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Results on Colombeau product of distributions

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Abstract. The differential $\mathbb{C}$-algebra $\mathcal{G}(\mathbb{R}^m)$ of generalized functions of J.-F. Colombeau contains the space $\mathcal{D}'(\mathbb{R}^m)$ of Schwartz distributions as a $\mathbb{C}$-vector subspace and has a notion of 'association' that is a faithful generalization of the weak equality in $\mathcal{D}'(\mathbb{R}^m)$. This is particularly useful for evaluation of certain products of distributions, as they are embedded in $\mathcal{G}(\mathbb{R}^m)$, in terms of distributions again. In this paper we propose some results of that kind for the products of the widely used distributions $x_\pm^a$ and $\delta^{(p)}(x)$, with x in $\mathbb{R}^m$, that have coinciding singular supports. These results, when restricted to dimension one, are also easily transformed into the setting of regularized model products in the classical distribution theory.

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1. Notation and definitions

We will recall first the basic definitions of Colombeau algebra $\mathcal{G}(\mathbb{R}^m)$, following their recent presentation in [7, Chapter 3].

Notation 1. If $\mathbb{N}_0$ stands for the nonnegative integers and $p = (p_1, p_2, \dots, p_m)$ is a multiindex in $\mathbb{N}_0^m$, we let $|p| = \sum_{i=1}^m p_i$ and $p! = p_1! \dots p_m!$. Then, if $x = (x_1, \dots, x_m)$ is in $\mathbb{R}^m$, we denote by $x^p = (x_1^{p_1}, x_2^{p_2}, \dots, x_m^{p_m})$ and $\partial^p{}_x = \partial^{|p|}/\partial x_1^{p_1} \dots \partial x_m^{p_m}$. Also, by $x < 0$ is meant: $x_1 \le 0, \dots, x_m \le 0$ and $x \ne 0$. Now for any q in $\mathbb{N}_0$, denote by $A_q(\mathbb{R}) = \{\varphi(x) \in \mathcal{D}(\mathbb{R}) : \int_{\mathbb{R}} x^j \varphi(x)\, dx = \delta_{0j}$ for $0 \le j \le q$, where $\delta_{00} = 1$, $\delta_{0j} = 0$ for $j > 0\}$. This also extends to $\mathbb{R}^m$ as an m-fold tensor product: $A_q(\mathbb{R}^m) = \{\varphi(x) \in \mathcal{D}(\mathbb{R}^m) : \varphi(x_1, \dots, x_m) = \prod_{i=1}^m \chi(x_i)$ for some χ in $A_q(\mathbb{R})\}$. Finally, we will denote by $\varphi_\varepsilon = \varepsilon^{-m}\varphi(\varepsilon^{-1}x)$, for any φ in $A_q(\mathbb{R}^m)$ and $\varepsilon > 0$.

Definition 1. Let $\mathcal{E}[\mathbb{R}^m]$ stand for the set of functions $f(\varphi, x) : A_0(\mathbb{R}^m) \times \mathbb{R}^m \to \mathbb{C}$ that are C^∞-differentiable with respect to x by a fixed 'parameter' φ, which, with the point-wise function operations, is clearly a $\mathbb{C}$-algebra. Then each generalized function of Colombeau is an element of the quotient algebra $\mathcal{G}(\mathbb{R}^m) = \mathcal{E}_M[\mathbb{R}^m] / \mathcal{I}[\mathbb{R}^m]$. Here the subalgebra $\mathcal{E}_M[\mathbb{R}^m]$ of $\mathcal{E}[\mathbb{R}^m]$ is the set of 'moderate' functions $f(\varphi, x)$ in $\mathcal{E}[\mathbb{R}^m]$ such that for each compact subset K of $\mathbb{R}^m$ and any p in $\mathbb{N}_0^m$ there is a q in $\mathbb{N}$ so that: for each φ in $A_q(\mathbb{R}^m)$ there are $c > 0$, $\eta > 0$ satisfying $\sup_{x \in K} |\partial^p f(\varphi_\varepsilon, x)| \le c\varepsilon^{-q}$ for $0 < \varepsilon < \eta$. In turn, the ideal $\mathcal{I}[\mathbb{R}^m]$ of $\mathcal{E}_M[\mathbb{R}^m]$ is the set of functions $f(\varphi, x)$ such that for each compact subset K of $\mathbb{R}^m$ and any p in $\mathbb{N}_0^m$ there is q in $\mathbb{N}$ so that: for every $r \geq q$ and each φ in $A_r(\mathbb{R}^m)$ there are $c > 0$, $\eta > 0$ satisfying $\sup_{x\in K} |\partial^p f(\varphi_\varepsilon, x)| \leq c\varepsilon^{r-q}$, for $0 < \varepsilon < \eta$.

