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On bounds of the drag for Stokes flow around a body without thickness

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Abstract. This paper is devoted to lower and upper bounds of the hydrodynamical drag T for a body in a Stokes flow.

We obtain the upper bound since the solution for a flow in an annulus and therefore the hydrodynamical drag can be explicitly derived. The lower bound is obtained by comparison to the Newtonian capacity of a set and with the help of a result due to J. Simon [10]. The chosen approach provides an interesting lower bound which is independent of the interior of the body.

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Introduction. Some problems like homogenization require an upper bound related to a "small" obstacle for the drag or the capacity (see for instance G. Allaire [1] about Stokes homogenization and D. Cioranescu and F. Murat [3] about Laplace homogenization).

In this paper, we derive lower and upper bounds independent on the interior of the body.

The method is based on the fact that the solution corresponding to a flow in an annulus can be explicitly obtained. Moreover, the hydrodynamical drag can be compared to the Newtonian capacity of a set; therefore a result due to J. Simon [10] can be used. The originality of the present paper comes from the lower bound estimate; this is not a common result in fluid mechanics. We obtain a lower bound which depends only on the section of the obstacle A. Therefore, it provides an interesting value for some obstacles with zero thickness.

Remark. J. Sanchez Hubert and E. Sanchez Palencia [9] have studied the drag for some obstacles without thickness in an unbounded domain by an asymptotic expansion method around the shape. In particular, they have studied the flow around an half plane in the case of a Navier Stokes stationary 2 D flow and they have proved that in this case the drag is not null. Here we derive a lower bound which is true for all bounded obstacle with a non null section and which gives an hydrodynamical drag (non null). □

Statement of the problem. Let A and D be respectively a compact set and an open connected set with sufficiently regular boundaries ($\mathcal{C}^2$ for example) such that $A \subset D$. Then we consider the domain $\Omega = D/A$.

We are interested in the hydrodynamical drag T which is defined as follows (see for instance [2])

$$T = -e_1 \cdot \int_{\partial A} (-p\, Id + \sigma(u_{\mathrm{S}}))n\, ds = \frac{1}{2}\int_\Omega \sigma(u_{\mathrm{S}})^2 \tag{1}$$

where $\sigma(y)^2 = \sum_{i\,j} \sigma_{ij}(y)^2$ and $\sigma_{ij} = \partial_i\, y_j + \partial_j\, y_i$ for all i and j and where (u_{S}, p) is the solution of the following Stokes problem

$$\begin{cases} -\Delta u_{\mathrm{S}} + \nabla p = 0 \text{ in } \Omega\,, \\ \nabla\,.\,u_{\mathrm{S}} = 0 \text{ in } \Omega\,, \\ u_{\mathrm{S}} = 0 \text{ on } \partial A\,, \\ u_{\mathrm{S}} = e_1 \text{ on } \partial D\,, \end{cases} \tag{2}$$

where e_1 is the first basis vector.

Remark. We denote n the unit interior normal vector (outside A on ∂A, inside D on ∂D). This describes the flow around a body A moving with a velocity $-e_1$, in spatial coordinates fixed with respect to A. □

Notations and main results. Let c be the center and r_A be the radius of the smallest ball B_A which contains A. Let R_D be the radius of the biggest ball centered in c and contained in D. We assume that $r_A < R_D$ and we denote $C_{r_A,R_D} = \{\, x \in \mathbb{R}^N : r_A < |x-c| < R_D \,\}$.

Using the variational formulation of (2), we will check in Lemma 6 that T decreases with Ω. This means that T decreases when D increases and A decreases. Thus we can bound T by the drag associated to the annuli which are respectively included in and containing Ω.

Since the annulus C_{r_A,R_D} is included in Ω, we obtain an upper bound of the drag associated to (2). More precisely we get the following result.

Proposition 1. *Let* (u_{S}, p) *be the solution of* (2).

If $N = 2$:

$$T \le \frac{4\pi(r_A^2 + R_D^2)}{(r_A^2 - R_D^2) + (r_A^2 + R_D^2)\,\log\frac{R_D}{r_A}}\,.$$

