On non-homogeneous viscous incompressible fluids. Existence of regular solutions

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Abstract. We consider the flow of a non-homogeneous viscous incompressible fluid which is known at an initial time. Our purpose is to prove that, when Ω is smooth enough, there exists a local strong regular solution (which is global for small regular data).

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Introduction

Let Ω be a bounded connected open subset of $\mathbb{R}^3$, $T > 0$ and $Q_T = \Omega\times]0,T[$. A non homogeneous fluid is described by its velocity $u = (u_1, u_2, u_3)$, its density ρ, its viscosity $\nu = \nu(\rho)$ and its pressure p. It is modelized by

$$(1)\qquad \begin{cases} \rho\,\partial_t u - \nabla\,.\,\big(\nu(\rho)(\nabla u + {}^t\nabla u)\big) + \rho\,(u\,.\,\nabla)u + \nabla p = \rho\, f, \\ \nabla\,.\,u = 0, \\ \partial_t\rho + u\,.\,\nabla\rho = 0, \end{cases}$$

$$(2)\qquad u = 0 \ \text{ on } \ \Sigma_T = \partial\Omega\times]0,T[,$$

$$(3)\qquad u|_{t=0} = u_0 \ \text{ and } \ \rho|_{t=0} = \rho_0 \ \text{ in } \ \Omega.$$

The aim of this work is to prove the existence of a local regular solution of (1)–(3) in Q_T, when f and u_0 are regular data and ρ_0 is supposed to be regular and strictly greater than 0, i.e.

$$0 < M_1 \leq \rho_0 \ \text{ in } \ \Omega.$$

When the viscosity does not depend on the density, S.A. Antonzev and A.V. Kajikov [1] proved the existence of weak solutions (see also J.L. Lions [7]). O.A. Ladyzenskaya and V.A. Solonnikov [5] proved the local existence of a strong regular solution and the global existence for small data.

When $\nu = \nu(\rho)$, E. Fernández-Cara and F. Guillén [3] obtained the existence of a weak solution for $u_0 \in L^2(\Omega)^3$, $\nabla\,.\,u_0 = 0$ and $u_0\,.\,n = 0$, $\rho_0 \in L^\infty(\Omega)$, $\rho_0 \geq 0$, $f \in L^1\big(0,T; L^2(\Omega)^3\big)$ and $\nu \in \mathcal{C}(\mathbb{R}_+)$ such that $\nu(s) \geq \beta > 0$ for all $s \in \mathbb{R}_+$ (see also P.L. Lions [8]). According to uniqueness, M. Kabbaj [4] gives a result for a regular strong solution of (1)–(3) when ρ is supposed to be in $\mathcal{C}^2(\overline{Q_T})$.

1. Existence result

In all the paper long, we suppose that

Ω is a bounded open subset of $\mathbb{R}^3$ with a $\mathcal{C}^2$ boundary,
$\rho_0 \in \mathcal{C}^1(\overline{\Omega})$ satisfies $M_2 \geq \rho_0(x) \geq M_1 > 0$ for all $x \in \Omega$,
$\nu \in \mathcal{C}^1(]0, +\infty[)$, $\nu(a) \geq \nu_1 > 0$ for all $a > 0$,
$f \in L^q(Q_T)^3$, $u_0 \in W^{2-2/q,q}(\Omega)^3$, $\nabla \cdot u_0 = 0$, $u_0|_{\partial\Omega} = 0$ with $q > 3$.

Under these hypotheses, one has the following result:

Theorem 1. *There exists $t \leq T$ such that the equations* (1)–(3) *have a solution* $(u, \nabla p, \rho)$ *which satisfies*

$$u \in \mathcal{W}_q^{2,1}(Q_t), \quad \nabla p \in L^q(Q_t)^3, \quad \rho \in \mathcal{C}^1(\overline{Q}_t).$$

Moreover, there exists $R > 0$ depending on Ω, ν, T, ρ_0, such that if

$$\|f\|_{L^q(Q_T)^3} + \|u_0\|_{W^{2-2/q,q}(\Omega)^3} \leq R,$$

then $(u, \nabla p, \rho)$ is a solution of (1)–(3) *for $t = T$.* □

Outline of the proof. We use a fixed point argument, decoupling the variables u and ρ. More precisely, let us consider $z \in \mathcal{W}_q^{2,1}(Q_T)$ satisfying $\nabla \cdot z = 0$, $z(0) = u_0$ in Ω and $z|_{\Sigma_T} = 0$.

In the first part, we prove that there exists a unique regular solution $(u, \nabla p, \rho)$ of the equations

$$\tag{4} \begin{cases} \rho\, \partial_t u - \nabla \cdot \big(\nu(\rho)(\nabla u + {}^t\nabla u)\big) + \rho\,(z \cdot \nabla)u + \nabla p = \rho\, f \ \text{ in } \ Q_T, \\ \nabla \cdot u = 0 \ \text{ in } \ Q_T, \\ \partial_t \rho + z \cdot \nabla \rho = 0 \ \text{ in } \ Q_T, \\ u(0) = u_0 \ \text{ and } \ \rho(0) = \rho_0 \ \text{ in } \ \Omega, \\ u|_{\Sigma_T} = 0. \end{cases}$$

In the second part, we prove that there exists R such that if $\|f\|_{L^q(Q_T)^3} + \|u_0\|_{W^{2-2/q,q}(\Omega)^3} \leq R$ or if T is small enough, then $z \mapsto u$ is a continuous map from a convex closed bounded subset of $\mathcal{W}_q^{2,1}(Q_T)$ with the topology of a Banach space $X_{q,T}$ defined below into itself, where $\mathcal{W}_q^{2,1}(Q_T) \subset X_{q,T}$ with compact imbedding, and by Schauder's theorem, we infer the existence of a fixed point.

Remark. The proof of Theorem 1 is based on results of O.A. Ladyzenskaya and V.A. Solonnikov [5].

2. Functional spaces and preliminaries

Let $\mathcal{D}(\Omega)$ be the space of $\mathcal{C}^\infty$ functions with compact support in Ω, $\mathcal{D}'(\Omega)$ the space of distributions on Ω and $\langle\ ,\ \rangle_\Omega$ the duality product between $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$.

For $1 \le r < +\infty$, $L^r(\Omega)$ is the space of distributions f on Ω for which $|f|^r$ is integrable. This space is endowed with the norm

$$\|f\|_r = \Big(\int_\Omega |f|^r\Big)^{\frac{1}{r}},$$

and $L^\infty(\Omega)$ is the space of distributions f on Ω locally integrable and satisfying

$$\|f\|_\infty = \operatorname{supess}|f| < +\infty.$$

For $1 \le s \le +\infty$, the Sobolev spaces are defined by

$$\begin{aligned} W^{1,s}(\Omega) &= \{v \in L^s(\Omega) :\ \nabla v \in L^s(\Omega)^3\},\\ W^{1,s}_0(\Omega) &= \text{ closure of } \mathcal{D}(\Omega) \text{ in } W^{1,s}(\Omega),\\ W^{-1,s}(\Omega) &= \Big\{v \in \mathcal{D}'(\Omega) : v = v_0 + \sum_{i=1}^{3} \partial_i v_i :\ v_i \in L^s(\Omega),\ i = 0,\dots,3\Big\}, \end{aligned}$$

and we denote $H^1(\Omega) = W^{1,2}(\Omega)$, $H^1_0(\Omega) = W^{1,2}_0(\Omega)$, $H^{-1}(\Omega) = W^{-1,2}(\Omega)$ and

$$\begin{aligned} \mathcal{V} &= \{v \in \mathcal{D}(\Omega)^3 : \nabla \cdot v = 0\},\\ V &= \{v \in H^1_0(\Omega)^3 : \nabla \cdot v = 0\}. \end{aligned}$$

Let us recall that V coincides with the closure of $\mathcal{V}$ in $H^1(\Omega)^3$ (cf. Temam [12]).