The Colombeau algebra $\mathcal{G}(\mathbb{R}^m)$ contains all distributions (and C^∞-differentiable functions) on $\mathbb{R}^m$, canonically embedded as a $\mathbb{C}$-vector subspace (respectively, a subalgebra) by the map $i : \mathcal{D}'(\mathbb{R}^m)\to\mathcal{G}(\mathbb{R}^m) : u \mapsto \tilde{u} = [\tilde{u}(\varphi, x)]$. The representatives here are given by $\tilde{u}(\varphi, x) = (u * \check{\varphi})(x)$, with $\check{\varphi}(x) = \varphi(-x)$ and φ in $A_q(\mathbb{R}^m)$. Equivalently, one writes $\tilde{u}(\varphi, x) = \langle u_y, \varphi(y - x)\rangle$. A basic example is the embedding $\tilde{\delta}$ in $\mathcal{G}(\mathbb{R}^m)$ of the Dirac δ-function given by a representative $\tilde{\delta}(\varphi, x) = \langle \delta_y, \varphi(y - x)\rangle = \varphi(-x)$, for any φ in $A_q(\mathbb{R}^m)$.

Definition 2. A generalized function f in $\mathcal{G}(\mathbb{R}^m)$ is said to admit some u in $\mathcal{D}'(\mathbb{R}^m)$ as *associated distribution*, which is denoted by $f \approx u$, if f has a representative $f(\varphi_\varepsilon, x)$ in $\mathcal{E}_M[\mathbb{R}^m]$ such that for any test-function $\psi(x)$ in $\mathcal{D}(\mathbb{R}^m)$ there exists q in $\mathbb{N}_0$ so that, for all $\varphi(x)$ in $A_q(\mathbb{R}^m)$,

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^m} f(\varphi_\varepsilon, x)\psi(x)dx = \langle u, \psi\rangle.$$

This definition is independent of the representative chosen and the distribution associated is unique if it exists; the image in $\mathcal{G}(\mathbb{R}^m)$ of every distribution is associated with that distribution ([1]). The concept of association is thus a faithful generalization of the equality of distributions in $\mathcal{D}'(\mathbb{R}^m)$.

Now by 'Colombeau product of distributions' is denoted the product of some distributions as they are embedded in Colombeau algebra $\mathcal{G}(\mathbb{R}^m)$ whenever the result admits an associated distribution in $\mathcal{D}'(\mathbb{R}^m)$ (see [5] for a comparison with other distribution products). This notion helps to bring the results 'down to the level' of distributions, connecting thus Colombeau theory with the classical distribution theory. Below we give some results on products of distribution with coinciding singularities in Colombeau algebra $\mathcal{G}(\mathbb{R}^m)$, or else — on their Colombeau product.

2. Preliminary results

The technical lemmas below will be needed later to prove our main results.

