# Invariant subspaces for some operators on locally convex spaces

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*Abstract.* The invariant subspace problem for some operators and some operator algebras acting on a locally convex space is studied.

*Keywords:* invariant subspace, locally convex space, locally bounded operator, universally bounded operator, compact operator

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## 1. Introduction

Let $X$ be a locally convex Hausdorff space over the complex field $\mathbb{C}$. Each system of seminorms $P$ inducing its topology will be called a *calibration* ([11]). We denote by $\mathcal{P}(X)$ the collection of all calibrations on $X$. Given $P \in \mathcal{P}(X)$, we call it *basic calibration* if the corresponding "semiballs" $U(\varepsilon, p) = \{x \in X : p(x) < \varepsilon\}$, $\varepsilon > 0$, $p \in P$, form a neighborhood base at 0. As it is easily seen, $P$ is basic if and only if for each $p_1, p_2 \in P$ there is some $p_0 \in P$ such that $p_i(x) \leq p_0(x)$, $i = 1, 2$. For any $P \in \mathcal{P}(X)$ we can generate a basic calibration $P' \in \mathcal{P}(X)$ by taking maxima of finite seminorms from $P$. For a given $P \in \mathcal{P}(X)$ we denote by $Q_P(X)$ the algebra of *quotient bounded* operators on $X$, i.e. the collection of all linear operators $T$ on $X$ for which

$$p(Tx) \leq c_p p(x), \qquad x \in X, \quad p \in P,$$

and by $B_P(X)$ the algebra of *universally bounded* operators on $X$, i.e. the set of all $T \in Q_P(X)$ for which $c = c_p$ is independent of $p \in P$ ([11]). The algebra $Q_P(X)$ is a unital locally m-convex algebra with respect to seminorms $\widehat{P} = \{\widehat{p}\}$ (see eg. [6]) where

$$\widehat{p}(T) = \sup\{p(Tx) : \ x \in X, \ p(x) \leq 1\}, \qquad p \in P,$$

and $B_P(X)$ is a unital normed algebra with respect to the norm

$$\|T\|_P = \sup\{\widehat{p}(T) : \ p \in P\}.$$

Let us define still some other families of linear operators. A linear operator $T$ on $X$ is *locally bounded*, or $T \in \mathcal{L}B(X)$, if there exists a neighborhood $U$ such that

$T(U)$ is bounded, and $T$ is *compact*, or $T \in \mathcal{K}(X)$, if there exists a neighborhood $U$ such that $T(U)$ is a relatively compact set. Let us denote

$$\mathcal{B}^0(X) = \cup\{B_P(X),\ P \in \mathcal{P}(X)\},$$

and by $\mathcal{L}(X)$ the set of all linear continuous operators on $X$ (similarly $\mathcal{L}(X,Y)$ for two spaces $X$ and $Y$). The following inclusions hold: $\mathcal{K}(X) \subset \mathcal{L}B(X) \subset \mathcal{B}^0(X) \subset \mathcal{L}(X)$ (the second inclusion which is not so obvious will be verified later, or see [11]).

Given any linear operator $T$ on $X$, we define the spectrum and the resolvent set of $T$ with respect to various algebras. For $T \in \mathcal{L}(X)$: $\lambda \in \rho(T)$ iff $(\lambda I - T)^{-1}$ exists in $\mathcal{L}(X)$, for $T \in Q_P(X)$: $\lambda \in \rho(Q_P, T)$ iff $(\lambda I - T)^{-1}$ exists in $Q_P(X)$ and similarly $\rho(B_P, T)$ for $T \in B_P(X)$. The corresponding complements in $\mathbb{C}$ will be denoted by $\sigma(T)$, $\sigma(Q_P, T)$ and $\sigma(B_P, T)$. Obviously, $\sigma(T) \subset \sigma(Q_P, T) \subset \sigma(B_P, T)$ for $T \in B_P(X)$. It is known that $\sigma(B_P, T)$ is bounded and closed for $T \in B_P(X)$ ([2]), but in general the above spectra can be unbounded. In the case when $\sigma(T)$ is bounded we denote

$$r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}.$$

By $\mathcal{R}(T)$ we shall denote the range of an operator $T$. Let $S$ be a map on $X$ which may be nonlinear. If there exist $P \in \mathcal{P}(X)$ and $c > 0$ such that

$$p(Sx) \le cp(x), \qquad x \in X, \quad p \in P,$$

$S$ will be called, as in [5], a *$P$-bounded map*.

