Generalized linearly ordered spaces and weak pseudocompactness

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Abstract. A space X is *truly weakly pseudocompact* if X is either weakly pseudocompact or Lindelöf locally compact. We prove that if X is a generalized linearly ordered space, and either (i) each proper open interval in X is truly weakly pseudocompact, or (ii) X is paracompact and each point of X has a truly weakly pseudocompact neighborhood, then X is truly weakly pseudocompact. We also answer a question about weakly pseudocompact spaces posed by F. Eckertson in [Eck].

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Introduction

All spaces considered in this paper will be non-empty Tychonoff spaces, and if X and Y are topological spaces, the symbol $X \cong Y$ will mean that they are homeomorphic. Let X be a set linearly ordered by $<$ and containing at least two elements. For $a, b \in X$ with $a < b$ let $(a, b) = \{x \in X : a < x < b\}$, $(\leftarrow, a) = \{x \in X : x < a\}$, $(a, \rightarrow) = \{x \in X : a < x\}$; these sets will be called *open intervals* in X. We also define the sets $[a, b) = \{x \in X : a \leq x < b\}$, and, in a similar way, $[a, b]$, $(a, b]$, $(\leftarrow, b]$, $[a, \rightarrow)$. A *linearly ordered topological space* (LOTS) X is a space whose topology is generated by the open intervals defined by a linear order relation $<$. A space X is a *generalized linearly ordered space* (GLOTS) if it is homeomorphic to a subspace of a LOTS. Let α, β be cardinals with $\omega \leq \alpha \leq \beta$. A topological space X is called $[\alpha, \beta]$*-compact* if every open cover $\mathcal{U}$ of X with $|\mathcal{U}| \leq \beta$ has a subcover of cardinality $< \alpha$. Thus, a space X is compact (resp., Lindelöf, countably compact) if and only if X is $[\omega, |X|]$-compact (resp., $[\omega_1, |X|]$-compact, $[\omega, \omega]$-compact). Note that $[\alpha, \beta]$-compactness is hereditary with respect to closed subsets.

In [GG] the authors introduced the concept of weak pseudocompactness. A space X is called *weakly pseudocompact* if there exists a compactification bX of X such that X is G_δ-dense in bX (which means that every nonempty G_δ-set in bX meets X). Obviously, all pseudocompact spaces are weakly pseudocompact, as well as all non-Lindelöf locally compact spaces. It turned out that many statements about weakly pseudocompact spaces include the combination "weakly pseudocompact or locally compact Lindelöf"; so, we will economize by saying that a space X is *truly weakly pseudocompact* if it satisfies one of these two properties. Thus, all locally compact spaces are truly weakly pseudocompact, and every truly weakly pseudocompact space which is not Lindelöf is weakly pseudocompact. Note that every weakly pseudocompact Lindelöf space is compact.

The following two basic and important statements are reformulations of results due to F. Eckertson [Eck]:

Theorem 0.1. *If X is a truly weakly pseudocompact space and A an open subset of X, then A is truly weakly pseudocompact.*

Theorem 0.2. *Let $\{ X_\xi : \xi \in A \}$ be a collection of topological spaces. The free topological sum $X = \bigoplus\{ X_\xi : \xi \in A \}$ is truly weakly pseudocompact if and only if each X_ξ is truly weakly pseudocompact.*

We are going to consider the local versions of these properties.

Definition 0.3. (1) A space X is *locally weakly pseudocompact* (*locally truly weakly pseudocompact*) at a point $x \in X$ if there is a basic system of open neighborhoods $\mathcal{N}$ of x in X whose all of elements are weakly pseudocompact (resp., truly weakly pseudocompact).

(2) A space X is *locally weakly pseudocompact* (*locally truly weakly pseudocompact*) if X is locally weakly pseudocompact (resp., locally truly weakly pseudocompact) at each of its points.

Observe that a space X is locally truly weakly pseudocompact at $x \in X$ if there is a truly weakly pseudocompact neighborhood V of x in X.

By Theorem 0.1, in the following assertions, (1) implies (2) and (2) implies (3).

(1) X is truly weakly pseudocompact.
(2) Every proper open subset of X is truly weakly pseudocompact.
(3) X is locally truly weakly pseudocompact.

In this article we are going to prove that in certain classes of spaces these implications may be reversed. Namely, we are going to prove that in the class of the Generalized Linearly Ordered Spaces (GLOTS) (2) $\Rightarrow$ (1), and in the class of paracompact GLOTS, (3) $\Rightarrow$ (1).

1. Spaces constructed from partitions of ordinals

Let κ be an ordinal number, and let S be a subset of $[0,\kappa]$. Let $\Phi = (L_1, L_2)$ be a partition of S (so $L_1 \cup L_2 = S$ and $L_1 \cap L_2 = \emptyset$). For each $\lambda \in L_1$ put $X_\lambda = \{\lambda\}$, and for each $\lambda \in L_2$ fix a Tychonoff space X_λ. Let X be the (disjoint) union of $\{ X_\lambda : \lambda \in S \}$. Define a topology on X by the following conditions: if $\lambda \in L_2$, then X_λ is an open subspace of X_Φ, and if $\lambda \in L_1$, then the sets $\bigcup\{X_\xi : \xi \in (\eta, \lambda] \cap S \}$, $\eta < \lambda$, form an open base at λ in X. We denote this topological space by X_Φ. Obviously, if $L_1 = \emptyset$, then X_Φ is a free topological sum, and if $L_2 = \emptyset$, then X_Φ is a subspace of $[0,\kappa]$. It is asked in [Eck] whether

X_Φ must be weakly pseudocompact if each X_λ is truly weakly pseudocompact, $S = [0, \kappa]$ and $L_1 = \{\kappa\}$ where $\kappa > \omega_1$. Later, Eckertson and Ohta [EO] answered this question in the affirmative when κ is a regular cardinal and $|X_\lambda| \le 1$ for every $\lambda \in S$. We are going to answer Eckertson's question for every ordinal number κ and without any restrictions on the cardinality of the spaces X_λ.

If $\Phi(S) = (L_1, L_2)$ is a partition of a subset S of an ordinal κ, and $T \subset S$, then we can consider the restriction of $\Phi(S)$ to T: $\Phi(T) = (L_1 \cap T, L_2 \cap T)$. A routine verification proves

Lemma 1.1. (1) *If for every $\lambda \in S$, Y_λ is a subspace of X_λ, then Y_Φ is a subspace of X_Φ.*

(2) *If $T \subset S$, then $X_{\Phi(T)}$ is a subspace of $X_{\Phi(S)}$.*

(3) *Given a partition $\Phi = (L_1, L_2)$ of a subset S of an ordinal κ and a space X_Φ, there exist*

(i) *a subset T of κ,*

(ii) *a partition $\Psi = (M_1, M_2)$ of T, and*

(iii) *for each $t \in T$ a space Y_t,*

such that $X_\Phi \cong Y_\Psi$, $t \in M_1$ implies that t is a limit ordinal in T and $t \in M_2$ implies that t is a non-limit ordinal in T.

From now on, κ, S and $\Phi = (L_1, L_2)$ will be respectively an ordinal number, a subset of $[0, \kappa]$ and a partition of S, and for each $\lambda \in L_2$, X_λ will be a topological space. Besides, every partition $\Phi = (L_1, L_2)$ will satisfy the following conditions: (i) $\xi \in L_1$ implies that ξ is a limit ordinal of S, (ii) $\xi \in L_2$ implies that ξ is a non-limit ordinal (see Lemma 1.1 (3)), and (iii) L_2 is cofinal in $[0, \kappa)$.

