On the quantification of uniform properties

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Abstract. Approach spaces ([4], [5]) turned out to be a natural setting for the quantification of topological properties. Thus a measure of compactness for approach spaces generalizing the well-known Kuratowski measure of non-compactness for metric spaces was defined ([3]). This article shows that approach uniformities (introduced in [6]) have the same advantage with respect to uniform concepts: they allow a nice quantification of uniform properties, such as total boundedness and completeness.

Keywords: uniform space, approach uniform space, totally bounded, precompact, complete, measure of total boundedness, measure of completeness

Classification: 54E15, 54B30, 54E35

1. Introduction

Suppose that (X, d) is a metric space and that $A \subset X$, then

$$\boldsymbol{\mu}_K(A) := \inf \left\{ \varepsilon \in \mathbf{R}^+ \mid \exists\, X_1, \dots, X_n \subset X : \max_{i=1}^{n} diam(X_i) \leq \varepsilon, A \subset \bigcup_{i=1}^{n} X_i \right\}$$

is called the *Kuratowski measure of non-compactness* of A. An interesting variant of this measure is the so-called *Hausdorff measure of non-compactness* defined by

$$\boldsymbol{\mu}_H(A) := \inf \left\{ \varepsilon \in \mathbf{R}^+ \mid \exists\, x_1, \dots, x_n \in X : A \subset \bigcup_{i=1}^{n} B(x_i, \varepsilon) \right\}.$$

It is easily seen that for any $A \subset X$ we have $\boldsymbol{\mu}_H(A) \leq \boldsymbol{\mu}_K(A) \leq 2 \cdot \boldsymbol{\mu}_H(A)$.

These measures express *to what extent* a metric space is compact. The Hausdorff measure can be extended to arbitrary approach spaces ([5]). This article shows that, in the setting of approach uniformities, the same can be done for total boundedness, completeness and uniform connectedness.

Recall that an approach uniform space (X, Γ) is a set X together with an ideal Γ of functions from $X \times X$ into $[0, \infty]$, satisfying the following conditions:

(AU1) $\forall \gamma \in \Gamma, \forall x \in X : \gamma(x, x) = 0$;

(AU2) $\forall \xi \in [0, \infty]^{X \times X} : \left(\forall \varepsilon > 0, \forall N < \infty : \exists\, \gamma_\varepsilon^N \in \Gamma \text{ s.t. } \xi \wedge N \leq \gamma_\varepsilon^N + \varepsilon\right) \Rightarrow \xi \in \Gamma$;

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(AU3) $\forall \gamma \in \Gamma, \forall N < \infty, \exists \gamma^N \in \Gamma$ s.t. $\forall x, y, z \in X : \gamma(x,z) \wedge N \leq \gamma^N(x,y) + \gamma^N(y,z)$;

(AU4) $\forall \gamma \in \Gamma : \gamma^s \in \Gamma$.

Equivalently, an approach uniformity can be described with a *uniform tower*, i.e. a family of filters $(\mathcal{U}_\varepsilon)_{\varepsilon \in \mathbf{R}^+}$ on $X \times X$, such that

(UT1) $\forall \varepsilon \in \mathbf{R}^+, \forall U \in \mathcal{U}_\varepsilon : \Delta_X \subset U$;

(UT2) $\forall \varepsilon \in \mathbf{R}^+, \forall U \in \mathcal{U}_\varepsilon : U^{-1} \in \mathcal{U}_\varepsilon$;

(UT3) $\forall \varepsilon, \varepsilon' \in \mathbf{R}^+ : \mathcal{U}_\varepsilon \circ \mathcal{U}_{\varepsilon'} \supset \mathcal{U}_{\varepsilon+\varepsilon'}$;

(UT4) $\forall \varepsilon \in \mathbf{R}^+ : \mathcal{U}_\varepsilon = \bigcup_{\alpha > \varepsilon} \mathcal{U}_\alpha$

or equivalently, a family $(\mathcal{U}_\varepsilon)_{\varepsilon \in \mathbf{R}^+}$ of semi-uniformities, satisfying (UT3) and (UT4).

If d is a pseudo-metric, then the collection $\Gamma(d) := \{\gamma \mid \gamma \leq d\}$ is an approach uniformity. It is referred to as the *metric approach uniformity* induced by d.

If $\mathcal{U}$ is a uniformity, then the trivial tower $(\mathcal{U})_\varepsilon$ ($\mathcal{U}$ on every level ε), is a uniform tower, defining an approach uniformity $\Gamma(\mathcal{U})$, which is referred to as the *uniform approach uniformity* induced by $\mathcal{U}$.

If (X, Γ) and (Y, Ψ) are approach uniform spaces, then a function $f : (X, \Gamma) \to (Y, \Psi)$ is called a *uniform contraction* iff $\forall \psi \in \Psi : \psi \circ (f \times f) \in \Gamma$.