Let $\mathcal{W}^{2,1}_q(Q_T)$ be the space of distributions $u \in L^q\big(0,T; W^{2,q}(\Omega)^3\big)$ such that $\partial_t u \in L^q(Q_T)^3$. This space, endowed with the norm

$$\|u\|^{(2,1)}_{q,Q_T} = \|\partial_t u\|_{L^q(Q_T)^3} + \|\nabla(\nabla u)\|_{L^q(Q_T)^{27}} + \|\nabla u\|_{L^q(Q_T)^9} + \|u\|_{L^q(Q_T)^3}$$

is a Banach space. All functions of $\mathcal{W}^{2,1}_q(Q_T)$ are in $\mathcal{C}_u\big(0,T; W^{2-2/q,q}(\Omega)^3\big)$, where $\mathcal{C}_u(0,T) = \mathcal{C}([0,T])$, so we can define $|||\ |||_T$ on $\mathcal{W}^{2,1}_q(Q_T)$ by

$$|||u|||_T = \|u\|^{(2,1)}_{q,Q_T} + \sup_{0\le t\le T} \|u\|_{W^{2-2/q,q}(\Omega)^3}.$$

Endowed with this norm, $\mathcal{W}^{2,1}_q(Q_T)$ is a Banach space. Let us recall that for all $u \in \mathcal{W}^{2,1}_q(Q_T)$ and all t, $0 \le t \le T$ we have (cf. V.A. Solonnikov [11]):

$$\|u(t)\|_{W^{2-2/q,q}(\Omega)^3} \le \|u_0\|_{W^{2-2/q,q}(\Omega)^3} + c\|u\|^{(2,1)}_{q,Q_t},$$

where c is independent of $t \in [0,T]$.

We denote by

$$|\!|\!|(u,\nabla p)|\!|\!|_T = |\!|\!|u|\!|\!|_T + \|\nabla p\|_{L^q(Q_T)^3}.$$

Finally, let $\mathcal{C}^\varepsilon(\overline{\Omega})$, $0<\varepsilon<1$, be the set of functions $f \in \mathcal{C}(\overline{\Omega})$ which satisfy $|f(x)-f(y)| \le c|x-y|^\varepsilon$ for all $x,y \in \overline{\Omega}$ and $\mathcal{C}^{1,\varepsilon}(\overline{\Omega})$ the set of functions $f \in \mathcal{C}^1(\overline{\Omega})$ which satisfy $|\nabla f(x)-\nabla f(y)| \le c'|x-y|^\varepsilon$ for all $x,y \in \overline{\Omega}$.

Let us now give an evolution case of De Rham's theorem (cf. J. Simon [10, Lemma 2, p. 1096]).

Lemma 2. *Let $h \in \mathcal{D}'\big(0,T;H^{-1}(\Omega)^3\big)$ satisfy $\langle h,v\rangle_\Omega = 0$ for all $v \in \mathcal{V}$. Then there exists $g \in \mathcal{D}'\big(0,T;L^2(\Omega)\big)$ such that $h = \nabla g$.* □

Now one gives a compactness result:

Lemma 3. *There exists $1 > \varepsilon_q > 0$ such that*

$$\mathcal{W}_q^{2,1}(Q_T) \subset \Big(L^q\big(0,T;\mathcal{C}^{1,\varepsilon_q}(\overline{\Omega})^3\big) \cap \mathcal{C}_u\big(0,T;\mathcal{C}(\overline{\Omega})^3\big)\Big) =: X_{q,T}$$

with compact imbedding. □

The proof is based on the following result (see J. Simon [9, Corollary 8, p. 90])

Lemma 4. *Let X and Y be two Banach spaces, $X \subset Y$ with corresponding compact imbedding and B a Banach space, $X \subset B \subset Y$, such that there exists C and θ, $0<\theta<1$ such that*

$$\|v\|_B \le C\|v\|_X^{1-\theta}\|v\|_Y^\theta \quad \forall\, v \in X.$$

Let $1 \le s_0 \le +\infty$, $1 \le s_1 \le +\infty$ and let $\mathcal{F}$ be a bounded subset of $L^{s_0}(0,T;X)$ such that $\partial_t\mathcal{F}$ is bounded in $L^{s_1}(0,T;Y)$. Then,

(i) *if $\theta(1-1/s_1) \le (1-\theta)/s_0$, $\mathcal{F}$ is relatively compact in $L^s(0,T;B)$ $\forall\, s < s_*$, where $1/s_* = (1-\theta)/s_0 - \theta(1-1/s_1)$;*

(ii) *if $\theta(1-1/s_1) > (1-\theta)/s_0$, $\mathcal{F}$ is relatively compact in $\mathcal{C}_u(0,T;B)$.* □

Proof of Lemma 3:

(i) One has $\mathcal{W}_q^{2,1}(Q_T) \subset L^q\big(0,T;\mathcal{C}^{1,\varepsilon_q}(\overline{\Omega})^3\big)$ with corresponding compact imbedding.

For $X = W^{2,q}(\Omega)^3$ and $Y = L^q(\Omega)^3$, since we have $W^{2,q}(\Omega)^3 \subset L^q(\Omega)^3$ with compact imbedding, using Lemma 4 (i), with $s_1 = s_0 = q$, we obtain for all $\theta \le 1/q$

$$\mathcal{W}_q^{2,1}(Q_T) \subset L^q\big(0,T;(W^{2,q}(\Omega)^3, L^q(\Omega)^3)_\theta\big) = L^q\big(0,T;H_q^{2(1-\theta)}(\Omega)^3\big)$$

with compact imbedding (cf. H. Triebel [13, Theorem 2, p. 317] and [11, p. 185]).

In addition we have (cf. H. Triebel [13, p. 328]) $H_q^{2(1-\theta)}(\Omega)^3 \subset \mathcal{C}^{1,\alpha}(\overline{\Omega})^3$ for $\alpha = 1-2\theta-3/q > 0$. Therefore we have

$$\mathcal{W}_q^{2,1}(Q_T) \subset L^q\big(0,T;\mathcal{C}^{1,\varepsilon_q}(\overline{\Omega})^3\big)$$

with compact imbedding, where $\varepsilon_q = 1-2\theta-3/q$ and $\theta < \inf\{1/q,(q-3)/2q\}$.

(ii) One has $\mathcal{W}_q^{2,1}(Q_T) \subset \mathcal{C}_u\big(0,T;\mathcal{C}(\overline{\Omega})^3\big)$ with corresponding compact imbedding.

Using Lemma 4 (ii) with $s_1 = s_0 = q$, we obtain for all $\theta > 1/q$

$$\mathcal{W}_q^{2,1}(Q_T) \subset \mathcal{C}_u\big(0,T;H_q^{2(1-\theta)}(\Omega)^3\big)$$

with compact imbedding.

In addition we have (cf. H. Triebel [13, p. 328]) $H_q^{2(1-\theta)}(\Omega)^3 \subset \mathcal{C}(\overline{\Omega})^3$ for all $\theta < 1 - 3/2q$. Since $1/q < 1 - 3/2q$ $(q > 3)$, we have

$$\mathcal{W}_q^{2,1}(Q_T) \subset \mathcal{C}_u\big(0,T;\mathcal{C}(\overline{\Omega})^3\big)$$

with compact imbedding. □

3. Transport equation

Proposition 5. *Let $z \in \mathcal{W}_q^{2,1}(Q_T)$ satisfy $\nabla \,.\, z = 0$ and $z|_{\Sigma_T} = 0$. Then for all $\rho_0 \in \mathcal{C}^1(\overline{\Omega})$, there exists a unique solution $\rho \in \mathcal{C}^1(\overline{Q}_T)$ of*

$$\begin{cases} \partial_t \rho + z \,.\, \nabla \rho = 0 \quad \text{in} \quad Q_T, \\ \rho|_{t=0} = \rho_0. \end{cases} \tag{5}$$

It satisfies

$$\min_{x\in\overline{\Omega}} \rho_0(x) \le \rho(y,t) \le \max_{x\in\overline{\Omega}} \rho_0(x) \quad \forall\,(y,t) \in Q_T,$$

and the following estimates, for all $t \le T$:

$$\|\nabla\rho\|_{L^\infty(Q_t)^3} \le \sqrt{3}\|\nabla\rho_0\|_{L^\infty(\Omega)^3} \exp\{\|\nabla z\|_{L^1(0,t;L^\infty(\Omega)^9)}\}, \tag{6}$$

$$\|\partial_t\rho\|_{L^\infty(Q_t)} \le \sqrt{3}\|\nabla\rho_0\|_{L^\infty(\Omega)^3}\|z\|_{L^\infty(Q_T)^3} \exp\{\|\nabla z\|_{L^1(0,t;L^\infty(\Omega)^9)}\}. \tag{7}$$

Let K be a closed bounded subset of $\mathcal{W}_q^{2,1}(Q_T) \cap L^2\big(0,T;V\big)$. Then the map $z \mapsto \rho$ is continuous on K endowed with the topology of $X_{q,T}$ with values in $\mathcal{C}_u\big(0,T;\mathcal{C}^1(\overline{\Omega})\big)$.