It is easy to prove the following useful observation.

Lemma 1.2. *Let $\{\alpha_\xi : \xi \le \gamma\}$ be an increasing sequence in κ such that*

(1) *if $\xi \le \gamma$ is a limit ordinal, then $\alpha_\xi = \sup\{\alpha_\lambda : \lambda < \xi\}$;*

(2) *if $\xi \le \gamma$ is not a limit ordinal, then α_ξ is not a limit ordinal; and*

(3) *$S \subset [\alpha_0, \alpha_\gamma]$.*

For each $\xi < \gamma$, let $Y_{\xi+1} = \bigcup\{X_\lambda : \alpha_\xi < \lambda \le \alpha_{\xi+1}\}$ be the subspace of X_Φ. Let $M_1 = \{\xi \le \gamma : \xi$ is a limit ordinal and $\alpha_\xi \in L_1\}$ and $M_2 = \{\xi \le \gamma : \xi$ is not a limit ordinal and $Y_\xi \ne \emptyset\}$. If $\Psi = (M_1, M_2)$, then $Y_\Psi \cong X_\Phi$.

Now we will prove some results that relate compactness-like properties of X_Φ with properties of S and of each X_λ.

Theorem 1.3. *Let α be a regular cardinal number. The space X_Φ is $[\alpha, \beta]$-compact if and only if the following two conditions hold:*

(1) *X_λ is $[\alpha, \beta]$-compact for every $\lambda \in S$, and*

(2) *S is $[\alpha, \beta]$-compact (as a subspace of $[0, \kappa]$).*

Proof: ($\Rightarrow$) If for some $\lambda \in S$, X_λ is not $[\alpha, \beta]$-compact, then $\lambda \in L_2$ and X_λ is a closed subset of X_Φ, so X_Φ is not $[\alpha, \beta]$-compact.

Let $p_\Phi\colon X_\Phi \to S$ be the mapping that takes each X_λ to λ; it is easy to check that p is continuous and closed. Since $[\alpha,\beta]$-compactness is preserved by continuous mappings, S must be $[\alpha,\beta]$-compact.

($\Leftarrow$) It is a well-known fact in folklore (with a standard proof) that $[\alpha,\beta]$-compactness is inverse invariant with respect to closed mappings with $[\alpha,\beta]$-compact preimages. The required statement now follows from the closedness of the mapping p_Φ. $\square$

Corollary 1.4. *The space X_Φ is locally compact if and only if the following two conditions hold:*

(1) *X_λ is locally compact for every $\lambda \in S$; and*
(2) *for each $\lambda \in L_1$ there exists $\eta < \lambda$ such that X_ξ is compact for every $\xi \in (\eta,\lambda] \cap S$ and $(\eta,\lambda] \cap S$ is closed in $(\eta,\lambda]$.*

The next theorem is the main result of this section, and it will be useful for our further analysis of weak pseudocompactness in GLOTS (keep in mind the conventions stated after Lemma 1.1).

Theorem 1.5. *Let X_λ be a truly weakly pseudocompact space for every $\lambda \in S$. Then the following assertions are equivalent:*

(1) *X_Φ is truly weakly pseudocompact;*
(2) *X_Φ is locally truly weakly pseudocompact;*
(3) *for each $\lambda \in L_1$, either*
    (i) *there exists $\eta < \lambda$ such that X_ξ is compact for every $\xi \in (\eta,\lambda] \cap S$ and $(\eta,\lambda] \cap S$ is closed in $(\eta,\lambda]$, or*
    (ii) *there is a cofinal set $J \subset S$ in λ such that X_j is not Lindelöf for every $j \in J$, or*
    (iii) *for every $\eta < \lambda$, $(\eta,\lambda] \cap S$ is not Lindelöf.*

Proof: The implication (1) $\Rightarrow$ (2) is immediate from Theorem 0.1.

(2) $\Rightarrow$ (3): Let $\lambda \in L_1$. There exists $\eta_0 < \lambda$ such that $V = \bigcup\{X_\xi : \xi \in (\eta_0,\lambda]\}$ is truly weakly pseudocompact. If V is locally compact, Corollary 1.4 immediately implies (i). If V is not locally compact, then for every $\eta > \eta_0$ the space $\bigcup\{X_\xi : \xi \in (\eta,\lambda]\}$ is not Lindelöf. By Theorem 1.3, we obtain (ii) or (iii).

(3) $\Rightarrow$ (1): We are going to prove, using transfinite induction on κ, that X_Φ is a truly weakly pseudocompact space.

If κ is finite, then X_Φ is a non-Lindelöf free topological sum of truly weakly pseudocompact spaces, so X_Φ is weakly pseudocompact by Theorem 0.2. Assume that (3) $\Rightarrow$ (1) holds for every ordinal number $< \kappa$. We are going to prove this implication for κ. If $\kappa = \lambda_0 + 1$, then X_Φ is an open subset of $X_\Psi \oplus X_\kappa$, where X_κ is a truly weakly pseudocompact space and $\Psi = (L_1 \cap [0,\lambda_0], L_2 \cap [0,\lambda_0])$. Because of the inductive hypothesis and Theorems 0.1 and 0.2, we conclude that X_Φ satisfies the requirement.

Suppose that κ is a limit ordinal.

Case 1: There exists a cofinal subset J of κ such that every X_j with $j \in J$ is not Lindelöf. Then we can construct a γ-sequence $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_\xi < \dots$ $(\xi < \gamma \leq \kappa)$ such that

(1) for each limit ordinal $\xi \leq \gamma$, $\alpha_\xi = \sup\{\alpha_\lambda : \lambda < \xi\}$,
(2) $\alpha_\gamma = \sup\{\alpha_\xi : \xi < \gamma\} = \kappa$,
(3) for each non-limit ordinal $\xi \leq \gamma$, α_ξ is a non limit ordinal, and
(4) for each $\xi < \gamma$, there exists $\lambda \in (\alpha_\xi, \alpha_{\xi+1}] \cap L_2$ such that X_λ is not Lindelöf.

For each $\xi < \gamma$ let $Y_\xi = \bigcup\{X_\lambda : \lambda \in (\alpha_\xi, \alpha_{\xi+1}] \cap S\}$. Let $N_1 = \{\lambda \leq \gamma : \lambda$ is a limit ordinal and $\alpha_\lambda \in L_1\}$ and $N_2 = \{\lambda < \gamma : \lambda$ is not a limit ordinal$\}$. Put $\Psi = (N_1, N_2)$. Now for every $\xi < \gamma$ the space Y_ξ is truly weakly pseudocompact by the inductive hypothesis, but Y_ξ is not Lindelöf (Theorem 1.3), so it is weakly pseudocompact and not compact. Besides, by Lemma 1.2, $X_\Phi \cong Y_\Psi$.

Thus, we may assume without loss of generality that for each $\lambda \in L_2$, X_λ is weakly pseudocompact and not compact.

