $\mathcal{A}$ such that $\cup\mathcal{B} \subset H^{\le}$, $\cup\mathcal{C} \subset H^{\ge}$.

The unit sphere in $\mathbf{R}^{n+1}$ and the set $\{x = (x_1, x_2, \dots, x_{n+1}) : \|x\| = 1,\ x_i \geq 0,\ 1 \leq i \leq n+1\}$ are denoted by S_n, S_n^+ respectively. For every subset I of $\{1, 2 \dots, n+1\}$, $\overline{I}$ denotes the complement of I in $\{1, 2, \dots, n+1\}$.

In [1] among other results we have obtained the following

Theorem 1. *Let $\{A_1, A_2, \dots, A_{n+1}\}$ be a family of compact convexly connected sets in general position in $\mathbf{R}^n$. Then*

(i) *for each proper subset I of $\{1, 2, \dots, n+1\}$, there exists exactly one hyperplane H such that*

$$H \text{ separates strictly the sets } \cup\{A_i : i \in I\},\ \cup\{A_j : j \in \overline{I}\} \tag{1}$$

and

$$d(A_1, H) = d(A_2, H) = \cdots = d(A_{n+1}, H); \tag{2}$$

(ii) *there exist exactly $2^n - 1$ hyperplanes satisfying* (2).

In this paper we obtain a generalization of the previous result. Also, we prove that for every nontrivial partition $(\mathcal{B}, \mathcal{C})$ of an $(n+1)$-family of compact convexly connected sets in general position in $\mathbf{R}^n$, the set of hyperplanes separating $\cup\mathcal{B}$ and $\cup\mathcal{C}$ is homeomorphic to S_n^+.

2. Basic results

We start with the following result which generalizes Lemma 3 in [1].

Lemma 2. *Let $[x_1, x_2, \dots, x_{n+1}]$ be an n-simplex in $\mathbf{R}^n$. Then for each $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{n+1}) \in S_n^+$ and for each proper subset I of $\{1, 2, \dots, n+1\}$ there exists exactly one hyperplane H such that*

$$H \text{ separates the sets } \{x_i : i \in I\} \text{ and } \{x_j : j \in \overline{I}\}, \tag{3}$$

$$d(x_i, H) = k\alpha_i \text{ for some } k \text{ and all } i,\quad 1 \leq i \leq n+1. \tag{4}$$

Proof: The distance from an arbitrary point x_0 to a hyperplane

$$H = \{x \in \mathbf{R}^n : \langle x, b\rangle = \lambda\} \tag{5}$$

is given by the Ascoli's formula (see [6, p. 21])

$$d(x_0, H) = \frac{|\langle x_0, b\rangle - \lambda|}{\|b\|}\,. \tag{6}$$

Since the pair $(b, \lambda) \in (\mathbf{R}^n \setminus \{0\}) \times \mathbf{R}$ for which the hyperplane H admits the representation (5) is unique up to a non-zero multiplicative constant, the conditions (3) and (4) are equivalent with

$$\langle x_i, b\rangle - \lambda = \beta_i,\quad 1 \leq i \leq n+1 \tag{7}$$

where

$$\beta_i = \begin{cases} \alpha_i, & \text{if } \ i \in I, \\ -\alpha_i, & \text{if } \ i \in \overline{I}. \end{cases}$$

Denoting by $(x_{i1}, x_{i2}, \ldots, x_{in})$ the coordinates of x_i, $1 \le i \le n+1$, and by $(b_1, b_2, \ldots, b_n)$ the coordinates of b, we are lead to an $(n+1) \times (n+1)$ linear system

$$x_{i1}b_1 + x_{i2}b_2 + \cdots + x_{in}b_n - \lambda = \beta_i, \quad 1 \le i \le n+1. \tag{8}$$