Proof: The existence and uniqueness of such a solution, and the estimates (6)–(7) are given by O.A. Ladyzenskaya and V.A. Solonnikov [5].

Let us remark that if $z \in \mathcal{W}_q^{2,1}(Q_T)$ satisfies $\nabla \,.\, z = 0$, $z_{|\Sigma_T} = 0$, there exists a unique $y(\tau,t,x)$ (cf. O.A. Ladyzenskaya and V.A. Solonnikov [5]) solution of

$$y^k(\tau,t,x) = x^k - \int_\tau^t z^k\big(y(\xi,t,x),\xi\big)\,d\xi. \tag{8}$$

In addition, for all τ, t, $y(\tau,t,.)$ is a one to one map on Ω with Jacobian equal to 1 (cf. V.A. Solonnikov [11]). The solution ρ of (5) satisfies

$$\rho(x,t)=\rho_0\big(y(0,t,x)\big).$$

Let us prove the continuity of the map $z\mapsto\rho$. It is well known, (see [5]) that if ρ_1 and ρ_2 are two solutions of (5) associated to z_1 and z_2 belonging to $\mathcal{W}_q^{2,1}(Q_T)$ and satisfying $z_1|_{\Sigma_T}=z_2|_{\Sigma_T}=0$, $\nabla\,.\,z_1=\nabla\,.\,z_2=0$, we have for all t, $0<t\le T$, the following estimate:

$$\|\rho_1-\rho_2\|_{L^\infty(Q_t)}\le\|\nabla\rho_2\|_{L^\infty(Q_t)^3}\int_0^t\|z_1-z_2\|_{L^\infty(\Omega)^3}\,d\tau.$$

So the map $z\mapsto\rho$ is continuous from K endowed with the topology of $X_{q,T}$ with values in $\mathcal{C}(Q_T)$.

Now, denoting $y_i(\xi)=y_i(\xi,t,x)$, we have:

$$\begin{aligned}\big|\partial_j\rho_2(x,t)-\partial_j\rho_1(x,t)\big|\le&\Big|\sum_k\big(\partial_k\rho_0\big(y_2(0)\big)-\partial_k\rho_0\big(y_1(0)\big)\big)\partial_j y_2^k(0)\Big|\\&+\Big|\sum_k\big(\partial_k\rho_0\big)\big(y_1(0)\big)\big(\partial_j y_1^k(0)-\partial_j y_2^k(0)\big)\Big|.\end{aligned}$$

Since $\rho_0\in\mathcal{C}^1(\overline{\Omega})$ and $y_2\in\mathcal{C}^1(\overline{Q}_T)^3$, we have:

$$\begin{aligned}&\|\nabla(\rho_2-\rho_1)\|_{L^\infty(Q_t)^3}\\&\qquad\le3\|\nabla y_2(0)\|_{L^\infty(Q_t)^9}\big\|\big(\nabla\rho_0\big)\big(y_1(0)\big)-\big(\nabla\rho_0\big)\big(y_2(0)\big)\big\|_{L^\infty(Q_t)^3}\\&\qquad\quad+3\|\nabla\rho_0\|_{L^\infty(\Omega)^3}\big\|\nabla\big(y_1(0)-y_2(0)\big)\big\|_{L^\infty(Q_t)^9}.\end{aligned}$$

To prove the continuity of the map $z\mapsto\rho$, since $\nabla\rho_0\in\mathcal{C}(\overline{\Omega})^3$, it is enough to prove that if $z_1\to z_2$ in $X_{q,T}$, then $y_1(0)\to y_2(0)$ in $\mathcal{C}_u\big(0,T;\mathcal{C}^1(\overline{\Omega})\big)$. To prove this property we will estimate $\|y_1(0)-y_2(0)\|_{L^\infty(Q_T)^3}$ and $\big\|\nabla\big(y_1(0)-y_2(0)\big)\big\|_{L^\infty(Q_T)^9}$ in terms of $\|z_1-z_2\|_{X_{q,T}}$.

Estimate of $\|y_2(0)-y_1(0)\|_{L^\infty(Q_t)^3}$. We have, according to (8):

$$\big|y_1^k(\tau)-y_2^k(\tau)\big|\le\int_\tau^t\big|z_1^k\big(y_1(\xi),\xi\big)-z_1^k\big(y_2(\xi),\xi\big)\big|+\big|z_1^k\big(y_2(\xi),\xi\big)-z_2^k\big(y_2(\xi),\xi\big)\big|\,d\xi.$$

Since $z_1\in L^1\big(0,T;\mathcal{C}^1(\overline{\Omega})^3\big)$, for all $x,y\in\Omega$ and almost all $t\in]0,T[$ we have:

$$|z_1(x,t)-z_1(y,t)|\le\|\nabla z_1(t)\|_{L^\infty(\Omega)^9}|x-y|.$$

In addition, taking into account that $y_2(\xi,t,.)$ is a one to one map on Ω and z_i is in $\mathcal{C}(\overline{Q})^3$, we obtain:

$$\begin{aligned}&\left|y_1^k(\tau)-y_2^k(\tau)\right|\\&\quad\le\int_\tau^t\|\nabla z_1(\xi)\|_{L^\infty(\Omega)^9}\left|y_1(\xi)-y_2(\xi)\right|d\xi+\int_\tau^t\|z_1^k(\xi)-z_2^k(\xi)\|_{L^\infty(\Omega)^3}\,d\xi.\end{aligned}$$

So, using Gronwall lemma, we obtain:

$$\|y_2(\tau)-y_1(\tau)\|_{L^\infty(Q_t)^3}\le ct\|z_1-z_2\|_{X_{q,t}}\exp\Big\{ct^{1/q'}\|z_1\|_{X_{q,t}}\Big\},$$

where q' satisfies $1/q+1/q'=1$.

Estimate of $\|\nabla\big(y_1(0)-y_2(0)\big)\|_{L^\infty(Q_t)^9}$. We have

$$\begin{aligned}\left|\partial_i y_1^k(\tau)-\partial_i y_2^k(\tau)\right|&\le\Big|\sum_\ell\int_\tau^t\left((\partial_\ell z_1^k)\big(y_1(\xi),\xi\big)-(\partial_\ell z_1^k)\big(y_2(\xi),\xi\big)\right)\partial_i y_1^\ell(\xi)\,d\xi\Big|\\&\quad+\Big|\sum_\ell\int_\tau^t\left((\partial_\ell z_1^k)\big(y_2(\xi),\xi\big)-(\partial_\ell z_2^k)\big(y_2(\xi),\xi\big)\right)\partial_i y_1^\ell(\xi)\,d\xi\Big|\\&\quad+\Big|\sum_\ell\int_\tau^t(\partial_\ell z_2^k)\big(y_2(\xi),\xi\big)\,\partial_i\big(y_1^\ell(\xi)-y_2^\ell(\xi)\big)\,d\xi\Big|.\end{aligned}$$

Since $z_1\in L^1\big(0,T;C^{1,\varepsilon_q}(\overline{\Omega})^3\big)$, for all $x,y\in\Omega$ and almost all $t\in]0,T[$ we have:

$$|\nabla z_1(x,t)-\nabla z_1(y,t)|\le b(t)|x-y|^{\varepsilon_q},$$

where $b(t)=\|\nabla z_1(t)\|_{\mathcal{C}^{\varepsilon_q}(\Omega)^9}$ is in $L^1(0,T)$. In addition, $y_2(\xi,t,.)$ is a one to one map on Ω and ∇z_i is in $L^1\big(0,T;\mathcal{C}(\overline{\Omega})^9\big)$, so we have, using the estimate of $\big|y_1(\tau)-y_2(\tau)\big|$:

$$\begin{aligned}\left|\nabla\big(y_1(\tau)-y_2(\tau)\big)\right|&\le c\Bigg(\|\nabla y_1\|_{L^\infty(Q_t)^9}\int_\tau^t\|\nabla\big(z_1-z_2\big)(\xi)\|_{L^\infty(\Omega)^9}\,d\xi\\&+\|\nabla y_1\|_{L^\infty(Q_t)^9}\int_\tau^t b(\xi)\Big(\int_\xi^t\|(z_1-z_2)(\zeta)\|_{L^\infty(\Omega)^3}\,d\zeta\\&\qquad\times\exp\Big\{c\int_0^t\|\nabla z_1(\zeta)\|_{L^\infty(\Omega)^9}d\zeta\Big\}\Big)^{\varepsilon_q}d\xi\\&+\int_\tau^t\|\nabla z_2(\xi)\|_{L^\infty(\Omega)^9}\left|\nabla\big(y_1(\xi)-y_2(\xi)\big)\right|d\xi\Bigg).\end{aligned}$$