If $N = 3$:

$$T \le D(r_A, R_D)$$

with

$$\begin{aligned} D(a,b) =& \frac{8\pi}{3}\,\beta^2\Big(\frac{1}{a^5} - \frac{1}{b^5}\Big) + \frac{16\pi}{3}\alpha\beta\Big(\frac{1}{a^3} - \frac{1}{b^3}\Big) + \frac{40\pi}{3}\alpha^2\Big(\frac{1}{a} - \frac{1}{b}\Big) \\ &+ \frac{64\pi}{3}\alpha\gamma(b^2 - a^2) + 4\pi\gamma^2(b^5 - a^5) \end{aligned}$$

and where the constants α, β, γ are explicitly given in Lemma 4. $\square$

Conversely, Ω is included in the annulus C_{R_A,T_D} where R_A is the radius of the biggest ball contained in A and T_D is the radius of a concentric ball which contains D. Therefore we obtain, for example in dimension 2, the following lower bound

$$T \geq \frac{4\pi(R_A^2 + T_D^2)}{(R_A^2 - T_D^2) + (R_A^2 + T_D^2)\log\frac{T_D}{R_A}}\,.$$

For a body A with zero thickness (that is with an empty interior), this inequality provides $T \geq 0$ which is not an interesting result!

A better lower bound depending only on the section of A is given in Theorem 2 hereafter. For some bodies with zero thickness, this result provides a nonzero lower bound.

Remark. Isoperimetric inequality for the capacity obtained by the Schwarz symmetrization (see for instance J. Mossino [8, p. 60]) cannot be used. In the case of an obstacle without thickness, this inequality leads to $T \geq 0$ which again is not an interesting result. $\square$

We denote r_D the radius of the smallest ball which contains D. Let us consider a hyperplane $P \subset \mathbb{R}^N$ and denote A_p the orthogonal projection of A on P. We get a lower bound of the energy which depends on the area σ_p of the biggest disk contained in A_p. In fact one has the following theorem.

Theorem 2. *Let* (u_S, p) *be the solution of* (2).
If $N = 2$:

$$\frac{2\pi}{log(\frac{4\pi r_D}{r_A} + e)} \leq T \qquad \textit{with } e = \pi(1 + \sqrt{1 + \frac{1}{\pi^2}})\,.$$

If $N = 3$:

$$\frac{4\pi(2\,r_D + h)\sqrt{\frac{\sigma_p}{\pi^3}}}{(2\,r_D + h) - \sqrt{\frac{\sigma_p}{\pi^3}}} \leq T \qquad \textit{with } h = r_A(1 + \sqrt{1 + \frac{\sigma_p}{\pi^3 r_A^2}})\,.$$

$\square$

Remarks. In both cases, $N = 2$ and $N = 3$, we can apply an orthogonal projection on every hyperplanes in $\mathbb{R}^N$. The biggest area provides the most interesting inequality.

Note that for $N = 2$ the biggest area coincides with the length of the diameter of A. $\square$

Before beginning we prove that in fact, in our problem, the hydrodynamical drag T is equal to the energy $E = \displaystyle\int_\Omega |\nabla u_S|^2$; namely we have the following preliminary result.

Lemma 3. *Let* (u_S, p) *be the solution of* (2). *Then,*

$$T = \int_\Omega |\nabla u_S|^2 \,.$$

PROOF: Let us suppose that u_S can be written in the form $u_S = v + h$ with $v = 0$ on $\partial\Omega$, $\nabla \cdot v = 0$ in Ω, $\nabla \cdot h = 0$ in Ω, $h = e_1$ in a neighbourhood of ∂D and $h = 0$ on ∂A.

Since we have

$$\frac{1}{2}\sigma(u_S)^2 - |\nabla u_S|^2 = \sum_{ij} \partial_j (u_S)_i \partial_i (u_S)_j \,, \tag{3}$$

with σ defined in (1), and recalling the definitions of h and v, we can show with the help of the Green formula that

$$\int_\Omega \sum_{ij} \partial_j v_i \partial_i (u_S)_j = 0$$

and

$$\int_\Omega \sum_{ij} \partial_j h_i \partial_i (u_S)_j = 0 \,.$$

Thus from the decomposition of u_S and the definition (1), the result holds.

Now, we just have to build the decomposition of u_S. Let Ω_1 an open set such that $A \subset \Omega_1$, $\overline{\Omega}_1 \subset D$ for which we define $\theta \in \mathcal{D}(D)$ with $\theta = 1$ in Ω_1.

We define h as follows

$$h = e_1 + \sum_{i=1}^{2} \Big(\frac{-\partial(\theta x_i)}{\partial x_i}\, e_1 + \frac{\partial(\theta x_i)}{\partial x_1} e_i\Big) \,.$$

Thus, $u_S = h + v$ satisfies the properties mentioned at the beginning of the proof. $\square$

1. Stokes upper bound

In this section we study the problem (2) for which we give some properties concerning the associated energy. First, we use an idea of G.G. Stokes [12] developed for example by G. Allaire [1] to obtain the exact solution of the problem (2) in an annulus $C_{a,b} = B_b / \overline{B_a}$ with B_a and B_b two concentric balls.