Let $T = \mathrm{cl}_{[0,\kappa]} S$. Let $M_1 = \{\lambda \in T : \lambda$ is a limit of $S\}$ and $M_2 = L_2$. For each $\lambda \in M_2$ let Z_λ be a compact space containing X_λ as a G_δ-dense subspace. By Lemma 1.1, the space Z_Ψ contains X_Φ as a subspace, and by Theorem 1.3, Z_Ψ is compact.

Let $\zeta \in M_1 \setminus L_1$. If $H_\zeta = \{\lambda \in M_2 : \lambda > \zeta\} \neq \emptyset$, we choose an element $z_\zeta \in Z_{\lambda_\zeta} \setminus X_{\lambda_\zeta}$ where λ_ζ is the first element in H_ζ. If $H_\zeta = \emptyset$, we choose an element $z_\zeta \in Z_0 \setminus X_0$. Let Y be the quotient space obtained from Z_Ψ by identifying each $\zeta \in M_1$ with z_ζ, and let $p\colon Z_\Psi \to Y$ be the natural projection. For every closed set $F \subset Z_\Psi$, the set $p^{-1}(p(F)) = F \cup \bigcup\{\, z_\zeta, \zeta : F \cap \{\zeta, z_\zeta\} \neq \emptyset \,\}$ is closed, because as it is easy to see, for every subset A of $M_1 \setminus L_1$, the set A has the same limit points in Z_Ψ as the set $\{\, z_\zeta : \zeta \in A \,\}$. It follows that p is closed, so Y is a Hausdorff compact space. We have $X_\Phi = p^{-1}(p(X_\Phi))$, so the restriction of p to X_Φ is quotient; since this restriction is also one-to-one, p embeds X_Φ in Y. Thus, Y is a compact extension of X, and we only need to check that X is G_δ-dense in Y. Let $G = \bigcap_{n<\omega} A_n$ be a nonempty G_δ set in Y where A_n is an open subset in Y for every $n < \omega$. Let $g \in G$. If $g = \{z\}$ and $z \in \bigcup\{Z_\lambda : \lambda \in M_2\}$, then there is $x \in X_\Phi \cap \bigcap_{n<\omega} p^{-1}(A_n)$. So, $p(x) \in X_\Phi \cap G$. If $g = \{\zeta, z_\zeta\}$, then $z_\zeta \in \bigcap_{n<\omega} p^{-1}(A_n)$, hence, again, there is $x \in X_\Phi \cap \bigcap_{n<\omega} p^{-1}(A_n)$, and $p(x) \in X_\Phi \cap G$.

Case 2: There is $s_0 \in S$ such that X_s is Lindelöf for every $s \geq s_0$. In this case $X_\Phi = X_{\Phi_1} \oplus X_{\Phi_2}$ where $\Phi_1 = (L_1 \cap [0, s_0], L_2 \cap [0, s_0])$ and $\Phi_2 = (L_1 \cap (s_0, \kappa], L_2 \cap (s_0, \kappa])$. By the inductive hypothesis, X_{Φ_1} is truly weakly pseudocompact. On the other hand, each X_s is locally compact (and Lindelöf) for every $s \geq s_0$.

Subcase 1: If assertion (i) in (3) holds for κ, then, by Corollary 1.4, we arrive to the conclusion that X_{Φ_2} is locally compact, and hence also truly weakly pseudocompact.

Subcase 2: Now let us assume that assertion (iii) in (3) holds for κ. We are going to prove that in this subcase we can reduce the proof to Case 1.

In this subcase, there must exist an increasing α-sequence $s_0 = x_0 < x_1 < \dots < x_\xi < \dots$ such that $\alpha = \mathrm{cof}(\kappa)$, $\sup\{x_\xi : \xi < \alpha\} = \kappa$ and $x_\xi \in \{\lambda < \kappa : \lambda$ is a limit ordinal$\} \setminus S$ for every $\xi < \alpha$.

If $\alpha > \aleph_1$, then we define $y_0 = x_0$ and if y_ξ has been defined for all $\xi < \eta < \alpha$, let $y_\eta = \sup\{y_\xi : \xi < \eta\}$ if η is a limit ordinal, and $y_\eta = x_{\xi_0+\omega_1}$ if $\eta = \xi_1 + 1$ and $y_{\xi_1} = x_{\xi_0}$. Also we define $Y_{\xi+1} = \bigcup\{X_\gamma : \gamma \in S \cap (y_\xi, y_{\xi+1})\}$ with the topology inherited from X_Φ. Let $\Psi = (M_1, M_2)$ where $M_1 = \{\alpha\}$ and $M_2 = \{\lambda < \alpha : \lambda$ is not a limit ordinal$\}$. We have that $Y_\Psi \cong X_{\Phi_2}$. Besides, for each $\gamma \in M_2$, Y_γ is not Lindelöf because it is the free topological sum of $\aleph_1$ nonempty spaces. So we can return to Case 1 and conclude that $Y_\Psi \cong X_{\Phi_2}$ is truly weakly pseudocompact.

Finally, if $\alpha \le \aleph_1$ then there must exist a cofinal set J in α such that $Z_j = \bigcup\{X_\xi : j < \xi < j+1\}$ is not Lindelöf for every $j \in J$, because otherwise $W_\gamma = \bigcup\{X_\xi : x_\gamma < \xi < x_{\gamma+1}\}$ is Lindelöf for every $\gamma < \alpha$; hence, by Theorem 1.3, $S \cap (x_\gamma, x_{\gamma+1})$ is Lindelöf. Thus $(s_0, \kappa]$ must be Lindelöf because $\alpha \le \omega_1$; but then assertion (iii) in (3) does not hold for κ; a contradiction. Then, as was made for $\alpha > \omega_1$ we can use the proof of Case 1 and conclude that X_{Φ_2} is truly weakly pseudocompact.

Therefore, in any case, X_Φ is truly weakly pseudocompact (Theorem 0.2). $\square$

Corollary 1.6. *Let X_λ be a truly weakly pseudocompact space for every $\lambda \in S$. Then the space X_Φ is weakly pseudocompact if and only if X_Φ is locally truly weakly pseudocompact and not Lindelöf.*

Problem. In the next paragraph we give a slight modification of a problem due to F. Eckertson [Eck].

Let κ and γ be two cardinal numbers such that $\gamma \le \kappa$. Suppose for every $\lambda < \kappa$, X_λ is a weakly pseudocompact space or locally compact Lindelöf space. Let $X_\kappa = \{\kappa\}$, and consider the following topology on $X = \bigoplus\{X_\lambda : \lambda \le \kappa\}$: A basic system of neighborhoods for $x \in X_\lambda$ in X, when $\lambda < \kappa$, is a basic system of neighborhoods for x in X_λ, and a basic system of neighborhoods at κ is the family of sets of the form $\bigcup\{X_\lambda : \lambda \in B\}$ with $B \subset \kappa + 1$, $\kappa \in B$ and $|B| < \gamma$. We denote this space by $X_{\kappa,\gamma}$.

When is $X_{\kappa,\gamma}$ a weakly pseudocompact space?