Using the Gronwall lemma, we deduce the estimate

$$\big|\nabla\big(y_1(0)-y_2(0)\big)\big| \le c\|\nabla y_1\|_{L^\infty(Q_t)^9} \exp\Big\{ct^{1/q'}\|z_2\|_{X_{q,t}}\Big\}\Big(t^{\varepsilon_q}\|z_1-z_2\|_{X_{q,t}}^{\varepsilon_q} \times \|b\|_{L^1(0,t)} \exp\Big\{ct^{1/q'}\varepsilon_q\|z_1\|_{X_{q,t}}\Big\} + t\|z_1-z_2\|_{X_{q,t}}\Big),$$

which yields

$$\begin{aligned}&\|\nabla\big(y_1(0)-y_2(0)\big)\|_{L^\infty(Q_t)^9}\\ &\quad\le c\|\nabla y_1\|_{L^\infty(Q_t)^9} \exp\Big\{ct^{1/q'}\|z_2\|_{X_{q,t}}\Big\}\Big(t^{\varepsilon_q}\|z_1-z_2\|_{X_{q,t}}^{\varepsilon_q}\\ &\quad\times \|b\|_{L^1(0,t)} \exp\Big\{ct^{1/q'}\varepsilon_q\|z_1\|_{X_{q,t}}\Big\} + t\|z_1-z_2\|_{X_{q,t}}\Big).\end{aligned}$$

With all these estimates, we deduce the continuity of the map $z \mapsto y(0)$, and the proof of Proposition 5 is complete. □

4. Existence and uniqueness of a solution of the uncoupled equations (4)

4.1 The result.

Proposition 6. *Let $z \in \mathcal{W}_q^{2,1}(Q_T)$ satisfy $\nabla \cdot z = 0$, $z|_{t=0} = u_0$ in Ω, $z|_{\Sigma_T} = 0$. Under the hypothesis of Theorem 1, there exists a unique*

$$u \in \mathcal{W}_q^{2,1}(Q_T), \quad \nabla p \in L^q(Q_T)^3, \quad \rho \in \mathcal{C}^1(\overline{Q}_T)$$

solution of (4).

It satisfies

$$\text{(9)}\quad \begin{aligned}&|||(u,\nabla p)|||_T\\ &\le c\mathcal{M}_1^{\frac{2}{1-\alpha}}(1+Te^{T/2}+Te^{T/2}\|z\|_{L^\infty(Q_T)^3}^{\frac{2}{1-\alpha}})\Big(\|f\|_{L^q(Q_T)^3}+\|u_0\|_{W^{2-2/q,q}(\Omega)^3}\Big),\end{aligned}$$

where $\mathcal{M}_1 = (\overline{M}_3^9 + \overline{M}_4)^3\overline{M}_3^{12}$, $M_3 = \|\nabla\rho\|_{L^\infty(Q_T)^3}$, $M_4 = \|\partial_t\rho\|_{L^\infty(Q_T)}$, $\overline{M}_i = M_i + 1$, $\alpha = 3(q-2)[3(q-2)+4q]^{-1}$ and c does not depend on T, M_3 and M_4.

The proof is given in several steps.

4.2 Simplified auxiliary equations. We consider here the following problem: Find a solution $(u, \nabla p)$ of

$$\text{(10)}\qquad \begin{cases}\rho\partial_t u - \nu(\rho)\Delta u + \nabla p = f \ \text{ in } \ Q_T,\\ \nabla \cdot u = 0 \ \text{ in } \ Q_T,\\ u|_{t=0} = u_0 \ \text{ in } \ \Omega,\\ u|_{\Sigma_T} = 0.\end{cases}$$

We have the following result:

Proposition 7. *Let $\rho \in \mathcal{C}^1(\overline{Q}_T)$, $\rho(x,t) \geq M_1 > 0$ for all $(x,t) \in Q_T$. Under the hypothesis of Theorem* 1, *there exist*

$$u \in \mathcal{W}_q^{2,1}(Q_T), \quad \nabla p \in L^q(Q_T)^3,$$

solving (10).

In addition, there exists at most one solution of (10) *in the space* $\Big(\mathcal{C}_u\big(0,T;L^2(\Omega)^3\big) \cap L^2\big(0,T;H^1(\Omega)^3\big)\Big) \times H^{-1}(Q_T)^3$.

Proof: *Existence.* The existence of a solution of (10) in $\Big(L^\infty\big(0,T;H^1(\Omega)^3\big) \cap H^1\big(0,T;L^2(\Omega)^3\big)\Big) \times L^2\big(0,T;H^{-1}(\Omega)^3\big)$ is well known (see for example [6]).

Uniqueness. Let $(u_1, \nabla p_1)$ and $(u_2, \nabla p_2)$ be two solutions of (10) in $\Big(\mathcal{C}_u\big(0,T;L^2(\Omega)^3\big) \cap L^2\big(0,T;H^1(\Omega)^3\big)\Big) \times H^{-1}(Q_T)^3$. Then $u = u_1 - u_2$, $\nabla p = \nabla(p_1 - p_2)$ is a solution of

$$\begin{cases} \rho \partial_t u - \nu(\rho)\Delta u + \nabla p = 0 \ \text{ in } \ Q_T, \\ \nabla \cdot u = 0 \ \text{ in } \ Q_T, \\ u|_{t=0} = 0 \ \text{ in } \ \Omega, \\ u|_{\Sigma_T} = 0. \end{cases}$$

For all $\varphi \in \mathcal{D}(0,T;\mathcal{V})$, we have in $W^{-1,1}(0,T)$

$$\langle \rho \partial_t u, \varphi \rangle_\Omega - \langle \nu(\rho)\Delta u, \varphi \rangle_\Omega = 0.$$

Since $\langle \nu(\rho)\Delta u, \varphi \rangle_\Omega = -\int_\Omega \nabla\big(\nu(\rho)\big) \cdot \nabla u \cdot \varphi - \int_\Omega \nu(\rho)\nabla u \cdot \nabla\varphi$ is in $L^1(0,T)$, we have $\langle \rho\partial_t u, \varphi\rangle_\Omega \in L^1(0,T)$. In addition, $\varphi \mapsto \int_\Omega \nu(\rho)\nabla u \cdot \nabla\varphi$ and $\varphi \mapsto \int_\Omega \nabla\big(\nu(\rho)\big) \cdot \nabla u \cdot \varphi$ are continuous in the space $L^2(0,T;V)$ with values in $L^1(0,T)$, so $\varphi \mapsto \int_\Omega \rho\partial_t u \cdot \varphi$ is continuous in $L^2(0,T;V)$ with values in $L^1(0,T)$. Therefore, we deduce that for all $v \in L^2(0,T;V)$, we have in $L^1(0,T)$:

$$\int_\Omega \rho\partial_t u \cdot v + \int_\Omega \nabla\big(\nu(\rho)\big) \cdot \nabla u \cdot v + \int_\Omega \nu(\rho)\nabla u \cdot \nabla v = 0.$$

In particular, for $v = u$, we obtain in $L^1(0,T)$:

$$\frac{1}{2}\int_\Omega \rho\partial_t|u|^2 + \nu_1 \int_\Omega |\nabla u|^2 \leq \int_\Omega \big|\nabla\big(\nu(\rho)\big) \cdot \nabla u \cdot u\big|.$$

In addition, we have $\partial_t\big(\rho|u|^2\big) = \rho\partial_t|u|^2 + \partial_t\rho|u|^2$, so we obtain

$$\int_\Omega \partial_t\big(\rho|u|^2\big) + \nu_1\int_\Omega |\nabla u|^2 \leq c\int_\Omega \rho|u|^2,$$

where c depends only on ν, ρ, and we deduce from the Gronwall lemma that $u = 0$. The De Rham theorem implies that $\nabla p = 0$, and the uniqueness follows.