Lemma 1. *For an arbitrary φ in $A_0(\mathbb{R})$, i.e. φ in $\mathcal{D}(\mathbb{R})$ with $\int_{\mathbb{R}} \varphi(t)\, dt = 1$, suppose that* supp $\varphi \subseteq [a, b]$*, for some a, b in $\mathbb{R}$. Then, for any p in $\mathbb{N}_0$, it holds*

$$\int_a^b \varphi(t) \int_a^t (y - t)^p \varphi^{(p)}(y)\, dy\, dt = \frac{(-1)^p\, p!}{2}\,. \tag{1}$$

Proof: On expanding the term $(y-t)^p$ on the l.h.s. of (1) and then on multiple integrating by parts, we get:

$$\begin{aligned}
I_p &\equiv \sum_{j=0}^{p}(-1)^j\binom{p}{j}\int_a^b t^j\,\varphi(t)\int_a^t y^{p-j}\varphi^{(p)}(y)\,dy\,dt\\
&= \sum_{j=0}^{p}(-1)^j\binom{p}{j}\int_a^b t^j\,\varphi(t)\sum_{k=0}^{p-j}(-1)^k\frac{(p-j)!}{(p-j-k)!}t^{p-j-k}\varphi^{(p-k-1)}(t)\,dt\\
&= \sum_{j=0}^{p}\sum_{k=0}^{p-j}(-1)^{j+k}\frac{p!}{j!(p-j-k)!}\int_a^b t^{p-k}\varphi(t)\varphi^{(p-k-1)}(t)\,dt\\
&= \sum_{k=0}^{p}(-1)^k\frac{p!}{(p-k)!}\,J_{p-k}\sum_{j=0}^{p-k}(-1)^j\binom{p}{j}.
\end{aligned}$$

Here we have denoted by $J_{p-k}=\int_a^b t^{p-k}\varphi(t)\varphi^{(p-k-1)}(t)\,dt$, where, if $k=p$, $\varphi^{(-1)}(t)$ stands for $\int_a^t\varphi(y)\,dy$. For any $q=p-k>0$, however, it holds (see [6, §21.5–1(b)]): $\sum_{j=0}^{q}(-1)^j\binom{q}{j}=0$. Whence $I_p=(-1)^p\,p!\,J_0$. As for the remaining term with $p-k=j=0$, we get, by our assumption,

$$\begin{aligned}
J_0 &= \int_a^b\varphi(t)\left(\int_a^t\varphi(y)\,dy\right)dt=\int_a^b\left(\int_a^t\varphi(y)\,dy\right)d\left(\int_a^t\varphi(y)\,dy\right)\\
&= \frac{1}{2}\left(\int_a^t\varphi(y)\,dy\right)^2\Bigg|_a^b = \frac{1}{2}\,.
\end{aligned}$$

This proves equation (1). □

Lemma 2. *Let u and v be distributions in $\mathcal{D}'(\mathbb{R}^m)$ such that $u(x)=\prod_{i=1}^m u^i(x_i)$, $v(x)=\prod_{i=1}^m v^i(x_i)$ with each u^i and v^i in $\mathcal{D}'(\mathbb{R})$, and suppose that their embeddings in $\mathcal{G}(\mathbb{R})$ satisfy $\tilde{u}^i.\tilde{v}^i\approx w^i$, for $i=1,\dots,m$. Then $\tilde{u}.\tilde{v}\approx w$, where $w=\prod_{i=1}^m w^i(x_i)$.*

Proof: Suppose we have confined ourselves to the subspace of test-functions $\psi(x)=\prod_{i=1}^m\psi_i(x_i)$, with each ψ_i in $\mathcal{D}(\mathbb{R})$. In view of the tensor-product structure of the distributions u,v in $\mathcal{D}'(\mathbb{R}^m)$ as well as that of the elements φ of $A_0(\mathbb{R}^m)$, by applying a Fubini-type theorem for tensor-product distributions (see [4, §4.3]), we get:

$$\begin{aligned}
\langle\tilde{u}(\varphi_\varepsilon,x)\tilde{v}(\varphi_\varepsilon,x),\psi(x)\rangle &= \prod_{i=1}^m\langle\tilde{u}^i(\chi_\varepsilon,x_i)\tilde{v}^i(\chi_\varepsilon,x_i),\psi_i(x_i)\rangle\\
&= \prod_{i=1}^m\Big(\langle w^i(x_i),\psi_i(x_i)\rangle+f^i(\varepsilon)\Big)\,.
\end{aligned}$$