2. LOTS and weak pseudocompactness

In this section we will prove some facts about the weak pseudocompactness in GLOTS. Let X be a GLOTS, and let Z be a LOTS that contains X. We may embed Z in a compact LOTS $\hat{Z}$ (see, e.g., [Eng, 3.12.3(b)]); furthermore, the closure of X in $\hat{Z}$ is a LOTS, so we may assume without loss of generality that Z is compact and X is dense in Z. In what follows we assume that Z with a linear order $<$ is fixed; the denotations (a, b), $[a \rightarrow)$, etc. will always refer to the intervals in Z. We will call *standard neighborhoods* of a point $x_0 \in X$ the intersections with

X of intervals in Z that contain x_0; obviously, standard neighborhoods of a point of X form a base at this point in X. To avoid unnecessary cases, we will assume that the order $<$ is actually defined on $Z \cup \{\leftarrow, \rightarrow\}$ so that $\leftarrow$ is the minimal and $\rightarrow$ is the maximal element in $Z \cup \{\leftarrow, \rightarrow\}$ (so both $\leftarrow$ and $\rightarrow$ are isolated points in the linearly ordered space $Z \cup \{\leftarrow, \rightarrow\}$).

Definition 2.1. Let X be a GLOTS, and let x_0 be an element of X.

We say that a point $x_0 \in X$ is a *point of true weak pseudocompactness* (*weak pseudocompactness, local compactness*) *at the left* (*at the right*) if there is an $x \in Z$ with $x < x_0$ ($x > x_0$) such that $X \cap (x, x_0]$ ($X \cap [x_0, x)$) is truly weakly pseudocompact (resp., weakly pseudocompact, locally compact).

Obviously, every point that is a point of local compactness both at the left and right is a point of local compactness of X. Note that because X is dense in Z, every open, locally compact subset of X is open in Z, so every point of local compactness lies in the interior of X in Z.

Example 2.2. It is possible that for some element x_0 in a LOTS X, X is not truly weakly pseudocompact in x_0 to its left (or to its right), but, nevertheless, X is weakly pseudocompact. Indeed, let $L_2 = \{\, \lambda < \omega_1 : \lambda \text{ is a non-limit ordinal}\,\}$. For each $\lambda \in L_2$, let $(X_\lambda, <_\lambda)$ be a non-compact, locally compact and Lindelöf LOTS without the first and last element, and let $(Y_\lambda, \prec_\lambda)$ be a weakly pseudocompact, non-compact LOTS without the first and last elements. We define $X = \{\omega_1\} \cup \bigcup_{\lambda\in L_2} X_\lambda \cup \bigcup_{\lambda\in L_2} Y_\lambda$ with the following order: $x < y$ if and only if

(1) $x, y \in X_\lambda$ for a $\lambda \in L_2$ and $x <_\lambda y$, or
(2) $x, y \in Y_\lambda$ for a $\lambda \in L_2$ and $x \prec_\lambda y$, or
(3) $x \in X_\lambda$, $y \in X_\xi$ and $\lambda < \xi$, or
(4) $x \in Y_\lambda$, $y \in Y_\xi$ and $\lambda > \xi$, or
(5) $x \in X_\lambda$ and $y \in Y_\xi$ with $\lambda, \xi \in L_2$, or
(6) $x \in \bigcup_{\lambda\in L_2} X_\lambda$ and $y = \omega_1$, or
(7) $y \in \bigcup_{\lambda\in L_2} Y_\lambda$ and $x = \omega_1$.

Then X is not weakly pseudocompact at the right in ω_1 (by Theorem 1.5). We are going to prove that X is weakly pseudocompact.

For each $\lambda \in L_1 = \{\, \lambda \le \omega_1 : \lambda \text{ is a limit ordinal}\,\}$, let $X_\lambda = \{\lambda_X\}$ and $Y_\lambda = \{\lambda_Y\}$ where $\lambda_X = \lambda_Y = \lambda$. Let $W_\lambda = X_\lambda \cup \{p_\lambda\}$ be the Alexandroff compactification of X_λ, and let $U_\lambda = bY_\lambda$ be a compactification of Y_λ in which Y_λ is embedded as a G_δ-dense subspace. Let q_λ be a point in $bY_\lambda \setminus Y_\lambda$ for each $\lambda \in L_2$.

Take $K_0 = W_\Phi$ and $K_1 = U_\Phi$ where $\Phi = (L_1, L_2)$, and let K be the quotient space obtained by using the following equivalent relation in $K_0 \cup K_1$: $x \sim y$ if and only if

(1) $x, y \in K_i$ ($i = 0, 1$) and $x = y$, or
(2) there exists $\lambda \in L_2$ with $x = p_\lambda$ and $y = q_\lambda$, or
(3) there exists $\lambda \in L_1 \setminus \{\omega_1\}$ such that $x, y \in \{\, \lambda_X, \lambda_Y, p_{\lambda+1}, q_{\lambda+1}\}$;
(4) $x = (\omega_1)_X$ and $y = (\omega_1)_Y$.

It is easy to prove that K is a compactification of X, and X is G_δ-dense in K.

Lemma 2.3. *If X is locally compact at the left at a point $x \in X$, then there is a $z \in Z \cup \{\leftarrow\}$ such that $z < x$ and $(z, x] \subset X$.*

Proof: If $(\leftarrow, x) = \emptyset$, then put $z = \leftarrow$. Otherwise, there is a point $z < x$ such that the closure in X of $X \cap (z, x)$ is compact. The set (z, x) is open in Z, and X is dense, so the closure of $X \cap (z, x)$ in Z contains (z, x). Obviously, this closure lies in X. □

On the other hand, if x is not locally compact at the right (left) in X, then for any $z \in Z$ such that $z > x$ ($z < x$), x is a limit point of $(Z \setminus X) \cap (x, z)$ (respectively, of $(Z \setminus X) \cap (z, x)$).

Lemma 2.4. *If Y is a truly weakly pseudocompact space and F is a closed subset of Y such that $Y \setminus F$ is contained in a σ-compact subspace of Y, then F is truly weakly pseudocompact.*

Proof: If Y is locally compact, then F is locally compact. Now assume that Y is weakly pseudocompact and not compact, and let K be a compactification of Y in which Y is G_δ-dense. Let $\tilde{K} = \mathrm{cl}_K F$. The space $\tilde{K}$ is a compactification of F. Let $G = \bigcap_{n<\omega} G_n$ be a G_δ-set in $\tilde{K}$, where each G_n is open in $\tilde{K}$, and suppose that $p \in G$. For each $n < \omega$ there exists an open subset A_n of K such that $A_n \cap \tilde{K} = G_n$. Then $A = \bigcap_{n<\omega} A_n$ is a nonempty G_δ-set in K. Hence, $A \cap Y \neq \emptyset$. Let $(K_n)_{n<\omega}$ be a sequence of compact subsets of Y such that $Y \setminus F \subset \bigcup_{n<\omega} K_n$, and let $B = A \cap \bigcap_{n<\omega}(K \setminus K_n)$. Since K_n is compact for every $n < \omega$, B is a G_δ-set in K. If $p \notin B$, then $p \in K_n$ for some n. So, $p \in Y \cap \mathrm{cl}_K F$; but F is closed in Y, whence $p \in F$. If $p \in B$, then $B \cap Y \neq \emptyset$, because B is a non-empty G_δ-set in K. But $B \cap (Y \setminus F) = \emptyset$, hence $\emptyset \neq B \cap F \subset A \cap F = G \cap F$. Thus, F is G_δ-dense in $\tilde{K}$, and therefore is weakly pseudocompact. □

Lemma 2.5. *Let X be a truly weakly pseudocompact GLOTS, and suppose x_0 is a point of X that is locally compact at the right. Then $X \cap (\leftarrow, x_0]$ is truly weakly pseudocompact.*