Therefore, we recall the following result for which all coefficients can be explicated.

Lemma 4. *The solution of* (2) *in* $C_{a,b}$ *is of the form*

$$
\left\{
\begin{array}{l}
u = x_1 r f(r) e_r + g(r) e_1\,,\\
p = x_1 h(r)\,,
\end{array}
\right.
\tag{4}
$$

for $r = |x| \in [a,b]$. *The functions* f, g, h *are given by the following equalities:*

if $N = 2$:

$$
\left\{
\begin{array}{l}
f(r) = \dfrac{\beta}{r^4} + \dfrac{\alpha}{r^2} + \gamma\,,\\[2mm]
g(r) = -\alpha \log r - \dfrac{\beta}{2r^2} - \dfrac{3}{2}\gamma\, r^2 + \eta\,,\\[2mm]
h(r) = \dfrac{2\alpha}{r^2} - 4\gamma\,,
\end{array}
\right.
$$

with

$$
\left\{
\begin{array}{l}
\alpha = -(a^2 + b^2)\gamma\,,\\
\beta = a^2 b^2 \gamma\,,\\
\eta = (-(a^2+b^2)\log a + \dfrac{b^2}{2} + \dfrac{3}{2}a^2)\gamma\,,\\[2mm]
\gamma = \dfrac{1}{(a^2 - b^2) + (a^2 + b^2)\log\frac{b}{a}}\,;
\end{array}
\right.
$$

if $N = 3$:

$$
\left\{
\begin{array}{l}
f(r) = \dfrac{\beta}{r^5} + \dfrac{\alpha}{r^3} + \gamma\,,\\[2mm]
g(r) = \dfrac{-\beta}{3\, r^3} + \dfrac{\alpha}{r} - 2\gamma\, r^2 + \eta\,,\\[2mm]
h(r) = \dfrac{2\alpha}{r^3} - 10\gamma\,,
\end{array}
\right.
$$

with

$$
\left\{
\begin{array}{l}
\alpha = -\big(a^3 + b^2 \times \dfrac{b^3 - a^3}{b^2 - a^2}\big)\,\gamma\,,\\[2mm]
\beta = a^2 b^2 \times \dfrac{b^3 - a^3}{b^2 - a^2}\,\gamma\,,\\[2mm]
\eta = \big(3a^2 + \dfrac{4}{3}b^2 \times \dfrac{b^3 - a^3}{ab^2 - a^3}\big)\,\gamma\,,\\[2mm]
\gamma\big(-\dfrac{1}{3}a^3 \times \dfrac{b^3 - a^3}{b^2 - a^2} - (a^3 + b^2\dfrac{b^3 - a^3}{b^2 - a^2}) - 2b^3 + b(3a^2 + \dfrac{4}{3}b^2 \times \dfrac{b^3 - a^3}{ab^2 - a^3})\big) = b\,.
\end{array}
\right.
$$

$\square$

Outline of the proof. Replacing the expression (4) of (u_S, p) in the system (2), we obtain the following system

$$\begin{cases} \dfrac{g'(r)}{r} + (N+1)f(r) + rf'(r) = 0\,, \\ h(r) - g''(r) - (N-1)\dfrac{g'(r)}{r} - 2f(r) = 0\,, \\ h'(r) - (N+3)f'(r) - rf''(r) = 0\,, \\ f(a) = g(a) = 0\,, \\ f(b) = 0 \text{ and } g(b) = 1\,. \end{cases}$$

We can check that f, g, h, previously defined, satisfy this system. For more details see for instance G. Allaire [1, appendix B]. □

With the help of Lemmas 3 and 4, we obtain the following explicit formula for the hydrodynamical drag T in an annulus $C_{a,b}$.

Lemma 5. *Let (u_S, p) be the solution of (2) in $C_{a,b}$.*

If $N = 2$:

$$T = \frac{4\pi(a^2+b^2)}{(a^2-b^2) + (a^2+b^2)\,log\frac{b}{a}}\,.$$

If $N = 3$:

$$\begin{aligned} T =& \frac{8\pi}{3}\beta^2\Big(\frac{1}{a^5} - \frac{1}{b^5}\Big) + \frac{16\pi}{3}\alpha\beta\Big(\frac{1}{a^3} - \frac{1}{b^3}\Big) + \frac{40\pi}{3}\alpha^2\Big(\frac{1}{a} - \frac{1}{b}\Big) \\ &+ \frac{64\pi}{3}\alpha\gamma(b^2 - a^2) + 4\pi\gamma^2(b^5 - a^5), \end{aligned} \tag{5}$$

where the constants α, β, γ are given in Lemma 4. □

Outline of proof. For $N = 2$ or 3, we have

$$u(r) = x_1\, r\, f(r)\, e_r + g(r)\, e_1\,,$$

so that

$$|\nabla u|^2 = \sum_{ij} (\partial_i u_j)^2 = \sum_{ij} \big(\partial_i(x_1 x_j f(r)) + \delta_{j1}\partial_i g(r)\big)^2\,,$$

where δ_{ij} is the Krönecker symbol.

