Continuity of order-preserving functions

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Abstract. Let the spaces $\mathbf{R}^m$ and $\mathbf{R}^n$ be ordered by cones P and Q respectively, let A be a nonempty subset of $\mathbf{R}^m$, and let $f : A \longrightarrow \mathbf{R}^n$ be an order-preserving function. Suppose that P is generating in $\mathbf{R}^m$, and that Q contains no affine line. Then f is locally bounded on the interior of A, and continuous almost everywhere with respect to the Lebesgue measure on $\mathbf{R}^m$. If in addition P is a closed halfspace and if A is connected, then f is continuous if and only if the range $f(A)$ is connected.

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0. Introduction

Some well known results concerning continuity and local boundedness of real nondecreasing functions are extended on order-preserving functions acting between finite dimensional ordered vector spaces. For convenience and to fix the notation we collect some of the definitions and results on finite dimensional ordered spaces. A subset $P \subseteq \mathbf{R}^m$ is called a *cone* if it is closed under addition and closed under multiplication by nonnegative scalars. A cone P is said to be *pointed* if $P \cap (-P) = \{0\}$. A cone P induces on $\mathbf{R}^m$ an order relation $\leq$ defined by $x \leq y$ whenever $y - x \in P$. We shall say in such a case that the space $\mathbf{R}^m$ is ordered by the cone P. If P is pointed then $\mathbf{R}^m$ with the induced order is a partially ordered vector space.

Let A be a nonempty subset of the space $\mathbf{R}^m$ ordered by a cone P. Then A is said to be *order bounded* if it is contained in some *order interval* $[x, y] = \{z \in \mathbf{R}^m : x \leq z \leq y\}$, $x, y \in \mathbf{R}^m$. A is said to be *solid* if $x, y \in A$ implies $[x, y] \subseteq A$. The smallest solid set containing A is called the *solid cover* of A and equals

$$S(A) = \{x \in \mathbf{R}^m : a \leq x \leq b \ \text{ for some } \ a, b \in A\}.$$

A cone P in $\mathbf{R}^m$ is said to be *generating* if $P - P = \mathbf{R}^m$. It is well known that P is generating if and only if it has a nonempty interior int P (with respect to the norm topology of $\mathbf{R}^m$). This implies that P has a nonempty relative interior (in the subspace $P - P$). An element $e \in \mathbf{R}^m$ belongs to int P if and only if there exists $\epsilon > 0$ such that P contains the open ϵ-ball $B_\epsilon(e)$ or equivalently $B_\epsilon(0) \subseteq [-e, e]$. Since in addition the condition that $[x, y]$ is a neighborhood of 0 implies $-x \in$ int P, it follows that P is generating if and only if bounded subsets of $\mathbf{R}^m$ are order bounded.

The dual cone P^* of P is given by

$$P^* = \{y \in \mathbf{R}^m : \langle y, x\rangle \geq 0 \ \text{ for all } \ x \in P\},$$

where $\langle\, ,\rangle$ denotes the usual inner product of $\mathbf{R}^m$. Evidently $P^* \cap (-P^*) = P^\perp = (P-P)^\perp$, hence P^* is pointed if and only if P is generating.

The second dual P^{**} equals the closure cl P of P, [5, Corollary 11.7.2] or [3, 3.1.7]. It follows that the cone cl P is pointed if and only if the dual P^* is generating. We shall need some other equivalent descriptions of this property. Suppose that the cone cl P is not pointed. Then $\mathbf{R}x \subseteq \text{cl } P$ for some nonzero $x \in \mathbf{R}^m$. This implies (use for example [5, Theorem 6.1]) that P contains all affine lines $y + \mathbf{R}x$ with y from the relative interior of P. On the other hand, if P contains an affine line $v + \mathbf{R}u$, then $\mathbf{R}u \subseteq \text{cl } P$, hence cl P is not pointed. Thus, the cone cl P is pointed if and only if P contains no affine line.

It is easy to verify that P contains no affine line if and only if P induces on $\mathbf{R}^m$ an *almost archimedean* order, that is, if and only if $x, y \in \mathbf{R}^m$ and $lx \leq y$ for all integers l imply $x = 0$. Another useful description of this property is related to the norm topology. We may infer from [3, 3.2.8, 3.4.2] that P contains no affine line if and only if there exists some real $\alpha > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq \alpha\|y\|$. It follows immediately that if P contains no affine line, then order bounded subsets of $\mathbf{R}^m$ are bounded. The converse holds as well since $v + \mathbf{R}u \subseteq P$ with $u \neq 0$ implies that the order interval $[0, 2v]$ contains $v + \mathbf{R}u$ and is therefore not bounded.