Proof: If $X \cap (x_0, \rightarrow) = \emptyset$, then $X \cap (\leftarrow, x_0] = X$, and there is nothing to prove. Otherwise, fix a point $q \in X \cap (x_0, \rightarrow)$ so that $X \cap [x_0, q]$ is compact. If $X \cap [x_0, q]$ is finite, then $X \cap (\leftarrow, x_0]$ is open in X, and is truly weakly pseudocompact by Theorem 0.1. Otherwise we can find a sequence of points $\{x_n : n \in \omega\}$ in $X \cap (x_0, q)$ so that $a = \sup\{x_n : n < \omega\}$ does not coincide with any point in this sequence. In this case, $Y = X \cap (\leftarrow, a)$ is truly weakly pseudocompact by Theorem 0.1, and $X \cap [x_0, a) = \bigcup_{n<\omega}(X \cap [x_0, x_n])$ is σ-compact. Lemma 2.4 applied to $F = X \cap (\leftarrow, x_0]$ and Y yields the weak pseudocompactness of $X \cap (\leftarrow, x_0]$. □

Corollary 2.6. *If x_0 has a truly weakly pseudocompact neighborhood in X, and X is locally compact at the right (left) in x_0, then X is truly locally pseudocompact in x_0 at the left (right).*

Proof: Apply Lemma 2.4 to a standard neighborhood of x_0. □

Lemma 2.7. *Let $X = A \cup B$ be a space where $A \cap B$ is compact. Then we have:*

(1) *X is weakly pseudocompact if A and B are weakly pseudocompact,*
(2) *X is weakly pseudocompact if A is a weakly pseudocompact and non-compact space, and B is locally compact,*
(3) *X is locally compact if both A and B are locally compact,*
(4) *X is truly weakly pseudocompact if A and B are truly weakly pseudocompact.*

Proof: (3) is immediate, and (4) is a consequence of the three previous assertions. So we will prove (1) and (2).

(1): Let K_A and K_B be compactifications of A and B respectively, in which A and B are G_δ-dense (if A or B is compact, then $K_A = A$ or $K_B = B$). Let K be the free topological sum $K_A \oplus K_B$. We define in K the following equivalence relation: $a \sim b$ iff either $a = b$ in K or $a = b$ in X. The quotient space $K_0 = K/\sim$ is a compact T_2 space because the natural projection $p : K \to K_0$ is closed. The space $p(X)$ is G_δ-dense in K_0 and is homeomorphic to X.

(2): We can assume that B is not compact because otherwise we obtain the conclusion from (1). Let K_A be a compactification of A such that A is G_δ-dense in K_A, and let $q \in K_A \setminus A$. Let $K_B = B \cup \{p\}$ be the one point compactification of B where $p \notin B$. Let $K = K_A \oplus K_B$. We consider the following equivalence relation $\sim$ in K: $a \sim b$ iff either $a = b$ in K, or $a = q$ and $b = p$, or $a = b$ in X. The quotient space $K_0 = K/\sim$ is a compactification of X in which X is G_δ-dense. □

Corollary 2.8. *Let X be a GLOTS.*

(1) *If there exists $x_0 \in X$ such that both $X \cap (\leftarrow, x_0]$ and $X \cap [x_0, \rightarrow)$ are weakly pseudocompact, then X is weakly pseudocompact.*
(2) *Suppose there exists $x_0 \in X$ such that $X \cap (\leftarrow, x_0]$ is a weakly pseudocompact non-compact space, and $X \cap [x_0, \rightarrow)$ is locally compact. Then X is weakly pseudocompact.*
(3) *If X has a point x_0 such that both $X \cap [x_0, \rightarrow)$ and $X \cap (\leftarrow, x_0]$ are truly weakly pseudocompact, then X is truly weakly pseudocompact.*

Corollary 2.9. *If $x_0 \in X$, and x_0 is a point of true weak pseudocompactness from both right and left, then x_0 has a truly weakly pseudocompact open neighborhood in X.*

Proof: By definition, x_0 is a point of true weak pseudocompactness from both right and left if there exist $a, b \in Z$ such that $a < x < b$, and both $X \cap (a, x]$ and $X \cap [x, b)$ are truly weakly pseudocompact. Put $X' = X \cap (a, b)$; then X' is an open neighborhood of x_0 in X. The required statement follows from Corollary 2.8 applied to X'. □

Lemma 2.10. *Suppose every point of X has a truly weakly pseudocompact neighborhood. Then every truly weakly pseudocompact standard open set in X is contained in a truly weakly pseudocompact standard open set that is also closed in X.*

Proof: Let U be a standard open set in X, $U = X \cap (a, b)$ where $a \in Z \cup \{\leftarrow\}$ and $b \in Z \cup \{\rightarrow\}$. We will construct $a' \in (Z \setminus X) \cup \{\leftarrow\}$ and $b' \in (Z \setminus X) \cup \{\rightarrow\}$ so that $a' \leq a$, $b' \geq b$ and $U' = (a', b') \cap X$ is truly weakly pseudocompact; obviously the set U' of this form is clopen in X.

Let us first find a' so that $a' \in Z \setminus X$, $a' \leq a$ and $X \cap (a', b)$ is truly weakly pseudocompact.

We have the following possible cases:

Case 1. a does not belong to X.

Put $a' = a$.

Case 2. a belongs to X and a is not locally compact at any side.

Let V be a truly weakly pseudocompact neighborhood of a in X; since a is not a point of local compactness in X at any side, there are points $c, d \in Z \setminus X$ such that $a \in (c, d)$, $d \in (a, b)$ and $X \cap (c, d) \subset V$. Then $B = X \cap (c, d)$ is a clopen neighborhood of a, and B is truly weakly pseudocompact, because it is open in V. The set $C = (a, b) \setminus B$ is open in (a, b), hence truly weakly pseudocompact. Put $a' = c$; then $X \cap (a', b) = B \cup C$ is truly weakly pseudocompact by Theorem 0.2.

Case 3. a belongs to X, and a is locally compact at the right, but not locally compact at the left.

Let V be a truly weakly pseudocompact neighborhood of a in X. Since a is locally compact at the right, but not locally compact at the left, there are points c, d in Z such that $a \in X \cap (c, d) \subset V$, $c \in Z \setminus X$, $[a, d)$ is locally compact, and d is a point of local compactness of X. By Theorem 0.1, the set $X \cap (c, d)$ is truly weakly pseudocompact. By Lemma 2.3 applied to the sets (c, d) and (a, b), the sets $X \cap (c, a]$ and $[d, b)$ are truly weakly pseudocompact. Put $a' = c$; the set $X \cap (a', d] = (X \cap (a', a]) \cup (X \cap [a, d])$ is truly weakly pseudocompact by Corollary 2.8 (applied to the GLOTS $(a', d]$); by the same corollary, $X \cap (a', b) = (X \cap (a', d]) \cup (X \cap [d, b))$ is truly weakly pseudocompact.

Case 4. a belongs to X, a is locally compact at the left and not locally compact at the right.