## 2. Main results

Let us first prove two useful lemmas.

**Lemma 1.** *Let $p, q$ be two seminorms on $X$ such that: $q(x) \le 1$ for each $x \in X$ for which $p(x) < 1$. Then*

$$q(x) \le p(x), \quad x \in X.$$

Proof: Let $0 \le p(z) < q(z)$ for some $z \in X$. Then there is some $\lambda > 0$ such that $p(z) < \lambda < q(z)$, hence $p(z/\lambda) < 1$ and $q(z/\lambda) > 1$ which is a contradiction. □

**Lemma 2.** *Let $X$ be a Hausdorff locally convex space and $T_1, T_2 \in \mathcal{L}B(X)$, then there exists a common calibration $P' \in \mathcal{P}(X)$ such that $T_1, T_2 \in B_{P'}(X)$.*

Proof: We may take a basic calibration $P \in \mathcal{P}(X)$. Then there exist neighborhoods $U_1, U_2$ such that $T_i(U_i)$, $i = 1, 2$ are bounded. Without loss of generality we may assume that $U_i$ is the open semiball corresponding to the seminorm $p_i \in P$, $i = 1, 2$. For every $p \in P$ there are $\lambda_1^{(p)}, \lambda_2^{(p)} \ge 0$ such that

$\sup\{p(T_i x) : \ x \in U_i\} \le \lambda_i^{(p)}$, $i = 1, 2$. We assume firstly that $\lambda_i^{(p)} > 0$, $i = 1, 2$. For $x \in X$ for which $p_i(x) < 1$ it follows $p(T_i x/\lambda_i^{(p)}) \le 1$, $i = 1, 2$, and by Lemma 1 we obtain

$$p(T_i x) \le \lambda_i^{(p)} p_i(x), \quad x \in X, \quad i = 1, 2.$$

Since $P$ is a basic calibration there is some $p_0 \in P$ such that $p_i(x) \le p_0(x)$, $i = 1, 2$. Hence for $\lambda_p = \max\{\lambda_1^{(p)}, \lambda_2^{(p)}\}$ we have

$$p(T_i x) \le \lambda_p p_0(x), \quad p \in P, \quad x \in X, \quad i = 1, 2.$$

If one of $\lambda_i^{(p)}$ is zero, then $p(T_i x) = 0$ for each $x \in X$ and the above inequality trivially holds. Especially, we have $p_0(T_i x) \le \lambda_0 p_0(x)$, $x \in X$, $i = 1, 2$. Let us define $P' = \{p', p \in P\}$, where

$$p'(x) = \max\{p(x), \lambda_p p_0(x)\}, \quad x \in X.$$

We readily verify that $P'$ is again a calibration. Now, we can estimate for any $p' \in P'$ and $i = 1, 2$

$$p'(T_i x) = \max\{p(T_i x),\ \lambda_p p_0(T_i x)\} \le \lambda_p c_0 p_0(x) \le c_0 p'(x), \quad i = 1, 2,$$

where $c_0 = \max\{1, \lambda_0\}$. Hence $T_i \in B_{P'}(X)$, $i = 1, 2$. □

Taking $T_1 = T_2$ we obtain

**Corollary.** *Each $T \in \mathcal{L}B(X)$ is in $\mathcal{B}^0(X)$.*

If we take $T \in \mathcal{L}B(X)$, then $T \in B_P(X)$ for some $P \in \mathcal{P}(X)$ and hence $\sigma(B_P, T)$ is bounded and then $\sigma(T)$ is bounded, too. We shall first prove some generalizations of some results from [5].