Recall finally that a function f acting between ordered spaces is said to be *order-preserving* if $x \leq y$ implies $f(x) \leq f(y)$.

1. Solid sets

For further purposes we need some topological properties of solid subsets of finite dimensional ordered vector spaces.

Proposition 1. *Let A be a nonempty solid subset of the space $\mathbf{R}^m$ ordered by a generating cone P. Then the following statements holds.*

(1) *The interior of A is solid and satisfies* $\text{int } A = \text{int cl } A$.

(2) *There exist order intervals $[x_k, y_k]$, $k \in \mathbf{N}$, such that $y_k - x_k \in \text{int } P$ for all k, and*

$$\text{int } A = \bigcup_{k=1}^{\infty} [x_k, y_k] = \bigcup_{k=1}^{\infty} \text{cl}\,[x_k, y_k].$$

Proof: (1) Suppose that $x, y \in \text{int } A$ and $x \leq z \leq y$. There exists an element $e \in \text{int } P$ such that $x - e \in A$ and $y + e \in A$. It follows that $[x - e, y + e]$ is a neighborhood of z contained in A. Therefore $z \in \text{int } A$, and hence int A is solid. To prove that $\text{int } A = \text{int cl } A$ take an element $w \in \text{int cl } A$ and choose

$e \in \text{int } P$ such that $w \pm 2e \in \text{cl } A$. Then $[w-3e, w-e]$ and $[w+e, w+3e]$ are neighborhoods of $w-2e$ and $w+2e$ respectively, hence there exist elements

$$u \in A \cap [w-3e, w-e], \quad v \in A \cap [w+e, w+3e].$$

This implies that the neighborhood $[w-e, w+e]$ of w is contained in $[u, v]$. Since A is solid, $[u, v]$ is a subset of A and therefore $w \in \text{int } A$. It follows that $\text{int cl } A \subseteq \text{int } A$. The reverse inclusion is obvious.

(2) Let D be a countable dense subset of int A. We shall prove that int A is the union of order intervals of the form

$$[x, y], \quad x, y \in D, \quad y - x \in \text{int } P,$$

as well as the union of its closures $\text{cl}\,[x, y]$. Let $z \in \text{int } A$. There exists $e \in \text{int } P$ such that $z \pm e \in \text{int } A$. Take $\epsilon > 0$ such that $B_\epsilon(e) \subset \text{int } P$. Since D is dense in int A, there exist elements $x, y \in D$ such that

$$(z-e) - x \in B_\epsilon(0) \text{ and } y - (z+e) \in B_\epsilon(0).$$

It follows that $z - x \in \text{int } P$ and $y - z \in \text{int } P$, hence $z \in [x, y]$ and $y - x = (z-x) + (y-z) \in \text{int } P$. For the remaining part of the proof choose $\delta > 0$ such that

$$B_\delta(x) \subset \text{int } A, \quad B_\delta(y) \subset \text{int } A,$$

and use the fact that int A is solid to see that $\text{cl}\,[x, y] \subseteq [x, y] + B_\delta(0) \subseteq \text{int } A$. □

Lemma 2. *Let A be a solid subset of the space $\mathbf{R}^m$ ordered by a generating cone P, and let*

$$A_+ = \{x \in \text{cl } A : A \cap (x + \text{int } P) = \emptyset\},$$
$$A_- = \{x \in \text{cl } A : A \cap (x - \text{int } P) = \emptyset\}.$$

Then the boundary bd A *of A equals $A_+ \cup A_-$, and*

$$(A_+ - A_+) \cap \text{int } P = \emptyset, \quad (A_- - A_-) \cap \text{int } P = \emptyset.$$

Proof: If $v \in \text{int } A$, then there exists $e \in \text{int } P$ such that $v \pm e \in A$. It follows that $v \notin A_+ \cup A_-$, hence $A_+ \cup A_- \subseteq \text{bd } A$. To prove the reverse inclusion suppose that $x \in \text{bd } A \setminus (A_+ \cup A_-)$. This implies that there exist elements $y, z \in A$ such that $y \in x - \text{int } P$, $z \in x + \text{int } P$. Therefore $x \in [y, z] \subseteq A$ and

$$x - y + B_\epsilon(0) \subset P, \quad z - x + B_\epsilon(0) \subset P$$

for sufficiently small $\epsilon > 0$. It follows that $B_\epsilon(x) \subseteq [y, z]$, thus $x \in \text{int } A$. This contradicts $x \in \text{bd } A$, hence $\text{bd } A \subseteq A_+ \cup A_-$.