The category **AUnif** of approach uniform spaces and uniform contractions is a topological category. It contains **Unif** both reflectively and coreflectively and **pMET** coreflectively.

For every approach uniform space (X, Γ) and for any $x \in X$ we can consider the set

$$\mathcal{A}(x) := \{\gamma(x, \cdot) \mid \gamma \in \Gamma\} \subset [0, \infty]^X.$$

The family $(\mathcal{A}(x))_{x \in X}$ defines an approach space on X, which we shall call the *underlying approach space* of Γ.

If it is clear from the context we shall write X instead of (X, Γ) or $(X, (\mathcal{A}(x))_{x \in X})$.

Also recall that in any approach space $(X, (\mathcal{A}(x))_{x \in X})$ and for any filter $\mathcal{F}$ on X and any $x \in X$, we define

$$\lambda\mathcal{F}(x) := \sup_{\varphi \in \mathcal{A}(x)} \inf_{F \in \mathcal{F}} \sup_{y \in F} \varphi(y)$$

and

$$\alpha\mathcal{F}(x) := \sup_{\varphi \in \mathcal{A}(x)} \sup_{F \in \mathcal{F}} \inf_{y \in F} \varphi(y).$$

Let $\mathbf{F}(X)$ denote the set of all filters on X, and let $\mathbf{U}(X)$ denote the set of all ultra-filters on X.

Finally recall that given an approach space X, the *measure of compactness* of X (mentioned above, see [4]) is defined as

$$\begin{aligned}\boldsymbol{\mu}_c(X) &:= \sup_{\mathcal{F} \in \mathbf{F}(X)} \inf_{x \in X} \alpha\mathcal{F}(x) \\ &= \sup_{\mathcal{F} \in \mathbf{U}(X)} \inf_{x \in X} \lambda\mathcal{F}(x).\end{aligned}$$

2. Precompactness and total boundedness

Recall the following definitions concerning a semi-uniform space $(X,\mathcal{U})$.

$(X,\mathcal{U})$ is *totally bounded* iff $\forall U \in \mathcal{U}, \exists A_1, \dots, A_n \subset X$ such that $\bigcup_{i=1}^n A_i = X$ and $\forall i \in \{1, \dots, n\} : A_i \times A_i \subset U$.

$(X,\mathcal{U})$ is *precompact* iff $\forall U \in \mathcal{U}, \exists x_1, \dots, x_n \in X$ such that $\bigcup_{i=1}^n U(x_i) = X$.

If $(X,\mathcal{U})$ is totally bounded, then it is precompact. $(X,\mathcal{U})$ is totally bounded iff every ultrafilter is $\mathcal{U}$-Cauchy.

Definition 2.1. *Let X be an approach uniform space with tower $(\mathcal{U}_\varepsilon)_\varepsilon$. Then X is called ε-totally bounded (ε-precompact) if $\mathcal{U}_\varepsilon$ is totally bounded (precompact).*

Then $\boldsymbol{\mu}_{tb}(X) := \inf\{\varepsilon \mid X$ is ε-totally bounded$\}$ and $\boldsymbol{\mu}_{pc}(X) := \inf\{\varepsilon \mid X$ is ε-precompact$\}$ are called the measure of total boundedness and the measure of precompactness respectively.

Proposition 2.2. *Let X be an approach uniform space. Then*

$$\boldsymbol{\mu}_{pc}(X) \le \boldsymbol{\mu}_{tb}(X) \le 2 \cdot \boldsymbol{\mu}_{pc}(X).$$

PROOF: Since for any $\varepsilon \in \mathbf{R}^+$, if $\mathcal{U}_\varepsilon$ is totally bounded, then $\mathcal{U}_\varepsilon$ is precompact, it follows that $\boldsymbol{\mu}_{pc}(X) \le \boldsymbol{\mu}_{tb}(X)$. Conversely, if $\mathcal{U}_\varepsilon$ is precompact, then $\mathcal{U}_{2\varepsilon}$ is totally bounded. To see this, it suffices to observe that for any $U \in \mathcal{U}_{2\varepsilon}$, by (UT3), there exists some symmetric $V \in \mathcal{U}_\varepsilon$ such that $V \circ V \subset U$, and there exist $x_1, \dots, x_n \in X$ such that $\bigcup_{i=1}^n V(x_i) = X$, and then $V(x_i) \times V(x_i) \subset V \circ V \subset U$. $\square$

Proposition 2.5 shows that $\boldsymbol{\mu}_{tb}(X) = \boldsymbol{\mu}_{pc}(X)$ if X is a metric approach uniformity.

Example 2.3. Let $\mathcal{U}_\varepsilon$ be the discrete uniformity on $\mathbf{R}$ if $\varepsilon < 1$ and let it be the trivial uniformity on $\mathbf{R}$ if $\varepsilon \ge 2$. If $1 \le \varepsilon < 2$, then put $\mathcal{U}_\varepsilon := \langle\{(x,y) \in \mathbf{R}^2 \mid xy = 0 \text{ or } x = y\}\rangle$.