**Lemma 3.** *Let $X$, $Y$ be Hausdorff locally convex spaces, $T \in \mathcal{L}(X, Y)$ and $K \in \mathcal{L}B(Y)$. Let $S$ be a map on $X$ such that for some $P' \in \mathcal{P}(X)$ and some $\varepsilon > 0$*

$$p'(Sx) \le (r(K) + \varepsilon)^{-1} p'(x), \quad p' \in P', \quad x \in X. \tag{1}$$

*If $T = KTS$, then $T = 0$.*

Proof: Let us choose any $P \in \mathcal{P}(Y)$. Then there exists a neighborhood of zero $U_0$ on $Y$ such that $K(U_0)$ is bounded. We may assume that $U_0$ is an open semiball corresponding to $p_0 \in P$. Let us denote $B = \overline{cob}K(U_0)$ the absolute convex closed hull of $K(U_0)$ and $Y_B = span(B)$ the linear span of $B$. This is a normed space with respect to the norm $\|.\|_B$, the Minkowski's functional of $B$. It is not hard to see that the topology induced by this norm is finer than the relative topology induced by $P$. Clearly, $K(Y) \subset Y_B$ since $U_0$ is absorbent and $K(U_0) \subset B$ and it follows $\|Kx\|_B \leq 1$ for each $x \in Y$ such that $p_0(x) < 1$. By Lemma 1 we obtain

$$\|Kx\|_B \leq p_0(x), \quad x \in Y, \tag{2}$$

hence the map $K : Y \to Y_B$ is continuous. Let us prove that $K_B := K|_{Y_B}$ is continuous on $Y_B$. Since $B$ is bounded there is some $\lambda > 0$ such that $B \subset \lambda U_0$, hence $K(B) \subset \lambda K(U_0) \subset \lambda B$. Consequently, for all $x \in Y_B$ such that $\|x\|_B < 1$ it follows that $\lambda^{-1}\|Kx\|_B \leq 1$ and by Lemma 1 we have

$$\|Kx\|_B \leq \lambda \|x\|_B, \quad x \in Y_B.$$

Denote by $J : Y_B \to Y$ the inclusion map, then clearly $K_B = KJ$. Since the norm topology on $Y_B$ is finer than the relative one, we obtain ([3]) $\sigma(K) - \{0\} = \sigma(K_B) - \{0\}$. Thus, $r(K) = r(K_B)$. Without loss of generality we may assume that $P'$ is a basic calibration and (1) again holds. By the supposed equality it follows that $Tx \in Y_B$ for each $x \in X$ and $T = K^n T S^n$ for all $n \in \mathbb{N}$. Fix any $x \in X$ and $n \in \mathbb{N}$, then by the continuity of $K_B$ and $T$ and by the inequalities (1) and (2) we can estimate

$$\begin{aligned}\|Tx\|_B &= \|K^{n+1}TS^{n+1}x\|_B = \|K_B^n KTS^{n+1}x\|_B \leq \|K_B^n\|_B . \|KTS^{n+1}x\|_B \\ &\leq \|K_B^n\|_B . p_0(TS^{n+1}x) \leq \|K_B^n\|_B . C . p_1'(S^{n+1}x) \\ &\leq C . \|K_B^n\|_B . (r(K) + \varepsilon)^{-(n+1)} p_1'(x),\end{aligned}$$

where $p_1' \in P'$. For the above $\varepsilon > 0$ take any $\delta \in (0, \varepsilon)$ and $n \in \mathbb{N}$ sufficiently large to yield $\|K_B^n\|_B < (r(K_B) + \delta)^n$. Then

$$\|Tx\|_B \leq C . (r(K) + \delta)^n . (r(K) + \varepsilon)^{-(n+1)} . p_1'(x).$$

Sending $n \to \infty$ we obtain $Tx = 0$ and since $x \in X$ is arbitrary we have $T = 0$. □

As in [5] we call $K \in \mathcal{LB}(X)$ *decomposable at* 0 if for each $\varepsilon > 0$ we have a decomposition $X = M \oplus N$, where $M$ and $N$ are nontrivial invariant subspaces of $K$ and $r(K|_M) < \varepsilon$.

Let us prove the following result for locally convex spaces.