Suppose now that $u - v \in \text{int } P$ for some $u \in \text{cl } A$ and $v \in \mathbf{R}^m$. Take $\epsilon > 0$ such that $B_\epsilon(u-v) \subset \text{int } P$ and choose an element $w \in A \cap B_\epsilon(u)$. Then

$$w - v = (u - v) + (w - u) \in u - v + B_\epsilon(0) \subset \text{int } P,$$

and therefore $w \in A \cap (v + \text{int } P)$. It follows that $v \notin A_+$, so $(A_+ - A_+) \cap \text{int } P = \emptyset$. The equality $(A_- - A_-) \cap \text{int } P = \emptyset$ can be proved similarly. □

We are prepared to prove an interesting measure-theoretic property of solid subsets of an ordered vector space $\mathbf{R}^m$. The result is a generalization of [4, Proposition].

Proposition 3. *Let A be a solid subset of the space $\mathbf{R}^m$ ordered by a generating cone P. Then its boundary* bd A *is of Lebesgue measure zero.*

Proof: By Lemma 2 it suffices to prove that A_+ and A_- are of Lebesgue measure zero. To this end suppose that A_+ is nonempty, take an element $e \in \text{int } P$, and denote by $p : \mathbf{R}^m \longrightarrow \mathbf{R}^m$ the orthogonal projection onto the subspace $e^\perp$ of $\mathbf{R}^m$. We claim that p is injective on A_+. Indeed, if $x, y \in A_+$ and $p(x) = p(y)$, then $x - y = te$ for some $t \in \mathbf{R}$. Since $t \neq 0$ implies $x - y \in \text{int } P$ or $y - x \in \text{int } P$, Lemma 2 shows that $t = 0$ and therefore $x = y$. It follows that there exists a function $h : p(A_+) \longrightarrow \mathbf{R}$ such that

$$A_+ = \{u + h(u)e : u \in p(A_+)\}.$$

We shall prove that every pair $u, v \in p(A_+)$ satisfies

$$|h(u) - h(v)| \leq \epsilon^{-1} \|u - v\|,$$

where $\epsilon > 0$ is such that $B_\epsilon(e) \subset \text{int } P$. By the way of contradiction, suppose that $h(u) - h(v) > \epsilon^{-1}\|u - v\|$ for some $u, v \in p(A_+)$. Then

$$w = (h(u) - h(v))^{-1}(u - v) \in B_\epsilon(0)$$

and therefore $e + w \in \text{int } P$. Since

$$x = u + h(u)e \in A_+, \quad y = v + h(v)e \in A_+,$$

and

$$x - y = (h(u) - h(v))(e + w) \in \text{int } P,$$

Lemma 2 yields the desired contradiction. The established inequality shows that h is continuous, therefore A_+ is of Lebesgue measure zero. The proof for A_- is similar. □

Remark. The condition that P is generating is not superfluous in Proposition 3. More precisely, if $\mathbf{R}^m$ is ordered by a nongenerating pointed cone P, then there exists a solid subset $A \subseteq \mathbf{R}^m$ such that bd A is not of Lebesgue measure zero. A can be constructed as follows. Take a subset $D \subseteq P^\perp$ such that D and $P^\perp \setminus D$ are both dense in $P^\perp$, choose a point e from the relative interior of P, and put

$$A = (D + P) \cup (P^\perp \setminus D + P + e).$$

Then A is solid, and bd A contains a cylinder $B_{\epsilon/2}(e/2) + P^\perp$, where $\epsilon > 0$ is such that $B_\epsilon(e) \cap (P - P) \subseteq P$. The proof is left to the reader.

2. Order-preserving functions

In this section we extend some well known results concerning properties of real nondecreasing functions on order-preserving functions acting between finite dimensional ordered vector spaces. We begin with local boundedness.