Let V be a standard neighborhood of a that is truly weakly pseudocompact. Since a is not locally compact at the right, there is $d \in Z \setminus X$ such that $d \in (a, b)$ and $[a, d) \subset V$. By Corollary 2.6 applied to the set V, the set $X \cap [a, d)$ is truly weakly pseudocompact. Furthermore, the set $X \cap [d, b) = X \cap (d, b)$ is truly weakly pseudocompact, because it is open in $X \cap (a, b)$. By Theorem 0.2, the set $X \cap [a, b) = (X \cap [a, d)) \cup (X \cap (d, b))$ is truly weakly pseudocompact.

Put $c = \inf\{ z \in Z \cup \{\leftarrow\} : (z, a] \subset X \}$; since Z is compact, c exists, and since X is locally compact at a at the left, $c < a$ by Lemma 2.3. If $c \notin X$, put $a' = c$. Then $X \cap (a', b) = (X \cap (c, a]) \cup (X \cap [a, b))$ is truly weakly pseudocompact by Corollary 2.8, because $X \cap (c, a]$ is obviously locally compact. Suppose $c \in X$.

Then the same argument shows that $X \cap (c, b)$ is truly weakly pseudocompact; furthermore, X is not locally compact at the left at c by Lemma 2.3.; we now can apply the argument as in Cases 2 and 3 to the interval (c, b) to find a'.

CASE 5. a belongs to X, and a is a point of local compactness of X.

The proof in this case differs from the proof in Case 4 only in the proof that $X \cap [a, b)$ is weakly pseudocompact. Let V be an open neighborhood of a such that the closure of V is compact. If $V \cap (a, b) = \emptyset$, then $[a, b)$ is the union of (a, b) and an isolated point a, hence truly weakly pseudocompact. Otherwise, fix $d \in V \cap (a, b)$. Then d is a point of local compactness in $X \cap (a, b)$, and by Corollary 2.6, the set $X \cap [d, b)$ is truly weakly pseudocompact. The set $X \cap [a, d]$ is compact, so by Corollary 2.8, $X \cap [a, b) = (X \cap [a, d]) \cup (X \cap [d, b))$ is truly weakly pseudocompact. The construction of a' in this case is the same as in Case 4.

Thus, we have found an $a' \in (Z \cup \{\leftarrow\}) \setminus X$ so that $a' \leq a$ and the interval $X \cap (a', b)$ is weakly pseudocompact. Applying a similar procedure to the right end of the interval, we will construct the interval (a', b') as required. □

Theorem 2.11. *Let X be a paracompact GLOTS. If X is locally truly weakly pseudocompact, then X is truly weakly pseudocompact.*

PROOF: Let $\mathcal{U}_0$ be a cover of X with truly weakly pseudocompact open sets, and let $\mathcal{U}$ be a locally finite refinement of $\mathcal{U}_0$. By Theorem 0.1, every element of $\mathcal{U}$ is truly weakly pseudocompact. Every element of $\mathcal{U}$ is a disjoint union of a family of intersections with X of intervals in Z; let $\mathcal{V}_0$ be the collection of all these intersections. Again, by Theorem 0.1, all sets in $\mathcal{V}_0$ are truly weakly pseudocompact. Obviously, the cover $\mathcal{V}_0$ is point-finite. Let $\mathcal{V}$ be an irreducible subcover of $\mathcal{V}_0$ (recall that a cover of a space is called *irreducible* if it has no proper subcover; every point-finite cover has an irreducible subcover, see, e.g., [Eng, 5.3.1]). We will now need the following lemma, well-known in folklore (and easily proved by an analysis of possible cases):

Lemma 2.12. *If I_1, I_2 and I_3 are three intervals in a linearly ordered space whose intersection is not empty, then one of the intervals is contained in the union of the others.*

Obviously, the same lemma is true for intersections of intervals with a subspace; it follows that every point in X is contained in at most two intervals in $\mathcal{V}$, so $\mathcal{V}$ is locally finite.

A standard "star" argument shows that X is a sum of its subspaces $\{X_\alpha : \alpha \in A\}$ each of which is covered by a countable subfamily $\mathcal{V}_\alpha$ of intervals in $\mathcal{V}$. By Theorem 0.2, it suffices to show that each X_α is truly weakly pseudocompact.

Replace each interval $X \cap (a, b)$ in $\mathcal{V}_\alpha$ by a clopen truly weakly pseudocompact interval $X \cap (a', b')$ as in Lemma 2.10, and let $\mathcal{V}'_\alpha$ be the family of intersections of these intervals with X_α. Since X_α is clopen in X, by Theorem 0.1, $\mathcal{V}'_\alpha$ is a countable cover of X_α with clopen truly weakly pseudocompact subsets.

Enumerate $\mathcal{V}'_\alpha$: $\mathcal{V}'_\alpha = \{U_i : i \in \omega\}$, and for each $n \in \omega$ put $W_n = U_n \setminus \bigcup_{i=0}^{n-1} U_i$. Since the sets U_n are clopen in X, the sets W_n are also clopen; moreover, W_n is open in U_n, hence truly weakly pseudocompact. We have $X_\alpha = \bigoplus\{W_n : n \in \omega\}$, and X_α is truly weakly pseudocompact by Theorem 0.2. □

Lemma 2.13. *Let κ be a limit ordinal number. Let $\{b_\lambda : \lambda \leq \kappa\}$ be an increasing κ-sequence of elements in Z satisfying the following conditions:*

(1) *if $\gamma < \kappa$ is a non-limit ordinal, then $b_\gamma \in Z \setminus X$;*
(2) *if $\gamma < \kappa$ is a limit ordinal, then $b_\gamma = \sup\{b_\lambda : \lambda < \gamma\}$;*
(3) *for every $\lambda < \gamma < \kappa$, $(b_\lambda, b_\gamma) \cap X$ is truly weakly pseudocompact; and*
(4) *if $\sup\{b_\lambda : \lambda < \kappa\} = b_\kappa \in X$, then either for each $\gamma < \kappa$ there exists $\gamma_0 > \gamma$ such that $(b_{\gamma_0}, b_{\gamma_0+1}) \cap X$ is not Lindelöf, or there is $a < b_\kappa$ such that $(a, b_\kappa]$ is truly weakly pseudocompact.*

Then $Y = X \cap (b_0, b_\kappa]$ is truly weakly pseudocompact.

Proof: Case 1. Assume that there exists $\gamma_0 < \kappa$ such that, for every $\gamma \in (\gamma_0, \kappa)$, the interval $(b_{\gamma_0}, b_{\gamma+1}) \cap X$ is Lindelöf. Then Y is the free topological sum of spaces $Y_1 = X \cap (b_0, b_{\gamma_0+1})$ and $Y_2 = X \cap (b_{\gamma_0+1}, b_\kappa]$. By condition (3), Y_1 is truly weakly pseudocompact. It remains to prove that Y_2 has also this property.