**Theorem 4.** *Let $X$ be a Hausdorff locally convex space and $Y$ a complete Hausdorff locally convex space, $T \in \mathcal{L}(X, Y)$, $K \in \mathcal{LB}(Y)$ and $S$ a $P$-bounded map on $X$ for some $P \in \mathcal{P}(X)$ and such that $T = KTS$. Then*

(i) *if $r(K) = 0$, then $T = 0$;*
(ii) *if $K \in \mathcal{K}(Y)$, then $T$ has finite rank;*
(iii) *if $K$ is decomposable at 0, then $\mathcal{R}(T)$ is not dense in $Y$.*

Proof: (i) Since $S$ is $P$-bounded we have $p(Sx) \leq cp(x)$, $x \in X$, $p \in P$, for some $c > 0$. Let us choose $\varepsilon > 0$ such that $\varepsilon < 1/c$. Then $p(Sx) \leq \varepsilon^{-1}p(x)$, $p \in P$, $x \in X$, and by Lemma 3, $T = 0$.

(ii) Now, let $\varepsilon > 0$ be such that $\varepsilon < (2c)^{-1}$. Since $K$ is compact, its spectrum $\sigma(K)$ is a compact set, it has no limit point other than 0 and each $\lambda \in \sigma(K)$, $\lambda \neq 0$, is an eigenvalue ([3]). For a locally bounded operator one can generalize the Riesz functional calculus to locally convex spaces (see [10]). Denote $\sigma_\varepsilon = \{\lambda \in \sigma(K) : |\lambda| < \varepsilon\}$ and by $P_\varepsilon$ the corresponding projector for which $P_\varepsilon K = KP_\varepsilon$ and $\sigma(K|_{R(P_\varepsilon)}) = \sigma_\varepsilon$. By the same calibration $P$ as in (i) we have: $p(Sx) \leq (2\varepsilon)^{-1}p(x) \leq (r(P_\varepsilon K) + \varepsilon)^{-1}p(x)$, $p \in P$, $x \in X$, since $P_\varepsilon T = P_\varepsilon^2 KTS = P_\varepsilon KP_\varepsilon TS$, by Lemma 3, $P_\varepsilon T = 0$, hence $T = (I - P_\varepsilon)T$. Thus, $\mathcal{R}(T)$ is contained in the finite-dimensional subspace $\mathcal{R}(I - P_\varepsilon)$.

(iii) Again choose $\varepsilon > 0$ as in (ii) and use the decomposition $X = M \oplus N$ where $r(K|_M) < \varepsilon$. Denote by $P_M : Y \to M$ the corresponding projector. As in (ii) we obtain $P_M T = 0$, and since $\mathcal{R}(T) \subset \mathcal{R}(I - P_M)$, the range of $T$ is not dense. □

As it is shown in [5], for two given operators $A$, $B$ with $\mathcal{R}(A) \subset \mathcal{R}(B)$ acting between Banach spaces there exists a bounded map $S$ (which need not be linear) such that $A = BS$. This result can be generalized to the case in which the final space is locally convex.

**Lemma 5.** *Let $X$, $Z$ be Banach spaces and $Y$ a Hausdorff locally convex space. Let $A \in \mathcal{L}(X,Y)$, $B \in \mathcal{L}(Z,Y)$ such that $\mathcal{R}(A) \subset \mathcal{R}(B)$. Then there exists a map $S$ (not linear in general) from $X$ into $Z$ such that $A = BS$ and such that for some $C > 0$*

$$\|Sx\| \leq C\|x\|, \quad x \in X.$$

The proof is the same as in [5] and we omit it.

**Theorem 6.** *Let $Y$ be a complete Hausdorff locally convex space, $K \in \mathcal{LB}(Y)$ and $M := \mathcal{R}(T) \subset Y$ for some continuous operator $T$ from a Banach space $X$ into $Y$ and let $M \subset K(M)$. Then the following statements hold:*

(i) *if $r(K) = 0$, then $M = \{0\}$;*
(ii) *if $K \in \mathcal{K}(Y)$, then $M$ is finite-dimensional;*
(iii) *if $K$ is decomposable at 0, then $M$ is not dense in $Y$.*

Proof: Since $\mathcal{R}(T) \subset \mathcal{R}(KT)$, by Lemma 5 there is some $\|.\|$-bounded map $S : X \to X$ such that $T = KTS$ and by Theorem 4 all statements follow immediately. □

We shall now consider some invariant subspace problems on locally convex spaces. Let us denote by $\mathcal{L}_b(X)$ the space $\mathcal{L}(X)$ endowed with the topology $\tau_b$ of uniform convergence on bounded sets.