Theorem 4. *Let the spaces $\mathbf{R}^m$ and $\mathbf{R}^n$ be ordered by cones P and Q respectively, and let A be a nonempty subset of $\mathbf{R}^m$.*

(1) *If P is generating and if Q contains no affine line, then every order-preserving function $f : A \longrightarrow \mathbf{R}^n$ is locally bounded on* $\operatorname{cl} A \cap \operatorname{int} S(A)$.

(2) *If $P = \mathbf{R}^m$ and if Q is pointed, then every order-preserving function $f : A \longrightarrow \mathbf{R}^n$ is constant and therefore locally bounded.*

(3) *In all other cases there exists an order-preserving function $g : \mathbf{R}^m \longrightarrow \mathbf{R}^n$ such that g is locally unbounded at every point of $\mathbf{R}^m$.*

Proof: (1) Suppose that P is generating and that Q contains no affine line. Let a function $f : A \longrightarrow \mathbf{R}^n$ be order preserving and let $x \in \operatorname{cl} A \cap \operatorname{int} S(A)$. There exists $e \in \operatorname{int} P$ such that the neighborhood $[x-e, x+e]$ of x is contained in $S(A)$. Since $x \pm e \in S(A)$, there exist elements $y, z \in A$ such that $y \leq x - e$ and $x + e \leq z$. It follows that f maps the neighborhood $[y, z]$ of x into the interval $[f(y), f(z)]$ of $\mathbf{R}^n$. Since Q contains no affine line the order interval $[f(y), f(z)]$ is bounded, hence f is locally bounded at x.

(2) Let $f : A \longrightarrow \mathbf{R}^n$ be order-preserving, and let $P = \mathbf{R}^m$. If $x, y \in A$, then $x \leq y$ and $y \leq x$, hence $f(x) \leq f(y)$ and $f(y) \leq f(x)$. If Q is pointed, this implies $f(x) = f(y)$.

(3) Let $\beta : \mathbf{N} \longrightarrow \mathbf{Q}$ be a bijection, and let $g_0 : \mathbf{R} \longrightarrow \mathbf{R}$ be defined by $g_0(\beta(n)) = n$ for all $n \in \mathbf{N}$ and $g_0(t) = 0$ for all $t \in \mathbf{R} \setminus \mathbf{Q}$. Then g_0 is locally unbounded at every point of $\mathbf{R}$. Consider now three cases.

For the first case suppose that P is not generating. Then there exists a nonzero element $y \in P^{\perp}$. Take an element $z \in \mathbf{R}^n \setminus \{0\}$, and define $g : \mathbf{R}^m \longrightarrow \mathbf{R}^n$ by

$$g(x) = g_0(\langle x, y \rangle) z, \qquad x \in \mathbf{R}^m.$$

It is straightforward to check that g is order-preserving and everywhere locally unbounded.

For the second case suppose that Q is not pointed. Take a nonzero $w \in Q \cap (-Q)$, and define $g : \mathbf{R}^m \longrightarrow \mathbf{R}^n$ by $g(x) = g_0(\|x\|) w$, $x \in \mathbf{R}^m$. It is evident that g is order-preserving, and easy to see that g is everywhere locally unbounded.

For the third case suppose that $P \neq \mathbf{R}^m$, and that Q is pointed and contains an affine line. Observe first that $P \neq \mathbf{R}^m$ implies $P^* \neq \{0\}$. Then take a nonzero $w \in P^*$ and linearly independent vectors $u, v \in \mathbf{R}^n$ such that Q contains the affine line $u + \mathbf{R}v$. Define $g : \mathbf{R}^m \longrightarrow \mathbf{R}^n$ by

$$g(x + tw) = tu + g_0(t)v, \quad x \in w^{\perp}, \ t \in \mathbf{R}.$$

Since the halfspace $P_w = w^\perp \oplus \mathbf{R}^+ w$ contains P, and since $r > 0$ implies $ru+sv \geq 0$ for all $s \in \mathbf{R}$, g is order-preserving. It is easy to see that g is locally unbounded at every point of $\mathbf{R}^m$. □

It is well known that the set of all points of discontinuity of a real nondecreasing function is at most countable. A similar result holds for vector-valued functions with real domains.