Then $\boldsymbol{\mu}_{tb}(\mathbf{R}) = 2$ and $\boldsymbol{\mu}_{pc}(\mathbf{R}) = 1$.

Proposition 2.4. *Let (X,Γ) be a uniform approach uniform space, say $\Gamma = \Gamma(\mathcal{U})$. Then the following are equivalent:*

(1) $\boldsymbol{\mu}_{tb}(X) = 0$,
(2) $\boldsymbol{\mu}_{pc}(X) = 0$,
(3) *$(X,\mathcal{U})$ is totally bounded.*

Quite remarkably, whereas, in the case of approach spaces it is only possible to give a canonical extension of the Hausdorff measure of non-compactness ([3]), in the case of approach uniformities the foregoing definitions of $\boldsymbol{\mu}_{tb}$ and $\boldsymbol{\mu}_{pc}$ give canonical extensions precisely of Kuratowski's measure of non-compactness and of Hausdorff's measure of non-compactness respectively.

Proposition 2.5. *Let (X,Γ) be a ∞p-metric approach uniform space, say $\Gamma = \Gamma(d)$. Then we have*

(a) $\boldsymbol{\mu}_{pc}(X) = \boldsymbol{\mu}_H(X)$,
(b) $\boldsymbol{\mu}_{tb}(X) = \boldsymbol{\mu}_K(X)$.

Consequently, the following are equivalent:

(1) $\boldsymbol{\mu}_{tb}(X) = 0$,
(2) $\boldsymbol{\mu}_{pc}(X) = 0$,
(3) (X,d) *is totally bounded.*

Proof: (a) To prove that $\boldsymbol{\mu}_{pc}(X) \leq \boldsymbol{\mu}_H(X)$, suppose that $\boldsymbol{\mu}_H(X) \leq \varepsilon$. Then for any $\alpha > \varepsilon$, there is some $A \subset X$ finite, such that $X = \bigcup_{a\in A} B(a,\alpha) = \{d < \alpha\}(A)$. Thus $\boldsymbol{\mu}_{pc}(X) \leq \varepsilon$.

Conversely, suppose that $\mathcal{U}_\varepsilon$ is precompact and $\alpha > \varepsilon$. Then there is some $A \subset X$ finite, such that $\{d < \alpha\}(A) = \bigcup_{a\in A} B(a,\alpha) = X$, and thus $\boldsymbol{\mu}_H(X) \leq \varepsilon$. Therefore $\boldsymbol{\mu}_{pc}(X) \geq \boldsymbol{\mu}_H(X)$.

(b) To see that $\boldsymbol{\mu}_{tb}(X) \leq \boldsymbol{\mu}_K(X)$, observe that for any cover $X_1,\dots,X_n$ of X such that $\max_{i=1}^n diam X_i \leq \varepsilon$, we have that $\forall\,\alpha > \varepsilon : X_i \times X_i \subset \{d < \alpha\}$, which means that $\mathcal{U}_\varepsilon$ is totally bounded. Conversely, if $\mathcal{U}_\varepsilon$ is totally bounded, then $\forall\,\alpha > \varepsilon : \exists\, X_1,\dots,X_n$ covering X, such that for each $i \in \{1,\dots,n\}$ $X_i \times X_i \subset \{d < \alpha\}$. Thus, if $\boldsymbol{\mu}_{tb}(X) \leq \varepsilon$, then $\boldsymbol{\mu}_K(X) \leq \varepsilon$. □

Proposition 2.6. *Let (X,Γ) be a p-metric approach uniform space, say $\Gamma = \Gamma(d)$. Then the following are equivalent:*

(1) $\boldsymbol{\mu}_{tb}(X) < \infty$,
(2) $\boldsymbol{\mu}_{pc}(X) < \infty$,
(3) (X,d) *is bounded.*

Proof: The equivalence of (1) and (2) is clear from Proposition 2.2. To prove that (3) $\Rightarrow$ (1), observe that if $d \leq M$ ($M \in \mathbf{R}$), then $\mathcal{U}_M$ is the trivial uniformity and thus totally bounded. To see that (1) $\Rightarrow$ (3), notice that $\boldsymbol{\mu}_{tb}(X) < \infty$ yields some ε such that $\mathcal{U}_\varepsilon$ is totally bounded; fix $\alpha > \varepsilon$ and choose $x_1,\dots,x_n$ such that $\bigcup_{i=1}^n B(x_i,\alpha) = X$. Then $d \leq diam\{x_1,\dots,x_n\} + 2\alpha < \infty$. □

The fact that the uniformly continuous image of a totally bounded uniformity is again totally bounded, is generalized by the following proposition.