Here, by assumption, one has the asymptotic evaluation $f^i(\varepsilon) = o(1)$ $(\varepsilon \to 0)$ for each $i = 1, \dots, m$. Thus

$$\lim_{\varepsilon \to 0} \langle \tilde{u}(\varphi_\varepsilon, x)\tilde{v}(\varphi_\varepsilon, x), \psi(x)\rangle = \prod_{i=1}^{m} \langle w^i, \psi^i\rangle = \langle w, \psi\rangle,$$

where $w = \prod_{i=1}^m w^i(x_i)$ is a uniquely determined distribution in $\mathcal{D}'(\mathbb{R}^m)$. Moreover, since $\psi(x) = \prod_{i=1}^m \psi_i(x_i)$ is running a dense subset of $\mathcal{D}(\mathbb{R}^m)$ ([4, §4.3]), it follows, by Definition 2, that the product $\tilde{u}.\tilde{v}$ in $\mathcal{G}(\mathbb{R}^m)$ admits w as associated distribution. $\square$

3. Main results

Proposition 1. *For an arbitrary p in $\mathbb{N}_0^m$, let $\tilde{\delta}^{(p)}(x)$ and $\tilde{x}_+^p$ be the embeddings in $\mathcal{G}(\mathbb{R}^m)$ of the distributions $\delta^{(p)}(x)$ and $x_+^p = \{x^p$ for $x \geq 0, = 0$ elsewhere$\}$ in $\mathcal{D}'(\mathbb{R}^m)$. Then*

$$\tilde{x}_+^p \,.\, \tilde{\delta}^{(p)}(x) \;\approx\; \frac{(-1)^{|p|}\, p!}{2^m}\, \delta(x). \tag{2}$$

Proof: In the one-variable case $(x \in \mathbb{R}, p \in \mathbb{N}_0)$, $\tilde{x}_+^p$ is represented by

$$\tilde{x}_+^p(\varphi_\varepsilon, x) = \varepsilon^{-1} \int_0^\infty y^p\, \varphi((y-x)/\varepsilon)\, dy = \int_{-x/\varepsilon}^\infty (x + \varepsilon t)^p\, \varphi(t)\, dt,$$

where the substitution $(y-x)/\varepsilon = t$ is made. Also, on differentiation in $\mathcal{D}'(\mathbb{R})$, we have

$$\tilde{\delta}^{(p)}(\varphi_\varepsilon, x) = (-1)^p \varepsilon^{-p-1} \langle \delta_y, \varphi^{(p)}((y-x)/\varepsilon)\rangle = (-1)^p \varepsilon^{-p-1} \varphi^{(p)}(-x/\varepsilon).$$

Now if $\operatorname{supp} \varphi(x) \subseteq [a, b]$ for some a, b in $\mathbb{R}$, then $\operatorname{supp} \varphi(-x/\varepsilon) \subseteq [-\varepsilon b, -\varepsilon a]$. Thus, replacing $x \to y = -x/\varepsilon$, we get for any $\psi(x)$ in $\mathcal{D}(\mathbb{R})$

$$\begin{aligned} &\langle \tilde{x}_+^p(\varphi_\varepsilon, x)\, \tilde{\delta}^{(p)}(\varphi_\varepsilon, x), \psi(x)\rangle \\ &= \frac{(-1)^p}{\varepsilon^{p+1}} \int_{-b\varepsilon}^{-a\varepsilon} \left(\int_{-x/\varepsilon}^b (x + \varepsilon t)^p \varphi(t)\, dt \right) \varphi^{(p)}(-x/\varepsilon)\psi(x)\, dx \\ &= \int_a^b \psi(-\varepsilon y)\, \varphi^{(p)}(y) \int_y^b (y-t)^p \varphi(t)\, dt\, dy. \end{aligned}$$