**Theorem 7.** *Let $X$ be a complete Hausdorff locally convex space and $\mathcal{A}$ an operator algebra in $\mathcal{L}(X)$, such that $\mathcal{A} = \mathcal{R}(S)$ for some continuous operator $S$*

*from a Banach space $Y$ into $\mathcal{L}_b(X)$. Let there exist an operator $K_1 \in \mathcal{K}(X)$ and an operator $K_2 \in \mathcal{L}B(X)$ which is decomposable at* 0*, such that*

$$\mathcal{A}K_1 \subset K_2\mathcal{A}.$$

*Then $\mathcal{A}$ has a nontrivial invariant subspace.*

Proof: If $\mathcal{A}$ had no invariant subspace then by a generalized Lomonosov's theorem (see [7]) there exists an $A_0 \in \mathcal{A}$ such that $A_0K_1z = z$, $z \neq 0$, $z \in X$. Define $Ty := (Sy)(z)$, $y \in Y$, and let us prove that $T \in \mathcal{L}(Y, X)$. Let us choose any $P \in \mathcal{P}(X)$, any $p \in P$ and any bounded set $M$ which contains $z \in X$. Then by the continuity of $S$ there is some $C_p^M > 0$ such that $q_p^M(Sy) := \sup\{p((Sy)x) : x \in M\} \leq C_p^M \|y\|$ and hence for any $y \in Y$

$$p(Ty) = p((Sy)z) \leq C_p^M \|y\|.$$

Obviously, $\mathcal{R}(T) = \mathcal{A}z = \{Az,\ A \in \mathcal{A}\}$. If $\mathcal{A}z = \{0\}$, then $V = span\{z\}$ is an invariant subspace for $\mathcal{A}$. If $\mathcal{A}z \neq \{0\}$ then $\mathcal{A}z$ is a range of a nonzero continuous operator $T$ and clearly, $\mathcal{A}z$ is invariant for $\mathcal{A}$. For any $A \in \mathcal{A}$ we have $Az = AA_0K_1z = K_2A_2z$ for some $A_2 \in \mathcal{A}$ and hence $\mathcal{A}z \subset K_2(\mathcal{A}z)$. By part (iii) of Theorem 6, $\mathcal{A}z$ is not dense in $X$, hence $\overline{\mathcal{A}z}$ is a proper invariant subspace for $\mathcal{A}$. □

**Corollary 8.** *Let $X$ be a complete Hausdorff locally convex space and $\mathcal{A} \neq \mathbb{C}.I$ a Banach algebra in $\mathcal{L}(X)$ with a norm topology finer then the topology $\tau_b$ inherited from $\mathcal{L}(X)$ and let there be some $K_1 \in \mathcal{K}(X)$ and $K_2 \in \mathcal{L}B(X)$, decomposable at* 0*, such that*

$$\mathcal{A}K_1 \subset K_2\mathcal{A}.$$

*Then $\mathcal{A}$ has a nontrivial invariant subspace.*

The algebra of universally bounded operators is a normed algebra with respect to the norm $\|.\|_P$ for each $P \in \mathcal{P}(X)$ and it is complete whenever $X$ is complete (see [11]). Thus, we have

**Corollary 9.** *Let $X$ be a complete Hausdorff locally convex space and $P \in \mathcal{P}(X)$ such that $B_P(X) \neq \mathbb{C}.I$ and let exist $K_1 \in \mathcal{K}(X)$ and $K_2 \in \mathcal{L}B(X)$, decomposable at* 0*, such that*

$$B_P(X)K_1 \subset K_2B_P(X).$$

*Then $B_P(X)$ has a nontrivial invariant subspace.*

**Theorem 10.** *Let $X$ be a complete Hausdorff locally convex space and $\mathcal{A} \neq \mathbb{C}.I$ an operator algebra in $\mathcal{L}(X)$. Let there be some continuous operator $T$ from a Banach space $Y$ into $\mathcal{L}_b(X)$ such that $\mathcal{A} = \mathcal{R}(T)$ and let there be some $K_1, K_2 \in \mathcal{K}(X)$ such that*