We have that $(b_{\gamma_0+1}, b_\kappa) \cap X$ is locally compact. Each $\gamma_0 < \gamma < \kappa$ can be represented as $\gamma = \lambda(\gamma) + n(\gamma)$ where $n(\gamma) < \omega$ and $\lambda(\gamma)$ is a limit ordinal. If γ is a limit ordinal and $b_\gamma \in X$, we put $Y_\gamma = \{\gamma\}$, $Y_{\gamma+1} = [b_\gamma, b_{\gamma+1}) \cap X$, and $Y_{\gamma+n+1} = (b_{\gamma+n}, b_{\gamma+n+1}) \cap X$ for every $0 < n < \omega$. Finally, for each $\gamma < \kappa$ with $b_{\lambda(\gamma)} \notin X$, put $Y_{\gamma+1} = (b_\gamma, b_{\gamma+1}) \cap X$. Let $L_1 = \{\lambda \in (\gamma_0 + 1, \kappa] : \lambda$ is a limit ordinal and $b_\lambda \in X\}$ and $L_2 = \{\lambda \in (\gamma_0 + 1, \kappa) : \lambda$ is not a limit ordinal$\}$. Put $S = L_1 \cup L_2$, $\Phi = (L_1, L_2)$ and consider each Y_γ with $\gamma \in L_2$ as a subspace of X. Observe that Y_Φ is locally compact at each point $y \in Y_\Phi \setminus \{b_\kappa\}$. If $b_\kappa \in X$, then Y_Φ is also locally truly weakly pseudocompact at b_κ (condition (4)). Because of Theorem 1.5, Y_Φ is truly weakly pseudocompact.

If Y_Φ is Lindelöf, then it is locally compact. We define the function $p : Y_\Phi \to Y_2$ as follows: $p(x) = x$ if $x \in Y_\gamma$ and $\gamma \in L_2$, and if $\gamma \in L_1$, $p(\gamma) = b_\gamma = p(b_\gamma)$. The mapping p is perfect and onto, so Y_2 is locally compact.

If Y_Φ is not Lindelöf, then there is a compact space K which contains Y_Φ as a G_δ-dense subspace. We consider the following equivalent relation in K: $a \sim b$ iff either $a = b$ or $a = \gamma$ and $b = b_\gamma$. The space $K_0 = K/\sim$ is a Hausdorff compactification of Y_2 where this last space is embedded as a G_δ-dense subspace.

Case 2. For each $\gamma < \kappa$ there exist $\gamma_0 > \gamma$ with $\gamma_0 \leq \kappa$ such that $(b_{\gamma_0}, b_{\gamma_0+1}) \cap X$ is not Lindelöf. In this case we can assume, without loss of generality, that for every $\gamma < \kappa$, $(b_{\gamma+1}, b_{\gamma+2}) \cap X$ is a weakly pseudocompact non-compact space. For each limit ordinal γ such that $b_\gamma \in X$, we have a neighborhood $(a, b) \cap X$ of b_γ which is truly weakly pseudocompact. Since $b_\gamma = \sup\{b_\lambda : \lambda < \gamma\}$, and each $(b_\lambda, b_{\lambda+1}) \cap X$ is not Lindelöf, then $(a, b) \cap X$ is a weakly pseudocompact and non-compact space; so, there exists a compact space K_γ in which $(a, b) \cap X$ is embedded as a G_δ-dense subset. For this kind of γ, we define $W_\gamma = \{\gamma\}$,

$W_{\gamma+1} = \mathrm{cl}_{K_\gamma}([b_\gamma, b_{\gamma+1}) \cap X)$ and $W_{\gamma+n+1} = (b_{\gamma+n}, b_{\gamma+n+1}) \cap X$. Besides, for each $\gamma < \kappa$ with $b_{\lambda(\gamma)} \notin X$, we take $W_{\gamma+1} = (b_\gamma, b_{\gamma+1}) \cap X$. Let $M_1 = \{\lambda < \kappa : \lambda$ is a limit ordinal and $b_\lambda \in X\}$ and $M_2 = \{\lambda < \kappa : \lambda$ is not a limit ordinal$\}$. Put $\Psi = (M_1, M_2)$ and consider W_Ψ. Every W_λ is a truly weakly pseudocompact space, and if $\lambda \in M_1$, then there is a cofinal set $J \subset S = M_1 \cup M_2$ in λ such that W_j is not Lindelöf for every $j \in J$. Then W_Ψ is truly weakly pseudocompact (Theorem 1.5). Since W_Ψ contains non-Lindelöf closed subspaces, W_Ψ is weakly pseudocompact and non-compact. Let K be a compactification of W_Ψ in which W_Ψ is G_δ-dense embedded. Consider in K the relation $\sim$ defined by: $a \sim b \Leftrightarrow$ either $a = b$ or $a = \gamma$ and $b = b_\gamma$. The projection $p : K \to K/\sim\, = K_0$ is a closed mapping, so K_0 is a Hausdorff compact space containing Y as a G_δ-dense subspace. Therefore, Y is weakly pseudocompact. □

Of course, the proof of the previous lemma remains valid (with obvious changes) if we consider a decreasing κ-sequence instead of an increasing one.

Corollary 2.14. *We obtain the same conclusion than that in Lemma* 2.13 *if we only change condition* (4) *in this lemma for condition*

(4′) *if* $\sup\{b_\lambda : \lambda < \kappa\} = b_\kappa \in X$, *then either for each* $\gamma < \kappa$ *there exists* $\gamma_0 > \gamma$ *such that* $(b_{\gamma_0}, b_{\gamma_0+1}) \cap X$ *is not Lindelöf, or there exist* $a, b \in Z$ *such that* $a < b_\kappa < b$, $(a, b) \cap X$ *is truly weakly pseudocompact, and* $[b_X, b] \cap X$ *is compact.*

PROOF: Condition (4) in Lemma 2.13 follows from (4′) by Lemma 2.5. □

Lemma 2.15. *Let* X *be a GLOTS which is not locally compact at any point. Then* X *is locally weakly pseudocompact at* $x_0 \in X$ *if and only if* x_0 *is weakly pseudocompact at its right and at its left in* X.

PROOF: The sufficiency follows from Corollary 2.8 (1).

(⇒): If $X \cap (\leftarrow, x_0) = \emptyset$, then $X \cap (\leftarrow, x_0]$ is compact. Otherwise, fix $a, b \in Z$ so that $a < x_0 < b$ and $(a, b) \cap X$ is truly weakly pseudocompact. We can find an increasing κ-sequence $\{a_\lambda : \lambda < \kappa\}$ of elements of Z such that

(1) $a \le a_0$,
(2) $x_0 = \sup\{a_\lambda : \lambda < \kappa\}$,
(3) if $\gamma < \kappa$ is a non-limit ordinal, then $a_\gamma \in Z$.
(4) if $\gamma < \kappa$ is a limit ordinal, then $a_\gamma = \sup\{a_\lambda : \lambda < \gamma\}$.

Since $X \cap (a, b)$ is weakly pseudocompact and not locally compact at any point, each $X \cap (a_\lambda, a_\gamma)$, $\lambda < \gamma < \kappa$, is truly weakly pseudocompact and not Lindelöf. By Lemma 2.13, $X \cap (a_0, x_0]$ is weakly pseudocompact.

Similarly, there is $b_0 > x_0$ such that $X \cap [x_0, b_0)$ is weakly pseudocompact. □

Corollary 2.16. *Let* X *be a GLOTS which is not locally compact at any point. Then* X *is weakly pseudocompact if and only if for every* $x \in X$, $(\leftarrow, x] \cap X$ *and* $[x, \rightarrow) \cap X$ *are weakly pseudocompact.*

PROOF: Again, we obtain the sufficiency using Corollary 2.8.1.