Theorem 5. *Let the space $\mathbf{R}^n$ be ordered by a cone Q containing no affine line, and let A be a nonempty subset of $\mathbf{R}$ with the standard order. Then the set of all points of discontinuity of an order-preserving function $f : A \longrightarrow \mathbf{R}^n$ is at most countable.*

Proof: Since Q contains no affine line, Q^* is generating and therefore contains a basis $\{b_1, \dots, b_n\}$ of $\mathbf{R}^n$. The functions $f_i : A \longrightarrow \mathbf{R}$ defined by

$$f_i(x) = \langle b_i, f(x) \rangle, \quad i = 1, \dots, n,$$

are order-preserving, hence the set $D(f_i)$ of all points of discontinuity of f_i is at most countable. Since the set of all points of discontinuity of f equals the union of all $D(f_i)$, $i = 1, \dots, n$, the proof is complete. □

It is natural to ask whether Theorem 5 can be extended to order-preserving functions with domain contained in an ordered space $\mathbf{R}^m$ with $m > 1$. We cannot expect that the set of points of discontinuity of such a function is countable. Indeed, if $\mathbf{R}^m$ is ordered by a cone $P \neq \mathbf{R}^m$, then the characteristic function of the set P_w from the proof of Theorem 4(3) is order-preserving and discontinuous at every point of the hyperplane $w^\perp$. However, this function is continuous almost everywhere with respect to the Lebesgue measure. Our next result clarifies the general situation. In fact, we use the results of the first section to transplant smoothly the proof of the main result from [4] to the general situation. For the sake of convenience to the reader we present a complete proof.

Theorem 6. *Let the spaces $\mathbf{R}^m$ and $\mathbf{R}^n$ be ordered by cones P and Q respectively, and let A be a nonempty subset of $\mathbf{R}^m$.*

(1) *If P is generating and if Q contains no affine line, then every order-preserving function $f : A \longrightarrow \mathbf{R}^n$ is continuous almost everywhere with respect to the Lebesgue measure on $\mathbf{R}^m$.*

(2) *If $P = \mathbf{R}^m$ and if Q is pointed, then every order-preserving function $f : A \longrightarrow \mathbf{R}^n$ is constant and therefore continuous.*

(3) *In all other cases there exists an order-preserving function $g : \mathbf{R}^m \longrightarrow \mathbf{R}^n$ such that g is discontinuous at every point of $\mathbf{R}^m$.*

Proof: (1) If $m = 1$, (1) is covered by Theorem 5 and by (2), so we assume that $m > 1$. Consider first the case $n = 1$, $Q = \mathbf{R}^+$. An order-preserving function $f : A \longrightarrow \mathbf{R}$ can be extended by

$$\overline{f}(x) = \sup\{f(a) : a \in A,\, a \leq x\}, \quad x \in S(A),$$

to an order-preserving function $\overline{f} : S(A) \longrightarrow \mathbf{R}$, hence we may suppose that A is solid. Moreover, it follows easily from Propositions 1(2) and 3 that we may assume $A = \mathrm{cl}\,[0, e]$ with $e \in \mathrm{int}\, P$. For every $x \in \mathrm{int}\,[0, e]$ set

$$g(x) = \inf\{f(x + te) - f(x - te), \quad 0 < t \in \mathbf{R}\}.$$

Observe that for sufficiently small $s > 0$ the neighborhood $[x - se, x + se]$ of x is contained in $\mathrm{int}\,[0, e]$, and that f maps this neighborhood into the real interval $[f(x - se), f(x + se)]$ containing $f(x)$. Therefore f is continuous at x if and only if $g(x) = 0$. Put

$$D_k = \{x \in \mathrm{int}\,[0, e] : g(x) \geq \frac{1}{k}\}, \quad k = 1, 2, \ldots ,$$

and note that the set D of all points of discontinuity of f satisfies

$$D \cap \mathrm{int}\,[0, e] = \bigcup_{k \in \mathbf{N}} D_k.$$

Thus, we have to prove that each D_k is of Lebesgue measure zero.

We claim that $D_k = \mathrm{cl}\ D_k \cap \mathrm{int}\,[0, e]$. Take any $x \in \mathrm{int}\,[0, e] \setminus D_k$, and pick $s > 0$ such that

$$[x - se, x + se] \subseteq [0, e], \quad f(x + se) - f(x - se) < \frac{1}{k}.$$

Note that every $y \in [x - (s/2)e, x + (s/2)e]$ satisfies

$$[y - \frac{s}{2}e, y + \frac{s}{2}e] \subseteq [x - se, x + se],$$

hence $g(y) \leq f(y + (s/2)e) - f(y - (s/2)e) < 1/k$, and so $y \notin D_k$. This implies that the neighborhood $[x - (s/2)e, x + (s/2)e]$ of x does not intersect D_k. Hence $x \notin \mathrm{cl}\ D_k$, and the claim follows.