$$\mathcal{A}K_1 \subset K_2\mathcal{A}.$$

*Then the commutant of $\mathcal{A}$ has a nontrivial invariant subspace.*

PROOF: If the commutant $\mathcal{A}'$ had no invariant subspace then by Lomonosov's theorem [7] there exist an operator $B \in \mathcal{A}'$ and a nonzero $z \in X$ such that $BK_1 z = z$. For any $A \in \mathcal{A}$ it follows: $Az = ABK_1 z = BAK_1 z = BK_2 A_1 z$ for some $A_1 \in \mathcal{A}$. Hence the linear manifold $\mathcal{A}z$ satisfies the inclusion $\mathcal{A}z \subset (BK_2)\mathcal{A}z$ and as in the above proof we see that $\mathcal{A}z = \mathcal{R}(T)$, where $T$ is a continuous operator from a Banach space. By part (ii) of Theorem 6 it follows that $\mathcal{A}z$ is finite-dimensional. Let us choose $A_0 \in \mathcal{A}$ such that $A_0 \neq \lambda I$. If $\mathcal{A}z = \{0\}$ then $A_0$ has a nontrivial nullspace $M \supset span\{z\}$. If $\mathcal{A}z \neq \{0\}$ then it is a finite-dimensional invariant subspace for $A_0$. Thus $A_0$ has a nontrivial eigenspace which is invariant for all operators commuting with $A_0$, and $\mathcal{A}'$ has a nontrivial invariant subspace. □

**Corollary 11.** *Let $X$ be a complete infra-barrelled locally convex space and $A \in \mathcal{L}(X)$, $A \neq \lambda I$ and such that for some $P \in \mathcal{P}(X)$ it satisfies the condition:*

$$p(A^n x) \leq C_p p(x), \quad x \in X,\ p \in P,\ C_p \geq 0,\ n \in \mathbb{N}.$$

*Let there be some $k \in \mathbb{N}$ and $K \in \mathcal{K}(X)$ such that*

$$AK = KA^k.$$

*Then $A$ has a nontrivial hyperinvariant subspace.*

PROOF: Let us choose any sequence $\{a_n\} \in l_1$ and define

$$S_n x = \sum_{j=0}^{n} a_j A^j x, \quad x \in X,\ n \in \mathbb{N}.$$

Given $\varepsilon > 0$, we can find for arbitrary $p \in P$ and any bounded set $M$, sufficiently large $m, n \in \mathbb{N}$, $m > n$, such that the following estimations hold

$$q_p^M(S_m - S_n) = \sup_{x \in M} p\Big(\sum_{j=n+1}^{m} a_j A^j x\Big) \leq C_p \sup_{x \in M} p(x). \sum_{j=n+1}^{m} |a_j| < \varepsilon.$$

Thus, $\{S_n\}$ is a Cauchy sequence in $\mathcal{L}_b(X)$, since it is quasicomplete ([9]) it is also sequentially complete and we have for each sequence $\{a_n\} \in l_1$ an operator $S = \sum a_j A^j \in \mathcal{L}(X)$. Denote $\mathcal{A} = \{S := \sum a_j A^j : \{a_j\} \in l_1\}$. Then by an estimation similar to the one given above we can prove that the map $\{a_j\} \to S$ is a continuous map of $l_1$ into $\mathcal{L}_b(X)$. So, $\mathcal{A}$ is a range of a continuous operator from a Banach space and clearly $\mathcal{A}$ is an algebra. In the same manner as in [5] we have $SK = KS_1$ where $S, S_1 \in \mathcal{A}$ and the conclusion follows by Theorem 10. □

Let us now generalize a result from [8].