Regularity of such a solution. Choose p such that $\int_\Omega p/\nu(\rho) = 0$. Dividing by $\nu(\rho) \geq \nu_1 > 0$ and denoting $P = p/\nu(\rho)$, $\lambda = \rho/\nu(\rho)$, the equation (10) can be rewritten in the following form

$$\begin{cases} \lambda \partial_t u - \Delta u + \nabla P = \dfrac{f}{\nu(\rho)} - \dfrac{\nu'(\rho)\nabla\rho}{\nu(\rho)} P, \\ \nabla \cdot u = 0, \\ u|_{t=0} = u_0, \\ u|_{\Sigma_T} = 0, \end{cases}$$

with $\lambda \in \mathcal{C}^1(\overline{Q}_T)$ and $\lambda \geq \lambda_1 > 0$.

Let us consider the following equation:

$$\begin{cases} \lambda \partial_t u' - \Delta u' + \nabla P' = \dfrac{f}{\nu(\rho)} - \dfrac{\nu'(\rho)\nabla\rho}{\nu(\rho)} P, \\ \nabla \cdot u' = 0, \\ u'|_{t=0} = u_0, \\ u'|_{\Sigma_T} = 0, \end{cases} \tag{11}$$

where $P \in L^2(Q_T)$ is defined above. Since $f \in L^2(Q_T)^3$, there exists a unique solution $(u', \nabla P')$ of (11) in $\Big(\mathcal{C}_u\big(0,T;L^2(\Omega)^3\big) \cap L^2\big(0,T;H^1(\Omega)^3\big)\Big) \times H^{-1}(Q_T)^3$. In addition (cf. O.A. Ladyzenskaya and V.A. Solonnikov [5]), $u' \in \mathcal{W}_2^{2,1}(Q_T)$, $\nabla P' \in L^2(Q_T)^3$. Now, since $\big(u, \nabla P\big)$ is solution of (11) we deduce that $u \in \mathcal{W}_2^{2,1}(Q_T)$, $\nabla P \in L^2(Q_T)^3$, and therefore $\nabla p \in L^2(Q_T)^3$. Then, from Lemma 9 (in appendix) we deduce that $p \in L^{\sigma_0}(Q_T)$, where $\sigma_0 = \min(q, 8/3)$. Therefore, since $f \in L^q(Q_T)^3$, we deduce from the equation (11) that $u \in \mathcal{W}_{\sigma_0}^{2,1}(Q_T)$ and $\nabla p \in L^{\sigma_0}(Q_T)^3$ (see O.A. Ladyzenskaya, V.A. Solonnikov [5]). Repeating this process a finite number of times, we obtain Proposition 7. $\square$

4.3 Auxiliary equations. We consider now the following problem: Find a solution $(u, \nabla p)$ of

$$\begin{cases} \rho \partial_t u - \nabla \cdot \big(\nu(\rho)(\nabla u + {}^t\nabla u)\big) + \nabla p = f \ \text{ in } \ Q_T, \\ \nabla \cdot u = 0 \ \text{ in } \ Q_T, \\ u|_{t=0} = u_0 \ \text{ in } \ \Omega, \\ u|_{\Sigma_T} = 0. \end{cases} \tag{12}$$

We have the following result:

Proposition 8. *Under the hypothesis of Proposition 7, there exist*

$$u \in \mathcal{W}_q^{2,1}(Q_T), \ \nabla p \in L^q(Q_T)^3,$$

solving (12).

It satisfies

$$|||(u,\nabla p)|||_T \leq \mathcal{M}_1\big(\|f\|_{L^q(Q_T)^3} + \|u\|_{L^q(Q_T)^3} + \|u_0\|_{W^{2-2/q,q}(\Omega)^3}\big), \tag{13}$$

where $\mathcal{M}_1 = c\big(\overline{M}_3^9 + M_4\big)\overline{M}_3^{12}$, $\overline{M}_i = (M_i + 1)$, $1 \leq i \leq 4$,

$$|||(u,\nabla p)|||_t \leq c\mathcal{M}_1^2\big(\|f\|_{L^q(Q_t)^3} + \|u_0\|_{W^{2-2/q,q}(\Omega)^3}\big)\exp\{c\mathcal{M}_1 t\}, \tag{14}$$

for all t, $0 \leq t \leq T$, *where* c *depends only on* ν, M_1, M_2, M_3 *and* M_4.

In addition, there exists at most one solution of (12) *in the space* $\Big(\mathcal{C}_u\big(0,T;L^2(\Omega)^3\big) \cap L^2\big(0,T;H^1(\Omega)^3\big)\Big) \times H^{-1}(Q_T)^3$.

Proof: *Existence.* The existence of a solution of (12) in $\Big(L^\infty\big(0,T;H^1(\Omega)^3\big) \cap H^1\big(0,T;L^2(\Omega)^3\big)\Big) \times L^2\big(0,T;H^{-1}(\Omega)^3\big)$ is known (see for example [6]). As in Proposition 7, we can prove that there exists at most one solution of (12) in $\Big(\mathcal{C}_u\big(0,T;L^2(\Omega)^3\big) \cap L^2\big(0,T;H^1(\Omega)^3\big)\Big) \times H^{-1}(Q_T)^3$.

Regularity of this solution. The first equation of (12) can be written in the form

$$\rho\partial_t u - \nu(\rho)\Delta u + \nabla p = f + \nabla\big(\nu(\rho)\big)\big(\nabla u + {}^t\nabla u\big).$$

Since $f + \nabla\big(\nu(\rho)\big)\big(\nabla u + {}^t\nabla u\big) \in L^2(Q_T)^3$, there exists (cf. Proposition 7) one solution $u' \in \mathcal{W}_2^{2,1}(Q_T)$, $\nabla p' \in L^2(Q_T)^3$ of

$$\begin{cases} \rho\partial_t u' - \nu(\rho)\Delta u' + \nabla p' = f + \nabla\big(\nu(\rho)\big)\big(\nabla u + {}^t\nabla u\big), \\ \nabla \cdot u' = 0, \\ u'|_{t=0} = u_0, \\ u'|_{\Sigma_T} = 0, \end{cases} \tag{15}$$

where $(u,\nabla p)$ is the solution of (12). In addition, this solution is unique in the space $\Big(\mathcal{C}_u\big(0,T;L^2(\Omega)^3\big) \cap L^2\big(0,T;H^1(\Omega)^3\big)\Big) \times H^{-1}(Q_T)^3$. Since the solution $(u,\nabla p)$ of (12) is a solution of (15), we deduce that the solution of (12) verifies $u \in \mathcal{W}_2^{2,1}(Q_T)$, $\nabla p \in L^2(Q_T)^3$. Therefore (cf. Lemma 9), there exists σ_0, $2 < \sigma_0 \leq q$ such that $u \in L^{\sigma_0}\big(0,T;W^{1,\sigma_0}(\Omega)^3\big)$. So we deduce from (15), since $f \in L^q(Q_T)^3$, that $u \in \mathcal{W}_{\sigma_0}^{2,1}(Q_T)$ and $\nabla p \in L^{\sigma_0}(Q_T)^3$. Repeating this process a finite number of times (until $\sigma_m = q$), we obtain the regularity.

Estimates. Choose p such that $\int_\Omega p = 0$. Then setting $P = p/\nu(\rho)$, (12) can be rewritten in the following way:

$$\begin{cases} \dfrac{\rho}{\nu(\rho)}\partial_t u - \Delta u + \nabla P = \dfrac{f}{\nu(\rho)} - \dfrac{\nu'(\rho)\nabla\rho}{\nu(\rho)}P + \dfrac{\nu'(\rho)\nabla\rho}{\nu(\rho)} \cdot (\nabla u + {}^t\nabla u), \\ \nabla \cdot u = 0, \\ u|_{t=0} = u_0, \\ u|_{\Sigma_T} = 0. \end{cases}$$

Since $\nu_2 \geq \nu(\rho) \geq \nu_1 > 0$ and $\nu_2' \geq \nu'(\rho)$, we have the following estimate (cf. O.A. Ladyzenskaya, V.A. Solonnikov [5]),

$$\begin{aligned} |||(u, \nabla P)|||_T \leq c\big(M_4 + \overline{M}_3^9\big)\Big(&\|f\|_{L^q(Q_T)^3} + M_3\|P\|_{L^q(Q_T)} + M_3\|\nabla u\|_{L^q(Q_T)^9} \\ &+ \|u\|_{L^q(Q_T)^3} + \|u_0\|_{W^{2-2/q,q}(\Omega)^3}\Big), \end{aligned}$$

where c depends on ν, M_1 and M_2 only. Then we obtain

$$\begin{aligned} |||(u, \nabla p)|||_T \leq c\big(M_4 + \overline{M}_3^9\big)\big(&\|f\|_{L^q(Q_T)^3} + M_3\|p\|_{L^q(Q_T)} + M_3\|\nabla u\|_{L^q(Q_T)^9} \\ &+ \|u\|_{L^q(Q_T)^3} + \|u_0\|_{W^{2-2/q,q}(\Omega)^3}\big), \end{aligned}$$

where c depends on ν, M_1 and M_2 only.