Proposition 2.7. *Let (X,Γ) and (Y,Ψ) be approach uniform spaces. If $f : (X,\Gamma) \to (Y,\Psi)$ is a surjective uniform contraction, then $\boldsymbol{\mu}_{tb}(Y) \leq \boldsymbol{\mu}_{tb}(X)$ and $\boldsymbol{\mu}_{pc}(Y) \leq \boldsymbol{\mu}_{pc}(X)$.*

Proof: This follows from the fact that if X is ε-totally bounded (or precompact), then Y is ε-totally bounded (or precompact). □

In the category **sUnif**, total boundedness is stable for initial structures. Therefore we obtain the following.

Proposition 2.8. *Let J be a set and let $\big(f_j : X \to Y_j\big)_{j\in J}$ be an initial* **AUnif**-*source, then $\boldsymbol{\mu}_{tb}(X) \leq \sup_{j\in J} \boldsymbol{\mu}_{tb}(Y_j)$.*

Proof: If $\forall\, j \in J : Y_j$ is ε_j-totally bounded, then each Y_j is $\sup_{j\in J}\varepsilon_j$-totally bounded. Consequently, X is $\sup_{j\in J}\varepsilon_j$-totally bounded. $\square$

As immediate consequences of the previous proposition, we obtain the following results.

Proposition 2.9. *Let X be an approach uniformity. If Y is a subspace of X, then $\boldsymbol{\mu}_{tb}(Y) \leq \boldsymbol{\mu}_{tb}(X)$.*

Proposition 2.10. *Let J be a set, and let for each $j \in J$, X_j be an approach uniformity. Then $\boldsymbol{\mu}_{tb}\Big(\prod\limits_{j\in J} X_j\Big) = \sup\limits_{j\in J} \boldsymbol{\mu}_{tb}(X_j)$.*

Proof: Since all projections $\pi_j : \prod_{j\in J} X_j \to X_j$ are surjective uniform contractions, we know that $\forall\, j \in J : \boldsymbol{\mu}_{tb}(X_j) \leq \boldsymbol{\mu}_{tb}(\prod X_j)$.

The converse inequality is exactly Proposition 2.8. $\square$

Precompactness is, however, not stable for initial structures. This is illustrated in the following example.

Example 2.11. Let $\mathbf{R}$ be equipped with the approach uniformity in Example 2.3, and consider its subspace $\mathbf{R}_0$. Then $\boldsymbol{\mu}_{pc}(\mathbf{R}) = 1$ and $\boldsymbol{\mu}_{pc}(\mathbf{R}_0) = 2$.

But we do have the following.

Proposition 2.12. *Let J be a set and let $\big(f_j : X \to Y_j\big)_{j\in J}$ be an initial* **AUnif**-*source, then $\boldsymbol{\mu}_{pc}(X) \leq 2 \cdot \sup_{j\in J} \boldsymbol{\mu}_{pc}(Y_j)$.*

Proof: If $\forall\, j \in J : Y_j$ is ε_j-precompact, then each Y_j is $\sup_{j\in J} 2\varepsilon_j$-totally bounded. Consequently, X is $2 \cdot \sup_{j\in J}\varepsilon_j$-totally bounded. $\square$

The measure of total boundedness behaves nicely with respect to completion. If (X, Γ) is an approach uniform space, then let $\widehat{X}$ denote the set of all minimal Cauchy filters on X, w.r.t. the uniform coreflection. For each $\gamma \in \Gamma$ and for all $\mathcal{M}, \mathcal{N} \in \widehat{X}$, define $\widehat{\gamma}(\mathcal{M}, \mathcal{N}) := \inf_{F\in\mathcal{M}\cap\mathcal{N}} \sup_{x,y\in F} \gamma(x, y)$. Then $\{\widehat{\gamma} \mid \gamma \in \Gamma\}$ is a basis for an approach uniformity on $\widehat{X}$, which is called the *completion* of X. The map $i : X \to \widehat{X} : x \mapsto \dot{x}$ is an embedding, and $\forall\, \gamma \in \Gamma : \widehat{\gamma} \circ i = \gamma$ ([7]).

Proposition 2.13. *Let X be an approach uniform space. Then $\boldsymbol{\mu}_{tb}(X) = \boldsymbol{\mu}_{tb}(\widehat{X})$.*

Proof: Since X is a subspace of $\widehat{X}$, we have that $\boldsymbol{\mu}_{tb}(X) \leq \boldsymbol{\mu}_{tb}(\widehat{X})$.