**Theorem 12.** *Let $X$ be a Hausdorff locally convex space, $A \in \mathcal{LB}(X)$ and $\{K_n\}_{n=0}^{\infty}$ a sequence of operators from $B_P(X)$ for some $P \in \mathcal{P}(X)$ such that $\|K_n\|_P \to 0$ and $K_0 \in \mathcal{K}(X)$. Let the following relations hold*

$$K_n A = A K_{n+1}, \quad n = 0, 1, \dots \ .$$

*Then $A$ has a nontrivial hyperinvariant subspace.*

Proof: By the above relations it immediately follows that $K_0 A^n = A^n K_n$ for $n = 0, 1, 2, \dots$ and clearly $K_0 A$ is compact, too. Denote $\mathcal{A} = \{A\}'$. If $\mathcal{A}$ had no invariant subspace, then by [7] there exists $A_1 \in \mathcal{A}$ such that $1 \in \sigma_p(A_1 K_0 A)$ (the point spectrum). Since $A_1 K_0 A$ is also compact, then $1 \in \sigma_p((A_1 K_0 A)^*)$, too ([3]). Thus, there is some $f \in X'$, $f \neq 0$ such that $(A_1 K_0 A)^* f = f$. Consequently, for each $n \in \mathbb{N}$:

$$K_n^* A_1^* A^* (A^*)^{n-1} f = (A^*)^n K_0^* A_1^* f = (A^*)^{n-1} f. \tag{3}$$

If $(A^*)^n f = 0$ for some $n \in \mathbb{N}$, then $ker(A^*) \neq \{0\}$ and then $\overline{\mathcal{R}(A)}^{\perp} \neq \{0\}$ ([9]) (where for $M \subset X : M^{\perp} = \{f \in X' : f(x) = 0, x \in M\}$). So, $\overline{\mathcal{R}(A)} \neq X$. In this case this set is a proper hyperinvariant subspace of $A$. If $g_n := (A^*)^{n-1} f \neq 0$ for each $n \in \mathbb{N}$, then (3) implies

$$K_n^* A_1^* A^* g_n = g_n, \quad n \in \mathbb{N}.$$

Let us prove that there exists some $P' \in \mathcal{P}(X)$ such that all $K_n$ and $A_1 A$ are in $B_{P'}(X)$. Clearly, $AA_1$ is also locally bounded, hence there is some neighborhood $U_0$ for which $AA_1(U_0)$ is bounded. We may assume that $U_0$ is the semiball corresponding to some $p_0 \in P$. Thus, we have $\sup\{p(AA_1 x) : x \in U_0\} \leq \lambda_p$, $p \in P$. Without loss of generality we may also assume that $\lambda_p > 0$ for each $p \in P$. By Lemma 1 we obtain

$$p(AA_1 x) \leq \lambda_p p_0(x), \quad x \in X,\ p \in P,$$

and especially also $p_0(AA_1 x) \leq \lambda_0 p_0(x)$, $x \in X$. At the same time we have

$$p(K_n x) \leq \|K_n\|_P . p(x), \quad x \in X,\ p \in P.$$

Let us define $P' = \{p'\}$, where

$$p'(x) = \max\{p(x), \lambda_p p_0(x)\}, \quad x \in X,\ p \in P.$$

It is easy to see that $P'$ is again a calibration on $X$ and for each $x \in X$ and $p' \in P'$ we can estimate

$$p'(AA_1 x) = \max\{p(AA_1 x), \lambda_p p_0(AA_1 x)\} \leq \lambda_p c_0 p_0(x) \leq c_0 p'(x),$$

where $c_0 = \max\{1, \lambda_0\}$, and by a simple verification we also have

$$p'(K_n x) \leq \|K_n\|_P . p'(x), \quad x \in X,\ p' \in P'.$$

Thus, all $K_n$ and $AA_1$ are in $B_{P'}(X)$ and $\|K_n\|_{P'} \leq \|K_n\|_P$ for each $n \in \mathbb{N}$. Let us take an arbitrary $n \in \mathbb{N}$. Since $g_n \in X'$, there is some $p'_n \in P'$ with the corresponding quotient space $X_n := X/kerp'_n$ (which is a normed space with respect to the norm $\|\hat{x}_n\|_n = p'_n(x)$, where $\hat{x}_n = x + kerp'_n$) such that $g_n \in (X_n)'$ (see [4]). For any $x \in X$ we can now estimate

$$|g_n(x)| = |g_n(AA_1K_n x)| \leq \|g_n\|_n p'_n(AA_1K_n x) \leq \|g_n\|_n \|AA_1\|_{P'} \|K_n\|_P . p'_n(x).$$

Taking supremum over all $x \in X$ for which $p'_n(x) = \|\hat{x}_n\|_n \leq 1$ we obtain

$$\|g_n\|_n \leq \|g_n\|_n \|AA_1\|_{P'} \|K_n\|_P,$$

hence

$$1 \leq \|AA_1\|_{P'} \|K_n\|_P.$$

Since $n \in \mathbb{N}$ is arbitrary and $\|K_n\|_P \to 0$, we have a contradiction. □

Finally, we give some generalization of some results from [1].