Using (15) we have (cf. Lemma 9):

$$\|p\|_{L^q(Q_t)} \leq c\overline{M}_3^3\big(\|f\|_{L^q(Q_t)^3} + \overline{M}_3\|\nabla u\|_{L^q(Q_t)^9} + \|\nabla u\|_{L^q(\Sigma_t)^9}\big),$$

so we obtain

$$\begin{aligned} |||(u, \nabla p)|||_T \leq &A_2\big(\|f\|_{L^q(Q_T)^3} + \|\nabla u\|_{L^q(\Sigma_T)^3}\big) + A_1\|\nabla u\|_{L^q(Q_T)^9} \\ &+ c\big(M_4 + \overline{M}_3^9\big)\big(\|u\|_{L^q(Q_T)^3} + \|u_0\|_{W^{2-2/q,q}(\Omega)^3}\big) \end{aligned} \tag{16}$$

with

$$A_1 = c\big(\overline{M}_4 + \overline{M}_3^9\big)\overline{M}_3^5$$
$$A_2 = c\big(\overline{M}_4 + \overline{M}_3^9\big)\overline{M}_3^4,$$

where c depends on ν, M_1 and M_2 only.

Using the following interpolation inequalities (cf. O.A. Ladyzenskaya and V.A. Solonnikov [5]), since $(a^q + b^q) \leq (a+b)^q$ we have

$$\|\nabla u\|_{L^q(Q_T)^9} \leq \alpha_1\|\nabla(\nabla u)\|_{L^q(Q_T)^{27}} + c\alpha_1^{-1}\|u\|_{L^q(Q_T)^3},$$

$$\|\nabla u\|_{L^q(\Sigma_T)^9} \leq \alpha_2\|\nabla(\nabla u)\|_{L^q(Q_T)^{27}} + c\alpha_2^{-\frac{q+1}{q-1}}\|u\|_{L^q(Q_T)^3},$$

for all $\alpha_i \in]0,1]$, and taking $\alpha_1 = \big(4A_1\big)^{-1}$ and $\alpha_2 = \big(4A_2\big)^{-1}$, we obtain:

$$A_1\|\nabla u\|_{L^q(Q_T)^9} \le \frac{1}{4}\|\nabla(\nabla u)\|_{L^q(Q_T)^{27}} + cA_1^2\|u\|_{L^q(Q_T)^3},$$
$$A_2\|\nabla u\|_{L^q(\Sigma_T)^9} \le \frac{1}{4}\|\nabla(\nabla u)\|_{L^q(Q_T)^{27}} + cA_2^{2q/(q-1)}\|u\|_{L^q(Q_T)^3},$$

where c depends on Ω and q only. With these estimates, (16) gives:

$$|||(u,\nabla p)|||_T \le c\Big(A_2\|f\|_{L^q(Q_T)^3} + A_1^2\|u\|_{L^q(Q_T)^3} + A_2^{2q/(q-1)}\|u\|_{L^q(Q_T)^3} + \big(M_4 + \overline{M}_3^9\big)\big(\|u\|_{L^q(Q_T)^3} + \|u_0\|_{W^{2-2/q,q}(\Omega)^3}\big)\Big).$$

Now, since A_1^2, A_2 and $A_2^{2q/(q-1)}$ are smaller than $c\big(\overline{M}_3^9 + \overline{M}_4\big)^3\overline{M}_3^{12}$, we deduce from the previous inequality the estimate (13).

To prove the estimate (14), let

$$y(t) = \int_0^t \|u(\tau)\|^q_{L^q(\Omega)^3}\,d\tau = \|u\|^q_{L^q(Q_t)^3}.$$

We have $y \in W^{1,1}(0,T)$, $y(0)=0$ and $y'(t) = \|u(t)\|_q^q$. In addition, for all $t' \le t$ we have:

$$\begin{aligned}
y'(t') &= \int_0^{t'} \frac{d}{d\tau}\|u(\tau)\|_q^q\,d\tau + \|u_0\|_q^q = \int_0^{t'}\int_\Omega \frac{d}{d\tau}\big(|u(\tau)|^2\big)^{\frac{q}{2}}\,d\tau + \|u_0\|_q^q\\
&\le q\|\partial_t u\|_{L^q(Q_{t'})^3}\|u\|^{q-1}_{L^q(Q_{t'})^3} + \|u_0\|_q^q\\
&\le q\mathcal{M}_1\|u\|^q_{L^q(Q_{t'})^3} + q\mathcal{M}_1\big(\|f\|_{L^q(Q_t)^3} + \|u_0\|_{W^{2-2/q,q}(\Omega)^3}\big)\|u\|^{q-1}_{L^q(Q_{t'})^3}\\
&\qquad + \|u_0\|^q_{W^{2-2/q,q}(\Omega)^3}.
\end{aligned}$$

Since $\mathcal{M}_1 > 1$, using Young's inequality we obtain:

$$y'(t') \le (2q-1)\mathcal{M}_1 y(t') + q^2\mathcal{M}_1\big(\|f\|_{L^q(Q_t)^3} + \|u_0\|_{W^{2-2/q,q}(\Omega)^3}\big)^q.$$

Integrating this equation from 0 to t we have:

$$y(t) \le q^2\mathcal{M}_1\big(\|f\|_{L^q(Q_t)^3} + \|u_0\|_{W^{2-2/q,q}(\Omega)^3}\big)^q \exp\{(2q-1)\mathcal{M}_1 t\}.$$

Now, taking into account that $\mathcal{M}_1 > 1$ we obtain:

$$\|u\|_{L^q(Q_t)^3} \le c\mathcal{M}_1\big(\|f\|_{L^q(Q_t)^3} + \|u_0\|_{W^{2-2/q,q}(\Omega)^3}\big)\exp\{c\mathcal{M}_1 t\}.$$

Using this estimate, (13) gives

$$|||(u,\nabla p)|||_t \le c\mathcal{M}_1^2\big(\|f\|_{L^q(Q_t)^3} + \|u_0\|_{W^{2-2/q,q}(\Omega)^3}\big)\exp\{c\mathcal{M}_1 t\},$$

and the proof is complete. $\square$

4.4 Proof of Proposition 6.

Existence of a regular solution. We prove the existence by successive approximations. Let $u^0 = 0$ and for all $m \geq 1$:

(17)
$$\begin{cases} \rho\partial_t u^m - \nabla \cdot \big(\nu(\rho)(\nabla u^m + {}^t\nabla u^m)\big) + \nabla p^m = \rho f - \rho\big(z \cdot \nabla\big)u^{m-1} \ \text{ in } \ Q_T, \\ \partial_t\rho + z \cdot \nabla\rho = 0 \ \text{ in } \ Q_T, \\ \nabla \cdot u^m = 0 \ \text{ in } \ Q_T, \\ u^m|_{t=0} = u_0 \ \text{ in } \ \Omega, \\ u^m|_{\Sigma_T} = 0, \\ \rho|_{t=0} = \rho_0 \ \text{ in } \ \Omega. \end{cases}$$

It is known (cf. Proposition 8) that there exists a unique solution of (17). Denoting $w^m = u^m - u^{m-1}$, $\nabla P^m = \nabla(p^m - p^{m-1})$ and $\mathcal{W}_m(t) = |||(w^m, \nabla p^m)|||_t$, we deduce from (14) the following estimate

$$\begin{aligned} \mathcal{W}_m^q(t) &\leq c\|\nabla w^{m-1}\|^q_{L^q(Q_t)^3} \leq c\int_0^t \|w^{m-1}\|^q_{W^{2,q}(\Omega)^3}\, d\tau \\ &\leq c\int_0^t \mathcal{W}^q_{m-1}(\tau)\, d\tau \leq c^{m-1}\frac{t^{m-1}}{(m-1)!}\mathcal{W}_1^q(t), \end{aligned}$$

which implies the convergence of the series $\sum \mathcal{W}_m(t)$ for all $t \leq T$. From this, it follows the convergence of u^m in $\mathcal{W}_q^{2,1}(Q_T)$ and ∇p^m in $L^q(Q_T)^3$.