Conversely, suppose that $\boldsymbol{\mu}_{tb}(X) \leq \varepsilon$. Then $\mathcal{U}_\varepsilon$ is totally bounded. We have to prove that $\forall\, \widehat{U} \in \widehat{\mathcal{U}}_\varepsilon, \exists\, B_1, \dots, B_n$ covering $\widehat{X}$, such that $\forall\, k \in \{1, \dots, n\}$: $B_k \times B_k \subset \widehat{U}$. Let $\widehat{U} \in \widehat{\mathcal{U}}_\varepsilon$, $\widehat{U} = \{\widehat{\gamma} < \alpha\}$ $(\alpha > \varepsilon)$ say. Choose $\widetilde{\gamma} \in \Gamma$ such that $\forall\, u, x, y, z \in X : \gamma(u, z) \wedge N \leq \widetilde{\gamma}(u, x) + \widetilde{\gamma}(x, y) + \widetilde{\gamma}(y, z)$ for some $N > 2\varepsilon$.

Put $U := \{\widetilde{\gamma} < \frac{\alpha+\varepsilon}{2}\}$. Since $(X,\mathcal{U}_\varepsilon)$ is totally bounded, there exist $A_1,\dots,A_n$ covering X such that $\forall\, k \in \{1,\dots,n\} : A_k \times A_k \subset U$. For every $k \in \{1,\dots,n\}$ put $B_k := \overline{i(A_k)}$. Clearly

$$\bigcup_{k=1}^{n} B_k = \bigcup_{k=1}^{n} \overline{i(A_k)} = \overline{i\left(\bigcup_{k=1}^{n} A_k\right)} = \overline{i(X)} = \widehat{X}.$$

On the other hand, if $\mathcal{M},\mathcal{N} \in B_k$, then $\exists\, x, y \in A_k$:

$$\widehat{\widetilde{\gamma}}(\mathcal{M},\dot{x}) < \frac{\alpha-\varepsilon}{4} \quad \text{and} \quad \widehat{\widetilde{\gamma}}(\mathcal{N},\dot{y}) < \frac{\alpha-\varepsilon}{4}$$

and then

$$\begin{aligned}\widehat{\gamma}(\mathcal{M},\mathcal{N}) &\le \widehat{\widetilde{\gamma}}(\mathcal{M},\dot{x}) + \widehat{\widetilde{\gamma}}(\dot{x},\dot{y}) + \widehat{\widetilde{\gamma}}(\mathcal{N},\dot{y}) \\ &< \frac{\alpha-\varepsilon}{4} + \frac{\alpha+\varepsilon}{2} + \frac{\alpha-\varepsilon}{4} \\ &= \alpha.\end{aligned}$$

Therefore $\forall\, k \in \{1,\dots,n\} : B_k \times B_k \subset \{\widehat{\gamma} < \alpha\}$. □

Compact uniform spaces are always precompact. Therefore it is natural to ask whether the measure of precompactness of an approach uniformity is related to the measure of compactness of the underlying approach space.

Proposition 2.14. *Let X be an approach uniform space. Then $\boldsymbol{\mu}_{pc}(X) \le \boldsymbol{\mu}_c(X)$.*

Proof: We shall show that if $(\mathcal{U}_\varepsilon)_\varepsilon$ is the tower on X, then $\mathcal{U}_{\boldsymbol{\mu}_c(X)}$ is precompact. Suppose it is not. Then there exist $\gamma \in \Gamma$ and $\mu > \boldsymbol{\mu}_c(X)$ such that $\forall\, A \in 2^{(X)}$: $\{\gamma < \mu\}(A) \neq X$. Consider the filter

$$\mathcal{F} := \{X \setminus \{\gamma < \mu\}(A) \mid A \in 2^{(X)}\}$$

and the ultra-filter $\mathcal{H}$ containing $\mathcal{F}$. Since

$$\mu > \boldsymbol{\mu}_c(X) = \sup_{\mathcal{G} \in \mathbf{U}(X)} \inf_{x \in X} \lambda\mathcal{G}(x),$$

there exist $x \in X$ and $H \in \mathcal{H}$ such that $\forall\, y \in H : \gamma(x,y) < \mu$. This means that $H \subset \{\gamma < \mu\}(x)$ and thus $\{\gamma < \mu\}(x) \in \mathcal{H}$, while $X \setminus \{\gamma < \mu\}(x) \in \mathcal{H}$ too, which is impossible. □

For metric spaces, we always have $\boldsymbol{\mu}_{pc}(X) = \boldsymbol{\mu}_c(X) = \boldsymbol{\mu}_H(X)$, but the inequality in Proposition 2.14 is strict in general. This becomes clear in the following example.

Example 2.15. Let $\mathcal{M}(X)$ be the set of all probability measures on a separable metrizable topological space X. Let $\mathcal{H}$ be a weakly compact subset of $\mathcal{M}(X)$ and let $\mathcal{K}$ be any subset of $\mathcal{M}(X)$ containing $\mathcal{H}$. Fix $\varepsilon > 0$, and consider the following subspace of $\mathcal{M}(X)$:

$$Y := \{(1-\varepsilon)P + \varepsilon Q \mid P \in \mathcal{H}, Q \in \mathcal{K}\}.$$

Then $\boldsymbol{\mu}_c(Y) \leq 2\varepsilon$ (see [5]). We shall show that $\boldsymbol{\mu}_{pc}(Y) \leq \varepsilon$.