Fix $r > 0$ and note that it suffices to prove that $D_k \cap B_r(0)$ is of Lebesgue measure zero. Assume that $D_k \cap B_r(0)$ is nonempty, and let $\epsilon > 0$.

For each fixed $x \in \mathrm{cl}\,[0, e]$ consider the real function $h : t \longmapsto f(x+te)$. Since h is nondecreasing and jumps for at least $1/k$ at every t satisfying $x + te \in D_k$, the set $D_k \cap (x + \mathbf{R}e)$ contains finitely many points or it is empty. Using the equality $\mathrm{cl}\ D_k \cap \mathrm{int}\,[0, e] = D_k$ and the fact that $(x + \mathbf{R}e) \cap \mathrm{bd}\,[0, e]$ contains at most two points, we see that the set $\mathrm{cl}\ D_k \cap (x + \mathbf{R}e)$ contains finitely many points or it is empty.

Remove from the line $x + \mathbf{R}e$ finitely many disjoint relatively open intervals of common length less than ϵ and containing $\mathrm{cl}\ D_k \cap (x + \mathbf{R}e)$. Denote by $R(x)$ the remaining set, observe that $d = \mathrm{dist}\ (R(x), D_k) > 0$ and put

$$T(x) = \{y \in \mathbf{R}^m : \mathrm{dist}\ (y, x + \mathbf{R}e) < \min\{d, 1\}\ \}.$$

From the open covering $\{T(x) : x \in K_r\}$ of the compact set $K_r = \mathrm{cl}\,[0,e] \cap \mathrm{cl}\, B_r(0)$ extract a finite subcovering $\{T_i = T(x_i) : i = 1, \ldots, p\}$. Accept $T_0 = \emptyset$ and set

$$U_i = T_i \setminus \bigcup_{j<i} T_j.$$

Let $E = e^{\perp}$, and observe that by construction $U_i \cap D_k$ is contained in a subset of Lebesgue measure less or equal $\epsilon \mu_E(U_i \cap E)$, where μ_E denotes the $(m-1)$-dimensional Lebesgue measure in the subspace $E \subset \mathbf{R}^m$. It follows that $D_k \cap B_r(0)$ is contained in a subset of Lebesgue measure less or equal

$$\epsilon \sum_{i=1}^{p} \mu_E(U_i \cap E) = \epsilon \mu_E(\bigcup_{i=1}^{p} U_i \cap E).$$

Since $U_i \cap E \subseteq B_{r+1}(0)$ for $i = 1, \ldots, p$, this implies that $D_k \cap B_r(0)$ is of Lebesgue measure zero.

Consider now the general case. Suppose that the conditions in (1) are satisfied and proceed similarly as in the proof of Theorem 5. Let $\{b_1, \ldots, b_n\}$ be a basis of $\mathbf{R}^n$ contained in Q^*, and $f_i : A \longrightarrow \mathbf{R}$, $i = 1, \ldots, n$, functions defined by $f_i(x) = \langle b_i, f(x) \rangle$. It follows from the first part of the proof of this theorem that each f_i is continuous almost everywhere with respect to the Lebesgue measure, hence f is continuous almost everywhere as well.

(2) and (3) follow from Theorem 4. □

Remark. We give here an application of the above results to utility theory. Let X be a nonempty set ordered by an irreflexive and transitive relation $<$. It is pointed out by the referee that in utility theory the existence and continuity of real-valued order-preserving functions defined on X are investigated (see for example [2, Chapters 2, 3] and [1]). Theorems 4 and 6 can be applied when X is a subset of $\mathbf{R}^m$ equipped with an irreflexive and transitive order relation $<$ which is compatible with the vector space structure of $\mathbf{R}^m$ (i.e., $x < y$ implies $x + z < y + z$ for all z and $\lambda x < \lambda y$ for all real $\lambda > 0$). Such a relation is induced by a cone $P = \{x \in \mathbf{R}^m : x > 0\} \cup \{0\}$ in an obvious way: $x < y$ whenever $x \neq y$ and $x \leq y$ with respect to P (or $y - x \in P \setminus \{0\}$). Since P is generating if and only if the order $<$ is directed (i.e., each pair of different incomparable elements have a strict upper bound), Theorems 4 and 6 give the following result:

Let $\mathbf{R}^m$ be ordered by a directed compatible irreflexive and transitive order relation $<$, and let X be a nonempty subset of $\mathbf{R}^m$ equipped with the induced order. Then every order preserving function $f : X \longrightarrow \mathbf{R}$ is locally bounded on the interior of X, and continuous almost everywhere with respect to the Lebesgue measure on $\mathbf{R}^m$.