By the Taylor theorem, we have $\psi(-\varepsilon y) = \psi(0) + (-\varepsilon y)\psi'(\eta y)$ for some $\eta \in [0, 1]$. Now the integrand function in the latter equation, that reads

$$y\psi'(\eta y)\varphi^{(p)}(y) \int_y^b (y-t)^p \varphi(t)\, dt = y\psi'(\eta y)\varphi^{(p)}(y) \frac{(-1)^p}{p+1} \left(t_-^{p+1} * \varphi(t) \right)(y),$$

is clearly a product of differentiable functions, and is thus integrable on the finite interval $[a,b]$. Therefore, by taking the limit as $\varepsilon\to 0$ and applying the Dirichlet formula for changing the order of integration (which is permissible here), we get

$$\begin{aligned}\lim_{\varepsilon\to 0}\langle \tilde{x}_+^p(\varphi_\varepsilon,x)\,\tilde{\delta}^{(p)}(\varphi_\varepsilon,x),\psi(x)\rangle &= \int_a^b \psi(0)\varphi^{(p)}(y)\int_y^b (y-t)^p\varphi(t)\,dt\,dy\\ &= \psi(0)\int_a^b \varphi(t)\int_a^t (y-t)^p\varphi^{(p)}(y)\,dy\,dt.\end{aligned}\tag{3}$$

Employing now Lemma 1, we obtain equation (2) for $m=1$.

Further, in the multi-variable case, in view of the tensor-product structure of the distributions x_+^p and $\delta^{(p)}(x)$ in $\mathcal{D}'(\mathbb{R}^m)$, we can apply Lemma 2 that yields

$$\tilde{x}_+^p.\tilde{\delta}^{(p)}(x)=\prod_{i=1}^m \tilde{x}_{i+}^{p_i}.\tilde{\delta}^{(p_i)}(x_i)\approx\prod_{i=1}^m\left(\frac{(-1)^{p_i}p_i!}{2}\,\delta(x_i)\right)=\frac{(-1)^{|p|}\,p!}{2^m}\,\delta(x),$$

which completes the proof. □

Corollary 1. *If $\tilde{x}_-^p$ is the embedding of the distribution x_-^p, then it holds for any p in $\mathbb{N}_0^m$*

$$\tilde{x}_-^p\,.\,\tilde{\delta}^{(p)}(x)\;\approx\;\frac{p!}{2^m}\,\delta(x).\tag{4}$$

Proof: For any p in $\mathbb{N}_0^m$, we have $x_-^p=(-x)_+^p$. The result in (4) therefore follows by replacing $x\to -x$ in equation (2) and taking into account that $\delta^{(p)}(-x)=(-1)^{|p|}\delta^{(p)}(x)$. □

Remark 1. Equations (2) and (4) are consistent in dimension one with the known formula in $\mathcal{D}'(\mathbb{R})$

$$x^p\,\delta^{(p)}(x)=\;(-1)^p\,p!\,\delta(x)\quad(p\in\mathbb{N}_0).\tag{5}$$

Indeed, taking in view that $x^p=x_+^p+(-1)^p x_-^p$, the equations in consideration combine to give (5).