Therefore, we easily derive

$$\begin{aligned} |\nabla u|^2 =& r^2 f(r)^2 + g'(r)^2 + x_1^2\big((N+2)f(r)^2 + r^2 f'(r)^2 \\ &+ 4r\, f(r)f'(r) + \frac{4}{r}f(r)g'(r) + 2f'(r)g'(r)\big)\,. \end{aligned} \tag{6}$$

From (6) and by using the explicit formulas for f and g defined in Lemma 3, we can deduce an explicit form of the drag in an annulus. Hereafter we distinguish the cases $N = 2$ and $N = 3$.

— For $N = 2$. Recalling that $x_1 = r\cos\theta$, $\displaystyle\int_0^{2\pi} \cos^2\theta = \pi$ and

$$\begin{cases} g'(r) = \dfrac{-\alpha}{r} + \dfrac{\beta}{r^3} - 3\gamma r\,, \\ rf(r) = \dfrac{\alpha}{r} + \dfrac{\beta}{r^3} + \gamma r\,, \\ r^2 f'(r) = \dfrac{-2\alpha}{r} - \dfrac{4\beta}{r^3}\,, \end{cases}$$

we can evaluate all the terms in the right hand side of

$$\begin{aligned} \|\nabla u\|^2_{L^2(C_{a,b})} &= \|g'(r)\|^2_{L^2(C_{a,b})} + \|rf(r)\|^2_{L^2(C_{a,b})} \\ &\quad + \int_{C_{a,b}} \big(r^2 f'(r)^2 + 4f(r)^2\big) r^3 \cos^2\theta \, dr\, d\theta \\ &\quad + \int_{C_{a,b}} \Big(4rf(r)f'(r) + 4\frac{f(r)}{r} g'(r) + 2f'(r)g'(r)\Big) r^3 \cos^2\theta \, dr\, d\theta\,. \end{aligned}$$

Thus we obtain, with the help of α, β, γ given in Lemma 4, the announced result.

— For $N = 3$. Recalling that $x_1 = r\cos\theta\sin\varphi$, $\displaystyle\int_0^{2\pi}\int_0^{\pi} \cos^2\theta \sin^3\varphi \, d\theta\, d\varphi = \frac{4}{3}\pi$ and

$$\begin{cases} g'(r) = \dfrac{-\alpha}{r^2} + \dfrac{\beta}{r^4} - 4\gamma r\,, \\ rf(r) = \dfrac{\alpha}{r^2} + \dfrac{\beta}{r^4} + \gamma r\,, \\ r^2 f'(r) = \dfrac{-3\alpha}{r^2} - \dfrac{5\beta}{r^4}\,, \end{cases}$$

we can evaluate all the terms in the right hand side of

$$\begin{aligned} \|\nabla u\|^2_{L^2(C_{a,b})} &= \|g'(r)\|^2_{L^2(C_{a,b})} + \|rf(r)\|^2_{L^2(C_{a,b})} \\ &\quad + \int_{C_{a,b}} \big(r^2 f'(r)^2 + 5f(r)^2\big) r^4 \cos^2\theta \sin^3\varphi \, dr\, d\theta\, d\varphi \\ &\quad + \int_{C_{a,b}} \Big(4rf(r)f'(r) + 4\frac{f(r)}{r} g'(r) + 2f'(r)g'(r)\Big) r^4 \cos^2\theta \sin^3\varphi \, dr\, d\theta\, d\varphi\,. \end{aligned}$$

Thus we obtain, with the help of α, β, γ given in Lemma 4, the announced result. $\square$

As a consequence of Lemma 5, we have the following result.