It is well known and easy to see that a real-valued nondecreasing function defined on a connected subset of $\mathbf{R}$ is continuous if and only if its range is connected. This result can be generalized as follows.

Theorem 7. *Let the spaces* $\mathbf{R}^m$ *and* $\mathbf{R}^n$ *be ordered by cones* P *and* Q *respectively, and let* A *be a nonempty connected subset of* $\mathbf{R}^m$.

(1) *If* P *is a closed halfspace and if* Q *contains no affine line, then an order-preserving function* $f : A \longrightarrow \mathbf{R}^n$ *is continuous if and only if its range* $f(A)$ *is connected.*

(2) *If* $P = \mathbf{R}^m$ *and if* Q *is pointed, then every order-preserving function* $f : A \longrightarrow \mathbf{R}^n$ *is constant.*

(3) *Under the additional conditions* $\operatorname{int} A \neq \emptyset$ *and* $Q \neq \{0\}$, *in all other cases there exists an order-preserving function* $g : A \longrightarrow \mathbf{R}^n$ *such that* $g(A)$ *is connected and* g *is not continuous.*

PROOF: (1) If f is continuous, then its range $f(A)$ is connected. To prove the reverse implication suppose that the conditions in (1) are satisfied. Then $H = \operatorname{bd} P \subseteq \mathbf{R}^m$ is a hyperplane contained in P, and $\mathbf{R}^m = H \oplus \mathbf{R}w$ holds for some $w \in P \setminus \{0\}$. Since Q is pointed, this implies that an order-preserving function $f : A \longrightarrow \mathbf{R}^n$ is constant on $(H + tw) \cap A$ for each fixed $t \in \mathbf{R}$. Observe that the subset

$$A_0 = \{t \in \mathbf{R} : (H + tw) \cap A \neq \emptyset\}$$

of $\mathbf{R}$ is connected, and define $f_0 : A_0 \longrightarrow \mathbf{R}^n$ by

$$f_0(t) = f(x + tw), \quad t \in A_0,\ x \in H,\ x + tw \in A.$$

Then f_0 is order-preserving with respect to the usual order in $A_0 \subseteq \mathbf{R}$, and has the same range as f. Note also that if f_0 is continuous, then f is continuous as well. Therefore, it suffices to show that the discontinuity of f_0 implies that $f_0(A_0)$ is disconnected.

Let f_0 be discontinuous at some point $t_0 \in A_0$. Then at least one of the numbers

$$\begin{aligned} d_+ &= \inf\{\|f_0(t) - f_0(t_0)\| : t \in A_0,\, t > t_0\}, \\ d_- &= \inf\{\|f_0(t) - f_0(t_0)\| : t \in A_0,\, t < t_0\} \end{aligned}$$

is strictly positive. Suppose that $d_+ > 0$, put

$$F = \{y \in \mathbf{R}^n : y \leq f_0(t_0)\},$$

and observe that $f_0(t_0) \in f_0(A_0 \cap (-\infty, t_0]) \subseteq F$. Since Q contains no affine line, there exists $\alpha > 0$ such that $0 \leq u \leq v$ implies $\|u\| \leq \alpha\|v\|$. Suppose that $f_0(t) = y + z$ for some $t \in A_0$ satisfying $t > t_0$, and for some $y \in F$. Then $z = f_0(t) - y \geq f_0(t) - f_0(t_0) \geq 0$, and therefore $\|z\| \geq \alpha\|f_0(t) - f_0(t_0)\| \geq \alpha d_+$. It follows that

$$f_0(A_0 \cap (t_0, +\infty)) \cap (F + B_{\alpha d_+}(0)) = \emptyset.$$

Since $f_0(A_0 \cap (t_0, +\infty))$ is nonempty and since $F + B_{\alpha d_+}(0)$ is open and contains cl F, the range $f_0(A_0)$ is disconnected. If $d_+ = 0$, then $d_- > 0$, and the proof is similar.

(2) is a part of Theorem 4.