Notation 2. Extending further the multiindex notation, consider now the ordered m-tuples $a=(a_1,\dots,a_m)$ in $\mathbb{R}^m$ with the vector operations there. We specify that $a+k$ stands for $(a_1+k,\dots,a_m+k)$ for any k in $\mathbb{Z}$ (integers) and that 0 denotes the zero-vector in $\mathbb{R}^m$. Then we shall use the short-hand notations $x^a=(x_1^{a_1},\dots,x_m^{a_m})$ and $\prod_{i=1}^m\Gamma(a_i)=\Gamma(a)$ ($=p!$ whenever $a-1=p\in\mathbb{N}_0^m$). Finally, we denote $\Omega=\{a\in\mathbb{R}:a\neq -1,-2,\dots\}$ and by Ω^m the m-fold tensor product $\Omega\times\ldots\times\Omega$. Now one has the following:

Proposition 2. *The product of the generalized functions $\tilde{x}_+^a$ and $\tilde{x}_-^b$ in $\mathcal{G}(\mathbb{R}^m)$ admits an associated distribution for any a, b in Ω^m such that $a+b+1=0$, and it holds*

$$\tilde{x}_+^a . \tilde{x}_-^b \approx \frac{\Gamma(a+1)\Gamma(b+1)}{2^m}\,\delta(x). \tag{6}$$

Proof: In the one-variable case ($x, a \in \mathbb{R}$), recall first the definition of the distribution x_+^a. If $a > -1$, then $x \mapsto x_+^a$ is locally-integrable, thus defining the distribution $\langle x_+^a, \psi\rangle = \int_0^\infty x^a \psi(x)\,dx$ ($\psi \in \mathcal{D}(\mathbb{R})$), and it also holds $x_+^a = (a+1)^{-1}\partial_x x_+^{(a+1)}$. Now we can define a distribution x_+^a for any a in Ω, choosing a k in $\mathbb{N}_0$ subject to the condition $a+k+1>0$, if we set

$$x_+^a = \frac{1}{(a+k)(a+k-1)\dots(a+1)}\partial_x^k x_+^{a+k} = \frac{\Gamma(a+1)}{\Gamma(a+k+1)}\partial_x^k x_+^{a+k}.$$

Suppose further that k in $\mathbb{N}_0$ is such that $k > \max\{-a-1, -b-1\}$. Then, to get the embedding in $\mathcal{G}(\mathbb{R}^m)$ of the distribution x_+^a we use the notion of derivative in Colombeau algebra, which gives:

$$\begin{aligned}
\tilde{x}_+^a(\varphi_\varepsilon, x) &= \varepsilon^{-1}\frac{\Gamma(a+1)}{\Gamma(a+k+1)}\ \partial_x^k \int_0^\infty (y^{a+k})\varphi((y-x)/\varepsilon)\,dy \\
&= (-1)^k \varepsilon^{-1-k}\frac{\Gamma(a+1)}{\Gamma(a+k+1)}\int_0^\infty y^{a+k}\varphi^{(k)}((y-x)/\varepsilon)\,dy \\
&= (-1)^k \varepsilon^{-k}\frac{\Gamma(a+1)}{\Gamma(a+k+1)}\int_{-x/\varepsilon}^d (x+\varepsilon u)^{a+k}\varphi^{(k)}(u)\,du,
\end{aligned}$$

where, it is assumed that supp $\varphi(x) \subseteq [c,d]$ for some c, d in $\mathbb{R}$ and the substitution $u = (y-x)/\varepsilon$ is made. Similarly, with the same choice of k, we have

$$\begin{aligned}
\tilde{x}_-^b(\varphi_\varepsilon, x) &= (-1)^k \varepsilon^{-1-k}\frac{\Gamma(b+1)}{\Gamma(b+k+1)}\int_{-\infty}^0 (-y)^{b+k}\varphi^{(k)}((y-x)/\varepsilon)\,dy \\
&= (-1)^k \varepsilon^{-k}\frac{\Gamma(b+1)}{\Gamma(b+k+1)}\int_c^{-x/\varepsilon} (-x-\varepsilon v)^{b+k}\varphi^{(k)}(v)\,dv.
\end{aligned}$$