Corollary 6. *Let $N > 3$, $a > 0$. Let $u_{a,b}$ be the solution in $C_{a,b}$ given in Lemma 4 and $(u_{a,\infty}, p_{a,\infty})$ be a solution of (2) in $C_{a,\infty} = \{x \in \mathbb{R}^3 : |x| > a\}$ given by*

$$u_{a,\infty} = x_1\, r\, f(r) e_r + g(r)\, e_1\,, \quad p_{a,\infty} = x_1\, h(r)\,, \tag{7}$$

where

$$\begin{cases} f(r) = \dfrac{3\,a^3}{4\,r^5} - \dfrac{3\,a}{4\,r^3}\,, \\ g(r) = \dfrac{-a^3}{4\,r^3} - \dfrac{3\,a}{4\,r} + 1\,, \\ h(r) = -\dfrac{3\,a}{2\,r^3}\,. \end{cases} \tag{8}$$

We have

$$||\nabla u_{a,b}||^2_{L^2(C_{a,b})^{3\times 3}} \to ||\nabla u_{a,\infty}||^2_{L^2(C_{a,\infty})^{3\times 3}}$$

as $b \to \infty$. □

Proof for $a = 1$. We first check that $(u_{a,\infty}, p_{a,\infty})$ is a solution of (2) in $C_{a,\infty}$ with

$$\mathrm{T} = ||\nabla u_{a,\infty}||^2_{L^2(C_{a,\infty})^{3\times 3}} = 6\pi\,.$$

Using Lemma 4 and the formula (5), we show that

$$||\nabla u_{a,b}||^2_{L^2(C_{a,b})^{3\times 3}} \to (40\pi/3 + 8\pi/3 - 48\pi/9) \times 9/16 = 6\pi$$

when $b \to +\infty$. The general case $a \neq 1$ is deduced from the previous proof. □

Remark. For $N = 2$ we have not the same type of convergence accordingly to the Stokes paradox (see for instance J.G. Heywood [7]). In this case we get the following convergence result $T \to 0$ when $b \to \infty$. □

Now let us give a result concerning the variation of E with respect to the domain Ω.

Lemma 7. *Let $\Omega\,, \Omega'$ be two open sets such that $\Omega \subset \Omega'$ (that is $A' \subset A$ and $D \subset D'$), (u_{S}, p) be the solution of (2) in $\Omega = D/A$ and (u'_{S}, p') be the solution of (2) in $\Omega' = D'/A'$. We have*

$$\int_\Omega |\nabla u_{\mathrm{S}}|^2 \geq \int_{\Omega'} |\nabla u'_{\mathrm{S}}|^2\,.$$

PROOF: Let $\tilde{u}_{\mathrm{S}}$ denote the following extension of u_S

$$\tilde{u}_{\mathrm{S}} = \begin{cases} u_{\mathrm{S}} & \text{in } \Omega, \\ 0 & \text{in } A - A', \\ e_1 & \text{in } D' - \overline{D}. \end{cases}$$

Then, we have $\tilde{u}_{\mathrm{S}} = u'_{\mathrm{S}} + \varphi$ where $\varphi \in H^1_0(\Omega')$ and $\nabla \,.\, \varphi = 0$ in Ω'. Therefore

$$\int_{\Omega'} |\nabla \tilde{u}_{\mathrm{S}}|^2 = \int_{\Omega'} |\nabla u'_{\mathrm{S}}|^2 + \int_{\Omega'} |\nabla \varphi|^2 + 2 \int_{\Omega'} \nabla \varphi \,.\, \nabla u'_{\mathrm{S}}$$

but

$$\int_{\Omega'} \nabla \varphi \,.\, \nabla u'_{\mathrm{S}} = - \int_{\Omega'} \varphi \Delta u'_{\mathrm{S}} = 0$$

so that

$$\int_{\Omega'} |\nabla \tilde{u}_{\mathrm{S}}|^2 \geq \int_{\Omega'} |\nabla u'_{\mathrm{S}}|^2 .$$

$\square$

With the help of this lemma and thanks to the expressions given in Lemma 5 we easily obtain the Stokes upper bound given in Lemma 1.

2. Stokes lower bound

In this section we derive the lower bound (*i.e.* we prove Theorem 2) with the help of a result of J. Simon [10]. The result is based on a personal communication with A. Ancona and is recalled in Lemma 9 hereafter.

Let u_L be the solution of

$$\begin{cases} -\Delta u_{\mathrm{L}} = 0 \text{ in } \Omega\,, \\ u_{\mathrm{L}} = 0 \text{ on } \partial D\,, \\ u_{\mathrm{L}} = 1 \text{ on } \partial A\,. \end{cases} \tag{9}$$

In the sequel, we shall denote by u_{L} the following extension of $\tilde{u}_{\mathrm{L}}$

$$\tilde{u}_{\mathrm{L}} = \begin{cases} u_{\mathrm{L}} & \text{in } \Omega, \\ 1 & \text{in } A. \end{cases}$$

We shall obtain a lower bound on the capacity depending on some geometrical parameters and use Lemma 14 in order to minimize the Stokes drag.

Let us recall the following result.