(3) Using a translation and a homothety in $\mathbf{R}^m$ (to construct the appropriate function $g : A \longrightarrow \mathbf{R}^n$) we may suppose that A contains the unit ball $B_1(0)$. Consider now four cases.

For the first case suppose that P is not generating. Denote by $p : \mathbf{R}^m \longrightarrow \mathbf{R}^m$ the orthogonal projection on the nontrivial subspace $P^\perp$, take a nonzero $w \in \mathbf{R}^n$, and define $g : \mathbf{R}^m \longrightarrow \mathbf{R}^n$ by $g(x) = \|p(x)\| w$ if $x \notin P - P$, and $g(x) = w$ if $x \in P - P$. It can be seen easily that g is order-preserving and discontinuous at 0. The subset $\{\|p(x)\| : x \in A\}$ of $\mathbf{R}$ is connected and contains the open interval $(0, 1)$, hence the range $g(A)$ is connected.

For the second case suppose that Q is not pointed. Take a nonzero $w \in Q \cap (-Q)$, and define $g : \mathbf{R}^m \longrightarrow \mathbf{R}^n$ by $g(x) = \|x\| w$ if $x \neq 0$, and $g(0) = w$. Evidently g is order-preserving and discontinuous at 0. Similarly as in the previous case we can see that the range of g is connected.

For the third case suppose that P is a closed halfspace, and that Q is pointed containing an affine line. Then $H =$ bd P is a hyperplane in $\mathbf{R}^m$, and $\mathbf{R}^m = H \oplus \mathbf{R}w$ for some $w \in P \setminus \{0\}$. Take an affine line $u + \mathbf{R}v$ contained in Q, such that u and v are linearly independent. Define $g : \mathbf{R}^m \longrightarrow \mathbf{R}^n$ by

$$g(x + tw) = tu + (\sin \frac{1}{t})v \quad \text{ if } \; x \in H,\; t \in \mathbf{R} \setminus \{0\},$$

and by $g(x) = 0$ if $x \in H$. Since $r > 0$ implies $ru + sv \geq 0$ for all $s \in \mathbf{R}$, g is order-preserving. It is easy to see that g is discontinuous at 0, and that the range $g(A)$ is connected.

For the last case suppose that P is generating, $P \neq \mathbf{R}^m$, and that P is not a closed halfspace. Denote by $L = P \cap (-P)$ the linearity space of P, and by $p : \mathbf{R}^m \longrightarrow \mathbf{R}^m$ the orthogonal projection on $L^\perp$. Observe that $k = \dim L^\perp > 1$, and that $P_0 = P \cap L^\perp$ is a pointed and generating cone in $L^\perp$. It follows that P_0 is contained in a maximal pointed cone P_1 in $L^\perp$. Since $L^\perp$ is totally ordered by P_1, there exists an isomorphism ϕ of $L^\perp$ (ordered by P_1) onto the space $\mathbf{R}^k$ ordered by the lexicographic order. For each $\epsilon > 0$ define $g_\epsilon : \mathbf{R}^k \longrightarrow \mathbf{R}$ by

$$g_\epsilon(x_1, \dots, x_k) = \begin{cases} \operatorname{sgn} x_1 & \text{if } \; x_1 \neq 0, \\ \operatorname{sgn} x_2 & \text{if } \; x_1 = 0, \; |x_2| > \epsilon, \\ \epsilon^{-1} x_2 & \text{if } \; x_1 = 0, \; |x_2| \leq \epsilon, \end{cases}$$

and note that g_ϵ is order-preserving. Since ϕ is a homeomorphism, there exists an $\epsilon > 0$ such that the open ϵ-ball $B_\epsilon(0)$ of $\mathbf{R}^k$ is contained in $\phi(p(A))$. Take a nonzero $w \in Q$, and define $g : A \longrightarrow \mathbf{R}^n$ by

$$g(x) = g_\epsilon(\phi(p(x)))w, \quad x \in A.$$

Observing that $p(P) \subseteq P_0$, and using the fact that ϕ and g_ϵ are order-preserving, we see that g is order-preserving as well. Since $\phi(p(A))$ contains $B_\epsilon(0) \subseteq \mathbf{R}^k$, it follows easily from the definition of g that the range $g(A)$ equals $\{tw : |t| \leq 1\}$ and is therefore connected. Since g_ϵ is discontinuous at every point $x \in \mathbf{R}^k$ satisfying $x_1 = 0$, g is discontinuous. $\square$

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