From the affine independence of the points $x_1, x_2, \ldots, x_{n+1}$, it follows that the determinant D of order $n+1$ having the general row $(x_{i1}, x_{i2}, \ldots, x_{in}, 1)$ is different from zero. This proves that the system (8) possesses the unique solution

$$\begin{cases} b_j = \frac{D_j}{D}, & 1 \le j \le n \\ \lambda = -\frac{D_{n+1}}{D}, \end{cases} \tag{9}$$

where D_j is the determinant of order $n+1$ having the general row $(x_{i1}, x_{i2}, \ldots, x_{i,j-1}, \beta_i, x_{i,j+1}, \ldots, x_{in}, 1)$ and D_{n+1} is the determinant of the same order, with the general row $(x_{i1}, \ldots, x_{in}, \beta_i)$. Since at least two β_i are distinct, from (7) it can be easily deduced that $b \neq 0$. All these show that there exists a unique hyperplane H which satisfies the conditions (3) and (4). Note that $d(x_i, H) = \frac{\alpha_i}{\|b\|}$ and that the points x_i, $i \in I$, lie in the closed halfspace $H^{\geq}$, while the points x_j, $j \in \overline{I}$, lie in $H^{\leq}$. □

Let a point α_0 lie on the surface S_n^+ with all coordinates equal. The proof of the following lemma repeats the previous proof (taking $I = \{1, 2, \ldots, n+1\}$).

Lemma 3. *Let $\Delta = [x_1, x_2, \ldots, x_{n+1}]$ be an n-simplex in $\mathbf{R}^n$. Then for every $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n+1}) \in S_n^+ \setminus \{\alpha_0\}$ there exists exactly one hyperplane H such that*

- (i) *the simplex Δ is contained in one of the closed half-spaces determined by H, and*
- (ii) *$d(x_i, H) = k\alpha_i$ for some k and all i, $1 \le i \le n+1$.*

The following generalization of Theorem 1 is our main result.

Theorem 4. *Let $\{A_1, A_2, \ldots, A_{n+1}\}$ be a family of compact convexly connected sets in general position in $\mathbf{R}^n$. Then*

- (i) *for each $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n+1}) \in S_n^+$ and for each proper subset I of $\{1, 2, \ldots, n+1\}$ there exists exactly one hyperplane H such that*

$$H \text{ separates the sets } \ \cup\{A_i : i \in I\} \ \text{ and } \ \cup\{A_j : j \in \overline{I}\} \tag{10}$$

and

$$d(A_i, H) = k\alpha_i \ \text{ for some } \ k \ \text{ and all } \ i, \quad 1 \le i \le n+1; \tag{11}$$

(ii) *if for each* $\alpha \in S_n^+$, $N(\alpha)$ *denotes the number of the hyperplanes* H *satisfying* (11), *then*

$$N(\alpha) = \begin{cases} 2^n, & \text{if } \alpha \in S_n^+ \setminus \{\alpha_0\} \\ 2^n - 1, & \text{if } \alpha = \alpha_0. \end{cases}$$

Proof: (i) A similar argument to that used in proving Corollary 7 in [1] permits us to suppose the compact sets A_i being convex. Let $A = A_1 \times A_2 \times \cdots \times A_{n+1}$. The elements of A are denoted by $\overline{x}, \overline{y}, \dots$. Let I be an arbitrary fixed proper subset of $\{1, 2, \dots, n+1\}$. By Lemma 2, for each $\overline{x} = (x_1, x_2, \dots, x_{n+1}) \in A$, $(x_i \in A_i,\ 1 \le i \le n+1)$ there exists a unique hyperplane, denoted by $H(\overline{x})$, such that

(12) $d(x_i, H(\overline{x})) = k\alpha_i$ for some k (dependent on $\overline{x}$) and all i, $1 \le i \le n+1$,

$\{x_i : i \in I\} \subset H^{\geq}(\overline{x})$ and $\{x_j : j \in \overline{I}\} \subset H^{\leq}(\overline{x})$. The equation of $H(\overline{x})$ is $\langle x, b\rangle = \lambda$, where $b = (b_1, b_2, \dots, b_n) \in \mathbf{R}^n \setminus \{0\}$ and $\lambda \in \mathbf{R}$ are given by the formulas (9).

We define the map $f : A \to 2^A$ by $f(\overline{x}) = P_1(\overline{x}) \times P_2(\overline{x}) \times \cdots \times P_{n+1}(\overline{x})$, $\overline{x} \in A$ and $P_i(\overline{x})$ defined by

$$(13) \qquad \begin{cases} \text{(a)} \quad P_i(\overline{x}) = \{x \in A_i : \langle x, b\rangle = \min\{\langle y, b\rangle : y \in A_i\}\}, & i \in I, \\ \text{(b)} \quad P_i(\overline{x}) = \{x \in A_i : \langle x, b\rangle = \max\{\langle y, b\rangle : y \in A_i\}\}, & i \in \overline{I}. \end{cases}$$

Since the sets A_i are compact, the sets $P_i(\overline{x})$ are nonempty. If $y_i \in P_i(\overline{x})$, $1 \le i \le n+1$, then $P_i(\overline{x})$ coincides with the intersection of the set A_i with the hyperplane through y_i parallel to $H(\overline{x})$. Thus each $P_i(\overline{x})$ is a compact convex set, and $f(\overline{x})$ is a compact convex set for each $\overline{x} \in A$. Using Lemma 4 in [1] it can be easily verified that f is upper semicontinuous.