Let $\alpha > \varepsilon$ and let C be a finite subset of $\mathcal{C}(X, [0,1])$. Since $\mathcal{H}$ is weakly compact, Proposition 2.14 implies that $\boldsymbol{\mu}_{pc}(\mathcal{H}) = 0$. Consequently, there exists some $\mathcal{G} \subset \mathcal{H}$ finite such that $\{d_C < \alpha - \varepsilon\}(\mathcal{G}) = \mathcal{H}$.

For any $(1-\varepsilon)P + \varepsilon Q \in Y$, consider $G \in \mathcal{G}$ such that $d_C(G, P) < \alpha - \varepsilon$. Then

$$\begin{aligned}
d_C((1-\varepsilon)P + \varepsilon Q, G) &= \sup_{f \in C} \left| (1-\varepsilon) \int f\, dP + \varepsilon \int f\, dQ - \int f\, dG \right| \\
&\leq \sup_{f \in C} \left| \int f\, dP - \int f\, dG \right| + \sup_{f \in C} \left| \varepsilon \int f\, dQ - \varepsilon \int f\, dP \right| \\
&= d_C(G, P) + \varepsilon d_C(P, Q) \\
&\leq \alpha
\end{aligned}$$

which proves that Y is ε-precompact.

The inequalities in previous propositions quantify a number of well-known classical results concerning uniform and metric spaces. Conversely, these results can be deduced from the **AUnif**-generalizations.

Corollary 2.16.

(a) *A subspace of a totally bounded uniform space is totally bounded.*
(b) *A subspace of a totally bounded ∞p-metric space is totally bounded.*
(c) *A product of uniform spaces is totally bounded iff each factor space is totally bounded.*
(d) *A finite product of ∞p-metric spaces is totally bounded iff each factor space is totally bounded.*
(e) *A uniform space is totally bounded if and only if its completion is totally bounded.*
(f) *A compact uniform space is totally bounded.*

Proof: (a) Let X be a totally bounded uniform space and let $Y \subset X$ be a subspace. Applying Propositions 2.9 and 2.4, we see that $\boldsymbol{\mu}_{tb}(Y) \leq \boldsymbol{\mu}_{tb}(X) = 0$, and therefore $\boldsymbol{\mu}_{tb}(Y) = 0$. Again using Proposition 2.4, we conclude that Y is totally bounded.

(b)–(f) Analogously, (b) follows from 2.5 and 2.9, (c) from 2.4 and 2.10, (d) from 2.5 and 2.10, (e) from 2.4 and 2.13, and (f) from 2.4 and 2.14. □

3. Completeness

For uniform spaces, completeness means that every Cauchy-filter has an adherence point. In the context of approach uniformities we can consider Cauchy-filters on every level $\varepsilon \in \mathbf{R}^+$. Denote the set of all ε-Cauchy filters on X (that is, the set of all filters that are Cauchy with respect to $\mathcal{U}_\varepsilon$) by $\mathbf{C}_\varepsilon(X)$. If every ε-Cauchy filter 'clusters up to ε', then we call the approach uniformity ε-complete.

Definition 3.1. *Let X be an approach uniform space. Then X is called ε-complete if $\forall\, \theta \geq \varepsilon : \sup_{\mathcal{F}\in\mathbf{C}_\theta(X)} \inf_{x\in X} \alpha\mathcal{F}(x) \leq \theta$.*

Then $\boldsymbol{\mu}_v(X) := \inf\{\varepsilon \mid X \text{ is } \varepsilon\text{-complete}\}$ is called the measure of completeness of X.

The letter v in the notation $\boldsymbol{\mu}_v$ stands for the German term *Vollständigkeit*.

For ∞p-metric approach uniformities, the measure of completeness is totally uninteresting. For uniform approach uniformities, however, it generalizes the well known concept of completeness for uniform spaces.

Proposition 3.2. *Let (X,Γ) be a uniform approach uniform space, say $\Gamma = \Gamma(\mathcal{U})$. Then the following are equivalent:*

(1) $\boldsymbol{\mu}_v(X) = 0$,
(2) *$(X,\mathcal{U})$ is complete.*

PROOF: For arbitrary $\varepsilon \in \mathbf{R}^+$, ε-Cauchy filters are exactly $\mathcal{U}$-Cauchy filters. If $\boldsymbol{\mu}_v(X) = 0$, then

$$\forall\, \mathcal{F} \text{ Cauchy }, \exists\, x \in X, \forall\, U \in \mathcal{U}, \forall\, F \in \mathcal{F} : F \cap U(x) \neq \emptyset.$$

Thus every Cauchy filter has a cluster point, and therefore converges.