Lemma 8. *The measure $-\Delta u_{\mathrm{L}}$ has its support ∂A and*

$$\mathrm{cap}_D A = \int_D d(-\Delta u_{\mathrm{L}})\,.$$

Proof: Let u_{L} be the extension defined above; u_{L} being in $H^1(D)$ we have

$$\langle -\Delta u_{\mathrm{L}}, \varphi\rangle = \int_D \nabla u_{\mathrm{L}}.\nabla\varphi\, dx = \int_\Omega \nabla\, u_{\mathrm{L}}.\nabla\,\varphi\, dx\,, \quad \forall\varphi \in H_0^1(D)\,.$$

Moreover $u_{\mathrm{L}} \in H^2(\Omega)$ so that the trace of the normal derivative on $\partial\Omega$ can be defined and the Green formula in Ω can be used leading to

$$\int_\Omega \nabla\, u_{\mathrm{L}}.\nabla\,\varphi\, dx = -\int_{\partial A} \frac{\partial u_{\mathrm{L}}}{\partial n}\varphi\, d\sigma\,.$$

This proves, with the help of the maximum principle, that $-\Delta u_{\mathrm{L}}$ is equal to the positive measure $-\dfrac{\partial u_{\mathrm{L}}}{\partial n} d\sigma_{\partial A}$. Since

$$\mathrm{cap}_D A = \int_D |\nabla(1-u_{\mathrm{L}})|^2 = -\int_{\partial D} \frac{\partial u_{\mathrm{L}}}{\partial n}\, d\sigma$$

and

$$\int_D d(-\Delta u_{\mathrm{L}}) = \langle -\Delta u_{\mathrm{L}}, 1\rangle = \int_\Omega \nabla\, u_{\mathrm{L}}.\nabla\, 1\, dx - \int_{\partial D} \frac{\partial u_{\mathrm{L}}}{\partial n}\, d\sigma\,,$$

we get the announced result. □

We now give the fundamental result of [10] for which we recall the proof for reader's convenience. Let A' be a compact set such that $A' \subset D$ with a sufficiently regular boundary. We have the following result.

Lemma 9. *Let f be a continuous map from ∂A onto $\partial A'$ $(= f(\partial A))$ such that the Green function G in D satisfies the following hypothesis*

$$G(f(x), f(y)) \geq G(x,y) \qquad \forall x, y \in \partial A. \tag{10}$$

Then

$$\mathrm{cap}_D A \;\geq\; \mathrm{cap}_D A'\,.$$

Proof: Let u'_{L} denote the solution of (9) with A' instead of A. We define a measure μ on D, whose support is $\partial A'$, by

$$\int_D \varphi\, d\mu = \int_D \varphi\circ f\, d(-\Delta u_{\mathrm{L}}), \qquad \forall\varphi \in \mathcal{C}^0(D)\,.$$

From Lemma 8, we have

$$\mathrm{cap}_D A = \int_D d(-\Delta u_{\mathrm{L}}) = \int_D d\mu$$

and

$$\mathrm{cap}_D A' = \int_D d(-\Delta u'_{\mathrm{L}}) \,.$$

Let $u^\star$ be the solution of

$$\begin{cases} -\Delta u^\star = \mu \text{ in } D\,, \\ u^\star = 0 \text{ on } \partial D\,. \end{cases}$$

By the integral representation theory, we have

$$u^\star(f(x)) = \int_D G(f(x), y')\, d\mu(y') \quad \forall x \in \partial\, A\,.$$

Then, using the definition of μ we get

$$u^\star(f(x)) = \int_D G(f(x), f(y))\, d(-\Delta u_{\mathrm{L}})(y)\,.$$

On the other hand,

$$\begin{cases} -\Delta u_{\mathrm{L}} = -\Delta u_{\mathrm{L}} \text{ in } D\,, \\ u_{\mathrm{L}} = 0 \text{ on } \partial D\,. \end{cases}$$

Similarly, we can show that

$$u_{\mathrm{L}}(x) = \int_D G(x, y)\, d(-\Delta u_{\mathrm{L}})(y)\,.$$

Since

$$u'_{\mathrm{L}}(f(x)) = u_{\mathrm{L}}(x) = 1$$

and since $-\Delta u_{\mathrm{L}}$ is a positive measure, the assumption (10) implies that

$$u^\star - u'_{\mathrm{L}} \geq 0 \text{ on } \partial A'\,.$$

Moreover, as $u^\star - u'_{\mathrm{L}} = 0$ on ∂D and $\Delta(u^\star - u'_{\mathrm{L}}) = 0$ in $\Omega' = D/A'$, the maximum principle gives