By the Fan-Glicksberg-Kakutani fixed point theorem (see [2] and [4]), there is a point $\overline{z} = (z_1, z_2, \dots, z_{n+1}) \in A$ such that $\overline{z} \in f(\overline{z})$. Let $\langle x, b^0\rangle - \lambda^0 = 0$ be the equation of the hyperplane $H(\overline{z})$, with $b^0 = (b_1^0, b_2^0, \dots, b_n^0)$ and λ^0 given by the formulas (9). For each $i \in I$, $z_i \in H^{\geq}(\overline{z})$ and by definition of f, $z_i \in P_i(\overline{z})$. Thus, we infer from (13a) that $A_i \subset H^{\geq}(\overline{z})$ for all $i \in I$.

Then, for each $i \in I$, we have

$$d(A_i, H(\overline{z})) = \min\left\{\frac{|\langle x, b^0\rangle - \lambda^0|}{\|b^0\|} : x \in A_i\right\} = \frac{\langle z_i, b^0\rangle - \lambda^0}{\|b^0\|} = d(z_i, H(\overline{z})).$$

In a similar manner, we obtain that $A_j \subset H^{\leq}(\overline{z})$ and $d(A_j, H(\overline{z})) = d(z_j, H(\overline{z}))$, for all $j \in \overline{I}$. Therefore $H(\overline{z})$ separates the sets $\cup\{A_i : i \in I\}$, $\cup\{A_j : j \in \overline{I}\}$ and by (12) the sets $\{d(A_1, H(\overline{z})), d(A_2, H(\overline{z})), \dots, d(A_{n+1}, H(\overline{z}))\}$, $\{\alpha_1, \alpha_2, \dots, \alpha_{n+1}\}$ are proportional.

In the second part of the proof we verify the uniqueness of the hyperplane H which satisfies (10) and (11), for an arbitrary fixed set of indices I.

By the way of contradiction, suppose that there exist two distinct hyperplanes $H' = \{x \in \mathbf{R}^n : \langle x, b'\rangle - \lambda' = 0\}$, $H'' = \{x \in \mathbf{R}^n : \langle x, b''\rangle - \lambda'' = 0\}$ satisfying (10) and (11). For each $i \in \{1, 2, \ldots, n+1\}$ let x_i' and x_i'' be points in A_i such that $d(A_i, H') = d(x_i', H')$, $d(A_i, H'') = d(x_i'', H'')$. Then, for a convenient choice of the pairs (b', λ'), (b'', λ'') we have

$$(14) \quad \begin{cases} \text{(a)} \quad \min\{\langle x, b'\rangle - \lambda' : x \in A_i\} = \langle x_i', b'\rangle - \lambda' = \alpha_i & \text{if } i \in I, \\ \text{(b)} \quad \max\{\langle x, b'\rangle - \lambda' : x \in A_j\} = \langle x_j', b'\rangle - \lambda' = -\alpha_j & \text{if } i \in \overline{I} \end{cases}$$

and

$$(15) \quad \begin{cases} \text{(a)} \quad \min\{\langle x, b''\rangle - \lambda'' : x \in A_i\} = \langle x_i'', b''\rangle - \lambda'' = \alpha_i & \text{if } i \in I, \\ \text{(b)} \quad \max\{\langle x, b''\rangle - \lambda'' : x \in A_j\} = \langle x_j'', b''\rangle - \lambda'' = -\alpha_j & \text{if } i \in \overline{I}. \end{cases}$$

Then, for each $i \in I$, by (14a) and (15a) it follows that $\langle x_i', b' - b''\rangle + \lambda'' - \lambda' \leq 0$ and $\langle x_i'', b' - b''\rangle + \lambda'' - \lambda' \geq 0$. Obviously H' and H'' cannot be parallel, hence $b' \neq b''$. The convexity of A_i implies that the hyperplane $H = \{x \in \mathbf{R}^n : \langle x, b' - b''\rangle + \lambda'' - \lambda' = 0\}$ intersects all sets A_i, $i \in I$. Using a similar argument we obtain that H intersects all sets A_j, $j \in \overline{I}$. Therefore H intersects each member of the family $\{A_1, A_2, \ldots, A_{n+1}\}$ which is in general position. The contradiction obtained completes the proof.