The uniqueness of a such solution is obvious.

Estimation. We have the following estimate of $|||(u, \nabla p)|||_T$ given in Proposition 8

$$|||(u, \nabla p)|||_T \leq \mathcal{M}_1\big(F + \|u\|_{L^q(Q_T)^3} + \|\big(z \cdot \nabla\big)u\|_{L^q(Q_T)^3}\big), \tag{18}$$

where $F = \|f\|_{L^q(Q_t)^3} + \|u_0\|_{W^{2-2/q,q}(\Omega)^3}$.

Now, let us estimate each term of the right hand side of this inequality. Multiplying the first equation of (4) by u and integrating on Ω, we obtain

$$\int_\Omega \rho\big[\frac{1}{2}\partial_t(u^2) + (z \cdot \nabla)u \cdot u\big] + \int_\Omega \nu(\rho)\big(\nabla u + {}^t\nabla u\big) \cdot \nabla u = \int_\Omega \rho f \cdot u.$$

Since $\nu(\rho) \geq \nu_1 > 0$, we obtain, summing this equation with the transport equation multiplied by $(1/2)|u|^2$:

$$\frac{d}{dt}\int_\Omega \rho|u|^2 + 2\nu_1\int_\Omega |\nabla u|^2 \leq 2\int_\Omega \rho f \cdot u.$$

So we deduce the following estimate:

$$\Big(\int_\Omega |u|^2\Big)(t) \leq ce^t\Big(\int_0^t \|f\|_2^2\, d\tau + \|u_0\|_2^2\Big),$$

where c depends on M_1 and M_2 only.

Now, using Hölder inequality, we have:

$$\Big(\int_\Omega |u|^2\Big)(t) \le ce^t\Big(\int_0^t \|f\|_q^2\,d\tau + \|u_0\|_q^2\Big) \\ \le ce^t\Big(\int_0^t \|f\|_q^q\,d\tau\Big)^{2/q} + ce^t\|u_0\|_q^2,$$

and therefore

$$\|u\|_2 \le ce^{t/2}\big(\|f\|_{L^q(Q_T)^3} + \|u_0\|_q\big). \tag{19}$$

Using the fact that

$$\|u\|_q \le c\big(\|u\|_{W^{2,q}(\Omega)^3}\big)^\alpha \|u\|_2^{1-\alpha}$$

with $\alpha = 3(q-2)[3(q-2)+4q]^{-1}$ (cf. O.A. Ladyzenskaya and V.A. Solonnikov [5]), the previous estimate gives:

$$\|u\|_{L^q(Q_T)^3} \le c|||(u,\nabla p)|||_T^\alpha\Big(\int_0^T \|u\|_2^q\,dt\Big)^{(1-\alpha)/q}. \tag{20}$$

Since (19) gives

$$\Big(\int_0^T \|u\|_2^q\Big)^{1/q} \le cTe^{T/2}F,$$

we obtain from (20):

$$\|u\|_{L^q(Q_T)^3} \le c|||(u,\nabla p)|||_T^\alpha (Te^{T/2}F)^{1-\alpha}. \tag{21}$$

Now, to estimate the last term of the right hand of (18), we remark that

$$\|(z\cdot\nabla)u\|_{L^q(Q_T)^3} \le \|z\|_{L^\infty(Q_T)^3}\|\nabla u\|_{L^q(Q_T)^9}.$$

Since

$$\|\nabla u\|_{L^q(Q_T)^9} \le c\big(\|u\|_{L^q(0,T;W^{2,q}(\Omega)^3)}\big)^{1/2}\|u\|_{L^q(Q_T)^3}^{1/2},$$

we obtain

$$\|(z\cdot\nabla)u\|_{L^q(Q_T)^3} \le c\|z\|_{L^\infty(Q_T)^3}|||(u,\nabla p)|||_T^{1/2}\|u\|_{L^q(Q_T)^3}^{1/2} \\ \le c\|z\|_{L^\infty(Q_T)^3}|||(u,\nabla p)|||_T^{\frac{\alpha+1}{2}}(Te^{T/2}F)^{\frac{1-\alpha}{2}}.$$

We deduce from this estimate and from the estimates (18) and (21)

$$|||(u,\nabla p)|||_T \le c\mathcal{M}_1\big(F + |||(u,\nabla p)|||_T^\alpha(Te^{T/2}F)^{1-\alpha} \\ + \|z\|_{L^\infty(Q_T)^3}|||(u,\nabla p)|||_T^{\frac{\alpha+1}{2}}(Te^{T/2}F)^{\frac{1-\alpha}{2}}\big), \tag{22}$$

where $\mathcal{M}_1 = c\big(\overline{M}_3^9 + \overline{M}_4\big)^3\overline{M}_3^{12}$. Using the following Young's inequalities:

$$\mathcal{M}_1 |||(u,\nabla p)|||_T^{\alpha}(Te^{T/2}F)^{1-\alpha} \le \alpha\varepsilon|||(u,\nabla p)|||_T + (1-\alpha)\varepsilon^{-\frac{\alpha}{1-\alpha}}Te^{T/2}F\mathcal{M}_1^{\frac{1}{1-\alpha}},$$

and

$$\mathcal{M}_1\|z\|_{L^\infty(Q_T)^3}|||(u,\nabla p)|||_T^{\frac{\alpha+1}{2}}(TFe^{T/2})^{\frac{1-\alpha}{2}} \le \frac{1+\alpha}{2}\varepsilon|||(u,\nabla p)|||_T + \frac{1-\alpha}{2}\varepsilon^{-\frac{1+\alpha}{1-\alpha}}\big(\mathcal{M}_1\|z\|_{L^\infty(Q_T)^3}\big)^{\frac{2}{1-\alpha}}TFe^{T/2},$$

with ε small enough, we deduce from (22) the following estimate, since $\mathcal{M}_1 \ge 1$:

$$|||(u,\nabla p)|||_T \le cF\mathcal{M}_1^{\frac{2}{1-\alpha}}\big(1 + Te^{T/2} + Te^{T/2}\|z\|_{L^\infty(Q_T)^3}^{\frac{2}{1-\alpha}}\big),$$

and the proof is complete. □

5. Proof of Theorem 1

As we have seen (cf. Proposition 6), for all $z \in \mathcal{W}_q^{2,1}(Q_T)$ satisfying $z(0) = u_0$, $z|_{\Sigma_T} = 0$ and $\nabla\,.\,z = 0$, there exists a unique solution $u \in \mathcal{W}_q^{2,1}(Q_T)$, $\nabla p \in L^q(Q_T)^3$, $\rho \in \mathcal{C}^1(\overline{Q}_T)$ of (4).

Local existence. This proof is based on the Schauder theorem that can be found for example in N. Dunford, J.T. Schwartz [2, Theorem 5, p. 456].

In the first step, let us prove that there exist $T_M > 0$ and a convex compact subset K of X_{q,T_M} such that $z \mapsto u$ maps K into K.

For all $t \le T$, we have:

$$|||(u,\nabla p)|||_t \le cF\mathcal{M}_1^{\frac{2}{1-\alpha}}\big(1 + te^{t/2} + te^{t/2}|||z|||_t^{\frac{2}{1-\alpha}}\big)$$

with $\mathcal{M}_1 = c\big(\overline{M_3}^9 + \overline{M_4}\big)^3\overline{M_3}^{12}$ and

$$\overline{M_3} = 1 + M_3 \le 1 + \sqrt{3}\|\nabla\rho_0\|_{L^\infty(\Omega)^3}\exp\{ct^{1/q'}|||z|||_t\},$$
$$\overline{M_4} = 1 + M_4 \le 1 + \sqrt{3}\|\nabla\rho_0\|_{L^\infty(\Omega)^3}\|z\|_{L^\infty(Q_t)^3}\exp\{ct^{1/q'}|||z|||_t\},$$

where q' satisfies $1/q + 1/q' = 1$. Let q_1 be a real, $3 < q_1 < q$. Then

$$\|z\|_{L^\infty(Q_t)^3} \le c\sup_{0\le\tau\le t}\|z\|_{W^{2-2/q_1,q_1}(\Omega)^3} \le c\big(\|u_0\|_{W^{2-2/q_1,q_1}(\Omega)^3} + \|z\|_{q_1,Q_t}^{(2,1)}\big),$$

where c is a constant which does not depend on t (see V.A. Solonnikov [11]). Moreover

$$\|z\|_{q_1,Q_t}^{(2,1)} \le ct^{(q-q_1)/qq_1}\|z\|_{q,Q_t}^{(2,1)},$$

so, since $\|u_0\|_{W^{2-2/q_1,q_1}(\Omega)^3} \le c\|u_0\|_{W^{2-2/q,q}(\Omega)^3}$,

$$\overline{M_4} \le 1 + c\sqrt{3}\|\nabla\rho_0\|_{L^\infty(\Omega)^3}\big(\|u_0\|_{W^{2-2/q,q}(\Omega)^3} + t^{(q-q_1)/qq_1}\|z\|^{(2,1)}_{q,Q_t}\big)\exp\{ct^{1/q'}|||z|||_t\}.$$