Conversely, since every Cauchy filter converges, we have that

$$\sup_{\mathcal{F}\in\mathbf{C}_\varepsilon(X)} \inf_{x\in X} \alpha\mathcal{F}(x) = 0 \leq \varepsilon.$$

□

Proposition 3.3. *Let (X,Γ) be a ∞p-metric approach uniformity, say $\Gamma = \Gamma(d)$. Then $\boldsymbol{\mu}_v(X) = 0$.*

PROOF: Let $\varepsilon \geq 0$ arbitrary. Let $\mathcal{F}$ be an ε-Cauchy filter and let $\alpha > \varepsilon$. Then there is some $F \in \mathcal{F}$ such that $F \times F \subset \{d < \alpha\}$. For arbitrary $x \in X$ and $G \in \mathcal{F}$, we know that $F \cap G \neq \emptyset$, $y \in F \cap G$ say, and thus $d(x,y) < \alpha$. Consequently,

$$\sup_{\mathcal{F}\in\mathbf{C}_\varepsilon(X)} \inf_{x\in X} \sup_{G\in\mathcal{F}} \inf_{y\in G} d(x,y) \leq \varepsilon$$

which by arbitrariness of ε implies that $\boldsymbol{\mu}_v(X) = 0$. □

The measure of completeness generalizes different properties of completeness. If X is an approach uniform space with tower $(\mathcal{U}_\varepsilon)_\varepsilon$ and $\alpha \in \mathbf{R}^+$, then a subset $Y \subset X$ is called α-closed if it is closed with respect to the underlying topology of $\mathcal{U}_\alpha$.

Proposition 3.4. *Let X be an approach uniform space, and let $Y \subset X$. If Y is $\boldsymbol{\mu}_v(X)$-closed, then $\boldsymbol{\mu}_v(Y) \leq \boldsymbol{\mu}_v(X)$.*

Proof: Suppose $\boldsymbol{\mu}_v(X) \leq \varepsilon$. Every ε-Cauchy filter $\mathcal{F}$ in Y induces an ε-Cauchy filter $\mathcal{F}'$ in X. Consequently,

$$\forall\, \alpha > \varepsilon, \exists\, x \in X, \forall\, \gamma \in \Gamma, \forall\, F \in \mathcal{F}, \exists\, y \in F : \gamma(x,y) < \alpha$$

but since $\forall\, \gamma \in \Gamma, \forall\, \alpha > \varepsilon : \{\gamma(x,\cdot) < \alpha\} \cap Y \neq \emptyset$, we also have that

$$\forall\, \alpha > \varepsilon, \exists\, x \in Y, \forall\, \gamma \in \Gamma, \forall\, F \in \mathcal{F}, \exists\, y \in F : \gamma(x,y) < \alpha.$$

Therefore $\boldsymbol{\mu}_v(Y) \leq \varepsilon$. □

Proposition 3.5. *Let J be a set, and let for each $j \in J$, X_j be an approach uniform space. Then $\boldsymbol{\mu}_v\Big(\prod\limits_{j\in J} X_j\Big) = \sup\limits_{j\in J} \boldsymbol{\mu}_v(X_j)$.*

Proof: Suppose that $\forall\, j \in J : \boldsymbol{\mu}_v(X_j) \leq \varepsilon$. If $\mathcal{F}$ is an ε-Cauchy filter on $\prod_{j\in J} X_j$, then $\forall\, j \in J : \pi_j(\mathcal{F})$ is an ε-Cauchy filter on X_j. Since $\forall\, \theta > \varepsilon, \exists\, x_j \in X_j : \alpha\left(\pi_j(\mathcal{F})\right)(x_j) \leq \theta$, we have for $x = (x_j)_{j\in J}$ that $\forall\, \theta > \varepsilon : \alpha\mathcal{F}(x) \leq \theta$. Consequently $\boldsymbol{\mu}_v\left(\prod_{j\in J} X_j\right) \leq \varepsilon$.

Conversely, let $\boldsymbol{\mu}_v\left(\prod_{j\in J} X_j\right) < \varepsilon$ and let $\theta \geq \varepsilon$ and $j \in J$. Every θ-Cauchy filter $\mathcal{F}$ on X_j, generates a θ-Cauchy filter $\mathcal{F}'$ on $\prod_{j\in J} X_j$. Since $\inf\limits_{x\in\prod_{j\in J} X_j} \alpha\mathcal{F}'(x) \leq \theta$, considering $x_j = \pi_j(x)$ yields $\inf_{x_j\in X_j} \alpha\mathcal{F}(x_j) \leq \theta$. By arbitrariness of θ, this means that $\boldsymbol{\mu}_v(X_j) \leq \varepsilon$. □

We now investigate the relationship between the measure of completeness and other measures.