(ii) From (i) we deduce that there exist exactly $2^n - 1$ hyperplanes which satisfy (11) and separate the members of the family $\{A_1, A_2, \ldots, A_{n+1}\}$.

$N(\alpha_0) = 2^n - 1$ is the assertion (ii) in Theorem 1. If $\alpha \in S_n^+ \setminus \{\alpha_0\}$, arguing as above, Lemma 3 yields a unique hyperplane which leaves all sets A_i on the same side and which satisfies (11). $\square$

Let $\{A_1, A_2, \ldots, A_{n+1}\}$ be a family of compact convexly connected sets in general position in R^n. For each proper subset I of $\{1, 2, \ldots, n+1\}$ let $\mathcal{H}(I)$ denote the set of hyperplanes which separate the sets $\cup\{A_i : i \in I\}$ and $\cup\{A_j : j \in \overline{I}\}$. To each hyperplane $H \in \mathcal{H}(I)$ there corresponds a unique point $(b^H, \lambda^H) = (b_1^H, b_2^H, \ldots, b_n^H, \lambda^H) \in S_n$ such that $H = \{x \in \mathbf{R}^n : \langle x, b^H\rangle = \lambda^H\}$ and $\cup\{A_i : i \in I\} \subset H^{\geq}$. This correspondence permits to identify $\mathcal{H}(I)$ with a subset of S_n, namely $\{(b^H, \lambda^H) : H \in \mathcal{H}(I)\}$.

The following known results are needed in the proof of Theorem 7.

Lemma 5 [7, Theorem 1]**.** *If M is a compact convex set in $\mathbf{R}^n$, then the function $h : \mathbf{R}^n \to \mathbf{R}$ defined by $h(b) = \max\{\langle x, b\rangle : x \in M\}$ is continuous.*

Lemma 6 [3, p. 207, Lemma 3]**.** *Let X and Y be topological spaces, X compact and Y separated. If $f : X \to Y$ is a continuous bijection, then f is a homeomorphism.*

Theorem 7. *Let $\{A_1, A_2, \ldots, A_{n+1}\}$ be a family of compact convexly connected sets in general position in $\mathbf{R}^n$. Then for every proper subset I of $\{1, 2, \ldots, n+1\}$ the sets $\mathcal{H}(I)$ and S_n^+ are homeomorphic.*

Proof: Let I be a proper subset of $\{1, 2, \ldots, n+1\}$ arbitrarily fixed. Define $f : \mathcal{H}(I) \to S_n^+$ by $f(H) = \frac{1}{\|d_H\|} d_H$, where $d_H = (d(A_1, H), d(A_2, H), \ldots, d(A_{n+1}, H))$. By Theorem 4, f is a bijection. By Lemma 5, each component of f is continuous, hence f is continuous too. Then, taking into account the quoted identification, $\mathcal{H}(I) = f^{-1}(S_n^+)$ is a closed subset of the compact set S_n. So $\mathcal{H}(I)$ is compact and the assertion of Theorem 7 follows now from Lemma 6.

□

Remark. Theorems 4 and 7 can be reformulated obtaining analogous informations about the hyperplanes which strictly separate the members of the family $\{A_1, A_2, \ldots, A_{n+1}\}$. For instance we have:

Let $\{A_1, A_2, \ldots, A_{n+1}\}$ be a family of compact convexly connected sets in general position in $\mathbf{R}^n$. Then for each proper subset I of $\{1, 2, \ldots, n+1\}$ the set of hyperplanes strictly separating $\cup\{A_i : i \in I\}$ and $\cup\{A_j : j \in \overline{I}\}$ is homeomorphic to $\{(\alpha_1, \alpha_2, \ldots, \alpha_{n+1}) \in S_n : \alpha_i > 0,\ 1 \leq i \leq n+1\}$.

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