Therefore we have

$$|||(u,\nabla p)|||_t \le cH(t,|||z|||_t),$$

where $H(t,a)$ is continuous function in (t,a) defined by

$$H(t,a) = F\Big(1+\sqrt{3}\|\nabla\rho_0\|_{L^\infty(\Omega)^3}(1+\|u_0\|_{W^{2-2/q,q}(\Omega)^3} + t^{(q-q_1)/qq_1}a)\Big)^{\frac{42}{1-\alpha}} \times \exp\{\frac{42}{1-\alpha}ct^{\frac{1}{q'}}a\}\big(1+te^{t/2}+te^{t/2}a^{\frac{2}{1-\alpha}}\big).$$

Since $H(0,a) = H(0,a)$ for all $a \ge 0$, for $M = H(0,0)$, there exists T_M such that, if $|||z|||_{T_M} \le M$, then $|||(u,\nabla p)|||_{T_M} \le M$.

Let us denote

$$K = \{u \in \mathcal{W}^{2,1}_q(Q_{T_M}), u(0) = u_0,\ u|_{\Sigma_{T_M}} = 0,\ \nabla\,.\,u = 0, |||u|||_{T_M} \le M\}.$$

Then K is a convex compact subset of X_{q,T_M}, and $z \mapsto u$ maps K into K.

In the second step, let us prove that $z \mapsto u$ is continuous from K endowed with the topology of X_{q,T_M} into itself. Let z_1 and z_2 be two elements of K. Then we obtain, setting $z = z_1 - z_2$, $u = u_1 - u_2$, $\nabla p = \nabla(p_1 - p_2)$ and $\rho = \rho_1 - \rho_2$:

$$\begin{cases} \rho_1\partial_t u - \nabla\,.\,\big(\nu(\rho_1)(\nabla u + {}^t\nabla u)\big) + \rho_1\big(z_1\,.\,\nabla\big)u + \nabla p = G \text{ in } Q_T,\\ \nabla\,.\,u = 0 \text{ in } Q_T,\\ \partial_t\rho + z_1\,.\,\nabla\rho = -z\,.\,\nabla\rho_2 \text{ in } Q_T,\\ u|_{t=0} = 0 \text{ in } \Omega,\\ u|_{\Sigma_T} = 0,\\ \rho|_{t=0} = 0 \text{ in } \Omega, \end{cases}$$

where

$$G = \rho f - \rho\partial_t u_2 + \nabla\,.\,\big(\big(\nu(\rho_1)-\nu(\rho_2)\big)\big(\nabla u_2 + {}^t\nabla u_2\big)\big) - \rho_1\big(z\,.\,\nabla\big)u_2 + \rho\big(z_1\,.\,\nabla\big)u_2.$$

We deduce from (9) the following estimate:

$$|||(u,\nabla p)|||_{T_M} \le c\|G\|_{L^q(Q_{T_M})^3}\mathcal{M}_1^{\frac{2}{1-\alpha}}\big(1+Te^{T/2}+Te^{T/2}|||z_1|||_{T_M}^{\frac{2}{1-\alpha}}\big),$$

where $\|G\|_{L^q(Q_{T_M})^3}$ verifies

$$\begin{aligned}\|G\|_{L^q(Q_{T_M})^3} \leq & \|\rho\|_{L^\infty(Q_{T_M})}\Big(\|f\|_{L^q(Q_{T_M})^3} + \|\partial_t u_2\|_{L^q(Q_{T_M})^3} \\ & \qquad + \|z_1\|_{L^\infty(Q_{T_M})^3}\|\nabla u_2\|_{L^q(Q_{T_M})^9}\Big) \\ & + \|\nabla \cdot \big(\big(\nu(\rho_1) - \nu(\rho_2)\big)\big(\nabla u_2 + {}^t\nabla u_2\big)\big)\|_{L^q(Q_{T_M})^3} \\ & + \|\rho_1\|_{L^\infty(Q_{T_M})}\|z\|_{L^\infty(Q_{T_M})^3}\|\nabla u_2\|_{L^q(Q_{T_M})^9}.\end{aligned}$$

As we have proved in Proposition 5, if $z_2 \to z_1$ in X_{q,T_M}, then $\rho_2 \to \rho_1$ in the space $\mathcal{C}_u\big(0, T_M; \mathcal{C}^1(\overline{\Omega})\big)$. So, $\nu(\rho_2) \to \nu(\rho_1)$ in $\mathcal{C}_u\big(0, T_M; \mathcal{C}^1(\overline{\Omega})\big)$. From this, we obtain that if $z_2 \to z_1$ in X_{q,T_M}, then $\|G\|_{L^q(Q_{T_M})^3} \to 0$ and therefore $|||(u, \nabla p)|||_{T_M} \to 0$. This proves that the map $z \mapsto u$ is continuous. Using the Schauder fixed point theorem, we obtain that there exist $u \in K$ and $\nabla p \in L^q(Q_{T_M})^3$ solving (1)–(3).

Global existence. We have

$$|||(u, \nabla p)|||_T \leq cF\mathcal{M}_1^{\frac{2}{1-\alpha}}\big(1 + Te^{T/2} + Te^{T/2}\|z\|_{L^\infty(Q_T)^3}^{\frac{2}{1-\alpha}}\big).$$

Let $M > 0$ and suppose $|||z|||_T \leq M$. Then there exists $R > 0$ such that if $F = \|f\|_{L^q(Q_T)^3} + \|u_0\|_{W^{2-2/q,q}(\Omega)^3} \leq R$, then $|||u|||_T \leq M$. As in the proof of the local existence,

$$K = \{u \in \mathcal{W}_q^{2,1}(Q_T), u(0) = u_0,\ u|_{\Sigma_T} = 0,\ \nabla \cdot u = 0, |||u|||_T \leq M\}$$

is a convex compact subset of $X_{q,T}$, and $z \mapsto u$ maps continuously K into K. Therefore we deduce the existence of $u \in K$ and $\nabla p \in L^q(Q_T)^3$ solving (1)–(3). $\square$

Appendix

Lemma 9. *Let $f \in L^q(Q_T)^3$ and let $(u, \nabla p)$ be the unique solution of* (10) *satisfying*

$$u \in L^2\big(0, T; H^2(\Omega)^3\big),\ \partial_t u \in L^2(Q_T)^3,\ \nabla p \in L^2(Q_T)^3.$$

Suppose that $u \in L^s\big(0, T; W^{2,s}(\Omega)^3\big) \cap W^{1,s}\big(0, T; L^s(\Omega)^3\big)$ and $\nabla p \in L^s(Q_T)^3$ with $2 \leq s < q$. Choosing p such that $\int_\Omega p = 0$, there exists σ, $s < \sigma \leq q$ defined by

$$\sigma = \begin{cases} q & \text{if } s \geq 5, \\ \min\big(q, \frac{4s}{5-s}\big) & \text{if } 2 \leq s < 5, \end{cases}$$

such that

$$u \in L^\sigma\big(0, T; W^{1,\sigma}(\Omega)^3\big),\ \nabla u|_{\Sigma_T} \in L^\sigma(\Sigma_T)^9\ \ p \in L^\sigma(Q_T).$$

Moreover we have

$$\|p\|_{L^\sigma(Q_t)} \leq c_3\big(\|f\|_{L^\sigma(Q_t)^3} + \|\nabla u\|_{L^\sigma(Q_t)^9} + \|\nabla u\|_{L^\sigma(\Sigma_t)^9}\big),$$

where

$$c_3 = c\big[M_1^{-2}(M_2+1)M_3 + M_1^{-1}\big]M_2\big(1 + M_1^{-4}M_2^2M_3^2\big).$$

□

A proof of this lemma is given by M. Kabbaj [4]. □

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