Proposition 3.6. *Let X be an approach uniform space. Then $\boldsymbol{\mu}_v(\widehat{X}) \leq \boldsymbol{\mu}_v(X)$.*

Proof: Suppose that $\boldsymbol{\mu}_v(X) \leq \varepsilon$. Let $\mathcal{F}$ be an ε-Cauchy filter on $\widehat{X}$. Then $i^{-1}(\mathcal{F})$ is an ε-Cauchy filter on X and thus for any $\theta > \varepsilon$, there is some $x \in X$ such that

$$\sup_{\gamma\in\Gamma} \sup_{G\in i^{-1}(\mathcal{F})} \inf_{y\in G} \gamma(x,y) < \theta.$$

Consequently, for arbitrary $\gamma \in \Gamma$ and $F \in \mathcal{F}$ we have

$$\exists\, y \in i^{-1}(F) : \widehat{\gamma}(\dot{x},\dot{y}) = \gamma(x,y) < \theta.$$

Therefore

$$\inf_{\mathcal{M}\in\widehat{x}} \sup_{\gamma\in\Gamma} \sup_{F\in\mathcal{F}} \inf_{\mathcal{N}\in F} \widehat{\gamma}(\mathcal{M},\mathcal{N}) \leq \varepsilon.$$

□

Note that the converse inequality need not be true. Any non-complete uniform approach uniformity is a counter-example.

Proposition 3.7. *Let X be an approach uniform space. Then $\boldsymbol{\mu}_v(X) \leq \boldsymbol{\mu}_c(X)$.*

PROOF: Suppose $\boldsymbol{\mu}_c(X) \leq \varepsilon$. Then for any $\theta \geq \varepsilon$:

$$\sup_{\mathcal{F}\in\mathbf{C}_\theta(X)} \inf_{x\in X} \alpha\mathcal{F}(x) \leq \sup_{\mathcal{F}\in\mathbf{F}(X)} \inf_{x\in X} \alpha\mathcal{F}(x) \leq \varepsilon \leq \theta.$$

Therefore, $\boldsymbol{\mu}_v(X) \leq \varepsilon$. □

Proposition 3.8. *Let X be an approach uniform space. Then $\boldsymbol{\mu}_{pc}(X) \vee \boldsymbol{\mu}_v(X) \leq \boldsymbol{\mu}_c(X) \leq \boldsymbol{\mu}_{tb}(X) \vee \boldsymbol{\mu}_v(X)$.*

PROOF: The first inequality is a combination of Proposition 2.14 and Proposition 3.7. In order to prove the second inequality, suppose $\boldsymbol{\mu}_{tb}(X) \vee \boldsymbol{\mu}_v(X) \leq \varepsilon$. If $\mathcal{F}$ is an ultrafilter, then $\boldsymbol{\mu}_{tb}(X) \leq \varepsilon$ implies that $\mathcal{F}$ is ε-Cauchy. Since $\boldsymbol{\mu}_v(X) \leq \varepsilon$, we obtain $\inf\limits_{x\in X} \alpha\mathcal{F}(x) \leq \varepsilon$. Thus,

$$\boldsymbol{\mu}_c(X) = \sup_{\mathcal{F}\in\mathbf{U}(X)} \inf_{x\in X} \alpha\mathcal{F}(x) \leq \varepsilon.$$

□

Proposition 3.9. *Let X be an approach uniform space. Then $1/2\, \boldsymbol{\mu}_{tb}(X) \leq \boldsymbol{\mu}_c(\widehat{X}) \leq \boldsymbol{\mu}_{tb}(X) \vee \boldsymbol{\mu}_v(\widehat{X})$.*

PROOF: Observe that by Propositions 2.13 and 2.2,

$$\begin{aligned} 1/2\, \boldsymbol{\mu}_{tb}(X) &= 1/2\, \boldsymbol{\mu}_{tb}(\widehat{X}) \\ &\leq \boldsymbol{\mu}_{pc}(\widehat{X}) \\ &\leq \boldsymbol{\mu}_c(\widehat{X}) \\ &\leq \boldsymbol{\mu}_{tb}(\widehat{X}) \vee \boldsymbol{\mu}_v(\widehat{X}) \\ &\leq \boldsymbol{\mu}_{tb}(X) \vee \boldsymbol{\mu}_v(\widehat{X}). \end{aligned}$$

□

From the results in this section too, different classical theorems concerning uniform spaces can be deduced.

Corollary 3.10.

(a) *A closed subspace of a complete uniform space is complete.*
(b) *A product of uniform spaces is complete iff each factor space is complete.*
(c) *A uniform space is compact if and only if it is totally bounded and complete.*
(d) *A uniform space is totally bounded if and only if its completion is compact.*

PROOF: These theorems are consequences of Propositions 3.4, 3.5, 3.8 and 3.9 respectively. □

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(*Received* July 9, 1996, *revised* February 18, 1997)