**Theorem 13.** *Let $X$ be a Hausdorff locally convex space and $A \in \mathcal{L}(X)$, $A \neq \lambda I$. Let*

$$AK = \mu KA$$

*for some nonzero $K \in \mathcal{K}(X)$ and $\mu \in \mathbb{C}$. Then $A$ has a nontrivial hyperinvariant subspace.*

The proof of this theorem and of the following one is for a locally convex space the same as for a normed space and we omit it.

**Theorem 14.** *Let $X$ be a Hausdorff locally convex space and $A \in \mathcal{L}(X)$, $A \neq \lambda I$, and $\mathcal{M}$ a subspace of $\mathcal{L}(X)$ of finite dimension such that $A\mathcal{M} = \mathcal{M}A$ and such that $\mathcal{M} \cap \mathcal{K}(X) \neq \{0\}$. Then $A$ has a nontrivial hyperinvariant subspace.*

We shall give the following variant of generalization of a result from [1].

**Theorem 15.** *Let $X$ be a Hausdorff locally convex space and $A \in \mathcal{L}B(X)$, $B \in \mathcal{L}(X)$ and $K \in \mathcal{K}(X)$ nontrivial operators such that there exist $\lambda, \theta \in \mathbb{C}$, $|\lambda| < 1$ and $|\theta| \leq 1$ with the properties*

$$BA = \lambda AB \quad \textit{and} \quad BK = \theta KB.$$

*Then $A$ has a nontrivial invariant subspace.*

Proof: Since also $K \in \mathcal{L}B(X)$, by Lemma 2 there exists a calibration $P \in \mathcal{P}(X)$ such that $A, K \in B_P(X)$. If $A$ had no nontrivial invariant subspace then the same would be true for the algebra $\mathcal{A}$ generated by $A^k$, $k \in \mathbb{N}$. By [7] then there exist $S \in \mathcal{A}$ and $x \neq 0$ such that $SKx = x$. Since $S = \sum_{j=1}^n \lambda_j A^j$ for some $\{\lambda_j\} \subset \mathbb{C}$, we have $(\sum \lambda_j A^j)Kx = x$ and for each $m = 0, 1, 2\ldots$ also $B^m(\sum_{j=1}^n \lambda_j A^j)Kx = B^m x$. Taking into account the supposed relations we have

$$B^m A^j K = \lambda^{mj}\theta^m A^j K B^m, \quad m = 0, 1, 2, \ldots, \; j = 1, 2, \ldots$$

and we obtain

$$[(\lambda_1\lambda^m\theta^m A + \lambda_2\lambda^{2m}\theta^m A^2 + \cdots + \lambda_n\lambda^{mn}\theta^m A^n)K]B^m x = B^m x. \tag{4}$$

Denote by $T_m$ the operator in the square brackets. Then for each $p \in P$ and $y \in X$ we can estimate $p(T_m y) \leq M_{m,n} p(y)$, where

$M_{m,n} = |\lambda|^m |\theta|^m \|A\|_P[|\lambda_1| + |\lambda_2||\lambda|^m \|A\|_P + \cdots + |\lambda_n||\lambda|^{(n-1)m}\|A\|_P^{n-1}].\|K\|_P.$

Thus, $T_m \in B_P(X)$ and $\|T_m\|_P \to 0$ for $m \to \infty$. In virtue of (4) we obtain for any $p \in P$ and $x \in X$

$$p(B^m x) = p(T_m B^m x) \leq \|T_m\|_P . p(B^m x)$$

and if we choose $k \in \mathbb{N}$ such that $\|T_k\|_P < 1$, we have $p(B^k x) = 0$ for all $p \in P$. Consequently, $B^k x = 0$. So, $B$ has a nontrivial kernel which is an invariant subspace for $A$. □

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