Then, for any ψ in $\mathcal{D}(\mathbb{R})$,

$$\begin{aligned}
\langle \tilde{x}_+^a(\varphi_\varepsilon, x)\,\tilde{x}_-^b(\varphi_\varepsilon, x), \psi(x)\rangle &= \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+k+1)\Gamma(b+k+1)}\times \\
\times\varepsilon^{-2k}\int_{-d\varepsilon}^{-c\varepsilon}\psi(x)\int_{-x/\varepsilon}^d \varphi^{(k)}(u)&\int_c^{-x/\varepsilon}(x+\varepsilon u)^{a+k}(-x-\varepsilon v)^{b+k}\varphi^{(k)}(v)\,dv\,du\,dx \\
&\equiv \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+k+1)\Gamma(b+k+1)}\ I_{ab}(\varepsilon).
\end{aligned} \tag{7}$$

Here it is taken into account that $c \le -x/\varepsilon \le d$, and thus $-d\varepsilon \le x \le -c\varepsilon$.

Further, on making the substitution $w = -x/\varepsilon$ and taking in view the requirement $a+b+1=0$, we obtain

$$I_{ab}(\varepsilon) = \int_c^d \psi(-\varepsilon w) \int_w^d \varphi^{(k)}(u) \int_c^w (u-w)^{a+k}(w-v)^{b+k}\varphi^{(k)}(v)\,dv\,du\,dw.$$

Now, by the same argument as that used in Proposition 1 and by changing twice the order of integration, we get for $I_{ab} := \lim_{\varepsilon\to 0} I_{ab}(\varepsilon)$

$$\begin{aligned} I_{ab} &= \int_c^d \psi(0)\varphi^{(k)}(u) \int_c^u \int_c^w (u-w)^{a+k}(w-v)^{b+k}\varphi^{(k)}(v)\,dv\,dw\,du \\ &= \psi(0)\int_c^d \varphi^{(k)}(u) \int_c^u \varphi^{(k)}(v) \int_v^u (u-w)^{a+k}(w-v)^{b+k}\,dw\,dv\,du. \end{aligned}$$

Then the substitution $w \to t = (w-v)/(u-v)$, together with the relations $w-v = (u-v)t$, $u-w = (u-v)(1-t)$, and the definition of the first-order Euler integral ([6, §21.4–4]) yield

(8)

$$\begin{aligned} I_{ab} &= \psi(0)\int_c^d \varphi^{(k)}(u) \int_c^u (u-v)^{a+b+2k+1}\varphi^{(k)}(v)\left(\int_0^1 (1-t)^{a+k}t^{b+k}dt\right)dv\,du \\ &= \psi(0)\frac{\Gamma(a+k+1)\Gamma(b+k+1)}{(2k)!}\int_c^d \varphi^{(k)}(u)\int_c^u (u-v)^{2k}\varphi^{(k)}(v)\,dv\,du, \end{aligned}$$

where the requirement $a+b+1=0$ is again taken into account.

Hence, by equations (7) and (8), we have

$$\begin{aligned} \frac{\lim_{\varepsilon\to 0}\langle \tilde{x}_+^a(\varphi_\varepsilon, x)\,\tilde{x}_-^b(\varphi_\varepsilon, x), \psi(x)\rangle}{\Gamma(a+1)\Gamma(b+1)} &= \frac{\psi(0)}{(2k)!}\int_c^d \varphi^{(k)}(u)\int_c^u (u-v)^{2k}\varphi^{(k)}(v)\,dv\,du \\ &= \frac{(-1)^k\psi(0)}{(2k)!}\int_c^d \varphi(u)\,\partial_u^k\left(\int_c^u (u-v)^{2k}\varphi^{(k)}(v)\,dv\right)du \\ &= \frac{(-1)^k\psi(0)}{k!}\int_c^d \varphi(u)\int_c^u (u-v)^k\varphi^{(k)}(v)\,dv\,du = \frac{\psi(0)}{2}, \end{aligned}$$

where finally the result of Lemma 1, for the particular choice of k in $\mathbb{N}_0$, is applied. Thus, in the one-variable case, we get, by Definition 2,

$$\tilde{x}_+^a.\tilde{x}_-^b \approx \frac{\Gamma(a+1)\Gamma(b+1)}{2}\,\delta(x).$$