$$u^\star - u'_{\mathrm{L}} \geq 0 \text{ in } \Omega'\,.$$

This implies that $\frac{\partial}{\partial n}(u^\star - u'_{\mathrm{L}}) \geq 0$ on ∂D and, since $-\Delta u^\star = \mu$ in D and due to the definition of μ,

$$\int_D d(-\Delta(u_{\mathrm{L}} - u'_{\mathrm{L}})) \geq 0\,.$$

This ends the proof. $\square$

We now calculate the capacity of two concentric balls of radius a and b centered in 0 as well as the Green function in a ball B_R (see for instance R. Dautray and J.L. Lions [4, p. 503] and J. Simon [11, Theorem and Definition 80]). These expressions depend on the dimension N.

Lemma 10. *Let z be the solution of* (9) *in $C_{a,b}$.*
If $N=2$:

$$z(x)=\frac{\log\frac{b}{|x|}}{\log\frac{b}{a}},$$

thus

$$\mathrm{cap}_{B_b}\overline{B}_a=\frac{2\,\pi}{\log\frac{b}{a}}\,.$$

If $N=3$:

$$z(x)=\Big(\frac{1}{b}-\frac{1}{|x|}\Big)\,\Big(\frac{1}{b}-\frac{1}{a}\Big)^{-1},$$

thus

$$\mathrm{cap}_{B_b}\overline{B}_a=\frac{4\,\pi b\,a}{b-a}\,.$$

□

Lemma 11. *Let $G_{B_R}(x,y)$ be the Green function in a ball B_{R_1}.*
If $N=2$:

$$G_{B_R}(x,y)=\frac{1}{2\,\pi}\log\Big(\frac{|x^\star-y|\,|x|}{R\,|x-y|}\Big)\,,$$

and if $N=3$:

$$G_{B_R}(x,y)=\frac{1}{4\pi\,|x-y|}\Big(1-\frac{R\,|x-y|}{|x|\,|x^\star-y|}\Big) \tag{11}$$

with $x^\star=x\dfrac{R^2}{|x|^2}$. □

With the help of Lemmas 8–11 we can now derive a lower bound of the capacity. More precisely, we have the following result.

Theorem 12. *Let $\mathrm{cap}_D A$ be the Newtonian capacity corresponding to* (9).
If $N=2$:

$$\mathrm{cap}_D A\geq\frac{2\,\pi}{\log(2\,\pi\frac{r_D}{r_A}+e)}\,,\qquad \textit{where } e=\pi\,\Big(1+\sqrt{1+\frac{1}{\pi^2}}\Big),$$

and if $N=3$:

$$\mathrm{cap}_D A\geq\frac{4\,\pi(2\,r_D+h)\sqrt{\frac{\sigma_p}{\pi^3}}}{(2\,r_D+h)-\sqrt{\frac{\sigma_p}{\pi^3}}}\,,\qquad \textit{where } h=r_A\,\Big(1+\sqrt{1+\frac{\sigma_p}{\pi^3 r_A^2}}\Big)\,. \tag{12}$$

Proof: We give the proof for $N = 3$. The reader can find the case $N = 2$ in J. Simon [10, p. 22] with $\sigma_p = 2\,r_A$ the length of the diameter of A.

Let us consider a hyperplane $P \subset \mathbb{R}^N$ such that the center of the smallest ball which contains D belongs to it and denote A_p the projection of A on P. We define D_p the largest disk contained in A_p for which we build a compact set $K \subset A$ sufficiently regular ($\mathcal{C}^2$) such that $K_p = D_p$.

Using the same argument as in Lemma 7, we can show that $\mathrm{cap}_D A$ increases when D/A decreases. Therefore, if we obtain a lower bound for $\Omega = D/K$, this lower bound is valid for $\Omega = D/A$.

Let us choose coordinates such that $(h, 0, 0) \in D_p$,

$$D_p \subset \{(x_1, x_2, 0) : x_1 \geq h\}, \qquad \text{where } h = r_A\Big(1 + \sqrt{1 + \frac{\sigma_p}{\pi^3 r_A^2}}\Big)\,.$$

Let $A' = \overline{B}_{\sqrt{\sigma_p/\pi^3}}$ be the closed ball of radius $\sqrt{\sigma_p/\pi^3}$ centered in 0. Then, we define a continuous contractive map f from ∂K to $\partial A'$ ($= f(\partial K)$) such that:

f is composed of the projection on D_p, the identification of the boundary in one point and the homeomorphism defined in G. Godbillon [6, p. 33] which transforms the identified domain to the sphere $\partial\, B_{\sqrt{\sigma_p/\pi^3}}$.