Necessity: Let $x_0 \in X$. Since X is weakly pseudocompact and not locally compact at any point, there is $b_0 \in Z$ with $b_0 < x_0$ such that $(b_0, x_0] \cap X$ is weakly pseudocompact (Lemma 2.15). Moreover, $(\leftarrow, b_0) \cap X$ is truly weakly pseudocompact (Theorem 0.1), so $(\leftarrow, x_0] \cap X$ also satisfies this property (Theorem 0.2). But $(\leftarrow, x_0] \cap X$ is not Lindelöf because it is not locally compact, hence it must be weakly pseudocompact. The same argument works for $[x_0, \rightarrow) \cap X$. $\square$

Remark. The previous result does not hold for LOTS with points of local compactness, even zero-dimensional ones. Indeed, let ω^* be the set of natural numbers with the inverse natural order and let $X = \omega^* \cup [0, \omega_1)$ with the following order: $x < y$ iff either $x \in \omega^*$ and $y \in [0, \omega_1)$ or $x, y \in \omega^*$ and $x <_{\omega^*} y$ or $x, y \in [0, \omega_1)$ and $x <_{[0,\omega_1)} y$. X is a zero-dimensional weakly pseudocompact LOTS, but for every $x \in X$, $(\leftarrow, x]$ is Lindelöf and non-compact.

Definition 2.17. Let X be a GLOTS. and let o_0 and o_1 be the first and the last elements of Z. Put $Z_0 = Z \setminus \{o_0, o_1\}$, $\mathcal{L} = \{\, x \in X : (o_0, x] \subset X\}$ and $\mathcal{R} = \{\, x \in X : [x, o_1) \subset X\}$. We denote by a_X and b_X the supremum, in Z, of $\mathcal{L}$ and the infimum, in Z, of $\mathcal{R}$. Let $\mathfrak{R}_X$ be the set $\{a_X, b_X\}$. Of course, $\mathfrak{R}_X \subset Z$.

Remark.

(1) If $Z_0 \setminus X \neq \emptyset$, then $a_X \leq b_X$.
(2) The spaces $(\leftarrow, a_X] \cap X$ and $[b_X, \rightarrow) \cap X$ are locally compact.
(3) If $Z_0 \setminus X \neq \emptyset$, then there is an increasing (resp., decreasing) α-sequence of elements in $Z_0 \setminus X$ converging to b_X (resp., a_X).
(4) $X = ((\leftarrow, a_X] \cup [a_X, b_X] \cup [b_X, \rightarrow)) \cap X$.
(5) For every $x, y \in X$ with $x < y$, $(x, y) \cap X$ is truly weakly pseudocompact if and only if for all $a, b \in Z_0$ with $a < b$, $(a, b) \cap X$ is truly weakly pseudocompact.

Theorem 2.18. *Let X be a GLOTS. Then the following statements are equivalent:*

(1) *X is truly weakly pseudocompact;*
(2) *for every $x, y \in X$ with $x < y$, $(x, y) \cap X$ is truly weakly pseudocompact;*
(3) *for every $a, b \in Z_0$ with $a < b$, $(a, b) \cap X$ is truly weakly pseudocompact;*
(4) *for every $x \in X$, $(x, \rightarrow) \cap X$ and $(\leftarrow, x) \cap X$ are truly weakly pseudocompact;*
(5) *every proper open subset of X is truly weakly pseudocompact;*
(6) *there exists $x_0 \in X$ such that $(\leftarrow, x_0] \cap X$ and $[x_0, \rightarrow) \cap X$ are truly weakly pseudocompact.*

Proof: The implications (1) $\Rightarrow$ (5) $\Rightarrow$ (4) $\Rightarrow$ (2), and (3) $\Leftrightarrow$ (2) are trivial, so we need to prove (3) $\Rightarrow$ (1) $\Leftrightarrow$ (6).

(3) $\Rightarrow$ (1): If $Z_0 \subset X$, then X is locally compact and there is nothing to prove. If $Z_0 \setminus X \neq \emptyset$, then there exist two ordinals $\alpha, \kappa > 0$, a decreasing α-sequence

$\{a_\lambda : \lambda < \alpha\}$ and an increasing κ-sequence $\{b_\lambda : \lambda < \kappa\}$ of elements in Z_0 such that

(1) $a_0 < b_0$;
(2) $\inf\{a_\lambda : \lambda < \alpha\} = a_X$ and $\sup\{b_\lambda : \lambda < \kappa\} = b_X$;
(3) if $\gamma < \alpha$ (resp., $\gamma < \kappa$) is a non-limit ordinal, then $a_\gamma \in Z_0 \setminus X$ (resp., $b_\gamma \in Z_0 \setminus X$);
(4) if $\gamma < \alpha$ (resp., $\gamma < \kappa$) is a limit ordinal, then $a_\gamma = \inf\{a_\lambda : \lambda < \gamma\}$ (resp., $b_\gamma = \sup\{b_\lambda : \lambda < \kappa\}$).

The space X is equal to $X_0 \oplus X_1 \oplus X_2$ where $X_0 = X \cap (a_0, b_0)$, $X_1 = X \cap (\leftarrow, a_0]$ and $X_2 = X \cap [b_0, \rightarrow)$. By the assumption, X_0 is truly weakly pseudocompact. We are going to prove that X_2 is also truly weakly pseudocompact. This proof will work for X_1 too.

For each $\lambda < \gamma < \kappa$, (b_λ, b_γ) is truly weakly pseudocompact, and if $b_X \in X$, then b_X has a truly weakly pseudocompact neighborhood V; besides, $[b_X, \rightarrow)$ is locally compact. By Corollary 2.14, $X \cap (b_0, b_X]$ is truly weakly pseudocompact. Now, in order to conclude that X_2 is truly weakly pseudocompact, we only have to apply the fact that $[b_X, \rightarrow)$ is locally compact and Corollary 2.8.

Let us now prove (1) $\Leftrightarrow$ (7). Suppose that X is truly weakly pseudocompact. If X is not locally compact at any point and $x_0 \in X$, then $X \cap (\leftarrow, x_0]$ and $X \cap [x_0, \rightarrow)$ are truly weakly pseudocompact because of Corollary 2.16.

If X is locally compact at $x_0 \in X$, then there exists $a \in Z \cup \{\leftarrow, \rightarrow\}$ with $a < x_0$, such that $X \cap (a, x_0]$ is locally compact. If we denote by X_1 the space $X \cap (\leftarrow, x_0]$, then $b_{X_1} < x_0$ (see Definition 2.17). Now, we can construct an increasing κ-sequence converging to b_{X_1} and satisfying conditions (1)–(3) in Lemma 2.13 and (4′) in Corollary 2.14. So, $X \cap (\leftarrow, x_0]$ is truly weakly pseudocompact. In a similar way we can prove that $X \cap [x_0, \rightarrow)$ is truly weakly pseudocompact.

If $(\leftarrow, x_0]$ and $[x_0, \rightarrow)$ are truly weakly pseudocompact, then X is truly weakly pseudocompact by Corollary 2.8. $\square$

Problems 2.19. 1. *Is it true that the following assertions are equivalent for a non-Lindelöf LOTS X?*

(1) *X is weakly pseudocompact.*
(2) *X is locally truly weakly pseudocompact.*
(3) *X has an open cover consisting of truly weakly pseudocompact sets.*
(4) *There are $x, y \in X$ with $x < y$ such that $(x, \rightarrow)$ and $(\leftarrow, y)$ are truly weakly pseudocompact.*

2. *Is there a locally truly weakly pseudocompact space that is not truly weakly pseudocompact?*

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