To prove our result in $\mathcal{G}(\mathbb{R}^m)$, it only remains to apply Lemma 2: for any $a = (a_i, \dots a_m)$, $b = (b_i, \dots, b_m)$ in Ω^m such that $a+b+1=0$, we have

$$\tilde{x}_+^a.\tilde{x}_-^b = \prod_{i=1}^m \tilde{x}_{i+}^{a_i}.\tilde{x}_{i-}^{b_i} \approx \prod_{i=1}^m \left(\frac{\Gamma(a_i+1)\Gamma(b_i+1)}{2}\,\delta(x_i)\right) = \frac{\Gamma(a+1)\Gamma(b+1)}{2^m}\,\delta(x).$$

This finishes the proof. □

Remarks 2. The result of the above proposition, symmetric in the parameters a, b, can be rewritten taking into account the connection between them: replacing $b = -a - 1$ or, respectively, $a = -b - 1$ in (6), we have by [6, §21.4–1(c)] that, for any $a \in \mathbb{R}\backslash\mathbb{Z}$,

$$(9) \quad \tilde{x}_+^a . \tilde{x}_-^{-a-1} = \tilde{x}_+^{-a-1} . \tilde{x}_-^a \approx \frac{\Gamma(1+a)\Gamma(-a)}{2^m}\, \delta(x) = (-\pi/2)^m \operatorname{cosec}(\pi a)\delta(x).$$

3. The proofs of equations (2), (4) and (6) can be modified — in dimension one only — so as to obtain the same formulas for the regularized model product of the corresponding distributions (denoted by $[,]$; see [7, Chapter 2]). This is due to the fact that, replacing $\varphi(x)$ by $\rho\,(-x)$, where φ is in $A_0(\mathbb{R})$ (which is the only requirement on φ we have used), we get for any ψ in $\mathcal{D}(\mathbb{R})$: $\lim_{\varepsilon\to 0} \langle \tilde{u}(\varphi_\varepsilon, x)\, \tilde{v}(\varphi_\varepsilon, x),\, \psi(x)\rangle = \lim_{\varepsilon\to 0} \langle (u * \rho_\varepsilon)(v * \rho_\varepsilon), \psi\rangle = \langle [u, v],\, \psi\rangle$, where ρ satisfies exactly the requirements imposed on the mollifiers for general model products. Finally, we note that equations (2), (4) and (9) were derived in [2] and [3] for dimension one and for the particular choice of the mollifiers $\rho(x)$ being even functions of x.

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References

[1] Colombeau J.-F., *New Generalized Functions and Multiplication of Distributions*, North Holland Math. Studies 84, Amsterdam, 1984.

[2] Fisher B., *The product of distributions*, Quart. J. Oxford **22** (1971), 291–298.

[3] Fisher B., *The divergent distribution product* $x_+^\lambda x_-^\mu$, Sem. Mat. Barcelona **27** (1976), 3–10.

[4] Friedlander F.G., *Introduction to the Theory of Distributions*, Cambridge Univ. Press, Cambridge, 1982.

[5] Jelínek J., *Characterization of the Colombeau product of distributions*, Comment. Math. Univ. Carolinae **27** (1986), 377–394.

[6] Korn G.A., Korn T.M., *Mathematical Handbook*, McGraw-Hill Book Company, New York, 1968.

[7] Oberguggenberger M., *Multiplication of Distributions and Applications to Partial Differential Equations*, Longman, Essex, 1992.

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