More precisely, we have the following classic result.

Lemma 13. *The sphere S^m, $m \geq 1$, is homeomorphic to the space obtained from the ball D^m by the identification of its boundary S^{m-1} with one point.* $\square$

Let B_{R_1} be the ball of radius $R_1 = 2\,r_D + h$ centered in 0. Since D is included in B_{R_1}, we get

$$\mathrm{cap}_D K \;\geq\; \mathrm{cap}_{B_{R_1}} K\,. \tag{13}$$

Let $G_{B_{R_1}}$ be the Green function (11) in B_{R_1}. Suppose for the moment that it satisfies the hypothesis (10). Then, Lemma 9 gives

$$\mathrm{cap}_{B_{R_1}} K \;\geq\; \mathrm{cap}_{B_{R_1}} \overline{B}_{\sqrt{\frac{\sigma_p}{\pi^3}}} = \frac{4\,\pi(2\,r_D + h)\sqrt{\frac{\sigma_p}{\pi^3}}}{(2\,r_D + h) - \sqrt{\frac{\sigma_p}{\pi^3}}}\,.$$

The announced lower bound for $\mathrm{cap}_D A$ follows from (13).

It remains to prove that $G_{B_{R_1}}$ satisfies (10).

For any x, $y \in \partial K$, $x \neq y$, we obtain

$$\begin{aligned} |x^\star - y||x| \leq (|x^\star - x| + |x - y|)|x| &= {R_1}^2 - |x|^2 + |x - y||x| \\ &\leq {R_1}^2 - |x|^2 + 2r_A|x| \end{aligned} \tag{14}$$

and since $f(x)$, $f(y)$ belong to $\partial B_{\sqrt{\sigma_p/\pi^3}}$, we get

$$|f(x)^\star - f(y)||f(x)| \geq (|f(x)^\star| - |f(y)|)|f(x)| = {R_1}^2 - \frac{\sigma_p}{\pi^3}\,. \tag{15}$$

Since $|x| \geq x_1 > h$ and from (14) and (15), we have

$$|f(x)^\star - f(y)||f(x)| \geq |x^\star - y||x|\,. \tag{16}$$

From the definition of f, we have

$$|f(x) - f(y)| \leq |x - y|\,. \tag{17}$$

Due to the definition of $G_{B_{R_1}}(x,y)$ the maximum principle gives, see for instance D. Gilbart and N.S. Trudinger [5, p. 19],

$$G_{B_{R_1}}(x,y) \geq 0\,.$$

Thus, with (16) and (17) we get

$$\frac{1}{|f(y)-f(x)|}\Big(1 - \frac{R_1|f(x)-f(y)|}{|f(x)||f(x)^\star - f(y)|}\Big) \geq \frac{1}{|x-y|}\Big(1 - \frac{|x-y|R_1}{|x^\star - y||x|}\Big),$$

which proves (10).

Now let us give an easy result from which, with Lemma 3 and Theorem 12, we will deduce the Stokes lower bound.

Lemma 14. *Let* (u_{S}, p) *and* u_{L} *be respectively the solutions of* (2) *and* (9). *We have*

$$\int_\Omega |\nabla u_{\mathrm{S}}|^2 \geq \mathrm{cap}_D A\,.$$

Proof: Let $u_{\mathrm{S}}^1 \in H^1(\Omega)$ be the first component of u_{S}. We have $u_{\mathrm{S}}^1 = 1 - u_{\mathrm{L}} + \varphi$ with $\varphi \in H_0^1(\Omega)$, thus

$$\int_\Omega |\nabla u_{\mathrm{S}}^1|^2 = \int_\Omega |\nabla u_{\mathrm{L}}|^2 - 2\int_\Omega \nabla u_{\mathrm{L}}\,.\nabla\varphi + \int_\Omega |\nabla\varphi|^2\,.$$

Since

$$\int_\Omega \nabla u_{\mathrm{L}}.\,\nabla\varphi = 0$$

we get

$$\int_\Omega |\nabla u_{\mathrm{S}}|^2 \geq \int_\Omega |\nabla u_{\mathrm{S}}^1|^2 \geq \int_\Omega |\nabla u_{\mathrm{L}}|^2$$

which proves the announced result. □

Remark. In this paper we derive a lower bound for the Stokes drag. However, by comparison to the Stokes solution, it is easy to obtain a lower bound for the Navier-Stokes drag, namely we have the following inequality

$$\int_\Omega |\nabla u_{\mathrm{N.S}}|^2 \geq \int_\Omega |\nabla u_{\mathrm{S}}|^2\,.$$

□

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