Fixed points for multifunctions on generalized metric spaces with applications to a multivalued Cauchy problem

Adrian Petruşel

Abstract. The purpose of this paper is to prove an existence result for a multivalued Cauchy problem using a fixed point theorem for a multivalued contraction on a generalized complete metric space.

Keywords: generalized metric space, multivalued contraction, fixed points

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1. Introduction

In 1958 W.A.J. Luxemburg, using a fixed point theorem for a single-valued contraction on a generalized metric space, proved the existence and the uniqueness of solution of the following Cauchy problem:

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{1}$$

where t and x are real variables and f is a real function defined on the rectangle $|t - t_0| \leq a$, $|x - x_0| \leq b$, $a, b > 0$.

The purpose of this paper is to prove an existence result for a multivalued Cauchy problem using a fixed point theorem, for a multivalued contraction defined on a complete generalized metric space.

2. Preliminaries

The concept of a generalized metric space was introduced by Luxemburg and Jung as follows:

Definition 2.1 ([6], [9]). The pair (X, d) will be called a generalized metric space if X is an arbitrary nonempty set and d is a function $d : X \times X \to [0, \infty]$ which fulfills all the standard conditions for a metric.

In this paper, the generalized metric d is allowed to take the value $+\infty$ as well. In a generalized, just as in a metric space, we can define open and closed balls, convergence of sequences, completeness of the space, etc.

If (X, d) is a generalized metric space, $Y \subset X$, $x \in X$ and $\varepsilon > 0$ then:
$\delta(Y) = \sup\{d(a, b) \mid a, b \in Y\}$,
$D(Y, x) = \inf\{d(y, x) \mid y \in Y\}$,
$B_X(x, \varepsilon) = \{y \in X \mid d(x, y) < \varepsilon\}$,
$V(Y, \varepsilon) = \{x \in X \mid D(Y, x) \leq \varepsilon\}$,
$\mathcal{P}(X) = \{Y \mid Y \subseteq X\}$,
$P(X) = \{Y \in \mathcal{P}(X) \mid Y \neq \emptyset\}$,
$P_{cl}(X) = \{Y \in P(X) \mid Y = \overline{Y}\}$,
$P_{cp,cv}(X) = \{Y \in P(X) \mid Y$ compact and convex in $X\}$ (here X is a generalized normed space),

$$H(A, B) = \begin{cases} \inf\{\varepsilon > 0 \mid A \subset V(B, \varepsilon),\ B \subset V(A, \varepsilon)\}, & \text{if the infimum exists} \\ +\infty, & \text{otherwise.} \end{cases}$$

The pair $(P_{cl}(X), H)$ is a generalized metric space and H is called the generalized Hausdorff-Pompeiu distance induced by d.

Lemma 2.2 ([11])**.** *If (X, d) is complete generalized metric space then $(P_{cl}(X), H)$ is a complete generalized metric space.*

Definition 2.3 ([3])**.** Let (X, d) be a generalized metric space and $T : X \to P_{cl}(X)$ be a multivalued operator. Then, T is called an a-contraction if there exists a real number $a \in [0, 1[$ such that $x, y \in X$, $d(x, y) < \infty \Rightarrow H(T(x), T(y)) \leq ad(x, y)$.

Definition 2.4. Let (X, d) be a generalized metric space and $T : X \to P(X)$ a multivalued operator. Then $x^* \in X$ is called a fixed point for T if $x^* \in T(x^*)$. The set of all fixed points will be denoted by $\mathrm{Fix}T$.

The concept of semi-continuous mappings was introduced in 1932 by Bouligand and Kuratowski.

We consider here the notion of an upper semicontinuous multivalued operator.

Definition 2.5 ([7])**.** Let X, Y be two metric spaces. A multivalued operator $T : X \to P(Y)$ is called upper semicontinuous at $x_0 \in X$ if and only if for any neighborhood U of $T(x_0)$, there exists a neighborhood V of x_0 such that for each $x \in V$ we have $T(x) \subset U$. T is said to be upper semicontinuous (u.s.c.) on X if it is u.s.c. at any point $x_0 \in X$.

Definition 2.6. Let (X, d) be a generalized metric space and $T : X \to P_{cl}(X)$ be a multivalued operator. A sequence $(x_n)_{n \in \mathbf{N}} \subset X$ is called the sequence of successive approximations of T if and only if $x_0 \in X$ and $x_n \in T(x_{n-1})$, $\forall\, n \in \mathbf{N}^*$.

The following result is well known in the field of set-valued analysis (see [1]).

Proposition 2.7. *Let $\Omega \subset \mathbf{R} \times \mathbf{R}^n$ be an open set, $(t_0, x_0) \in \Omega$ and $F : \Omega \to P_{cp}(\mathbf{R}^n)$ an u.s.c. multivalued operator.*

Then there exist $I = [t_0 - a, t_0 + a] \subset \mathbf{R}$ (where $a > 0$) and $M > 0$ such that:

(i) $I \times B_{\mathbf{R}^n}(x_0, aM) \subset \Omega$,
(ii) $\|F(t,x)\| \leq M$ *on* $I \times B_{\mathbf{R}^n}(x_0, aM)$.

An important concept is that of integrably bounded multivalued operator.

Definition 2.8 ([4])**.** Let $(S, \mathcal{A}, \mu)$ be a complete σ-finite measure space and $(X, \|\cdot\|)$ be a separable Banach space. A multivalued operator $T : S \to P_{cl}(X)$ is said to be integrably bounded if and only if there is a function $r \in L^1(S)$ such that for all $v \in T(s)$ we have $\|v\| \leq r(s)$ a.e.

For $1 \leq p \leq \infty$ we define the set:

$$S_T^p := \{f \in L^p(\Omega, X) \mid f(s) \in T(s), \text{ a.e.}\},$$

i.e. S_T^p contains all selectors of T that belong to Lebesgue-Bochner space $L^p(\Omega, X)$.

It is easy to see that S_T^1 is a closed subset of $L^1(\Omega, X)$ and it is nonempty if and only if T is integrably bounded (see [2] and [4]).

Finally, the following theorem is a slight version of a result given in [10].

Theorem 2.9. *Let (X, d) be a complete generalized metric space and $T : X \to P_{cl}(X)$ be a multivalued a-contraction. We suppose that there is a sequence $(x_n)_{n \in \mathbf{N}} \subset X$ of successive approximations of T such that there exists an index $N(x_0) \in \mathbf{N}$ with the following property: $d(x_N, x_{N+l}) < \infty$, for all $l \in \mathbf{N}^*$. Then* $\operatorname{Fix} T \neq \emptyset$.

3. Main result

Consider the following multivalued Cauchy problem

$$\begin{cases} x'(t) \in F(t, x(t)) \\ x(t_0) = x^0 \end{cases} \tag{2}$$

where $F : \Omega \subset \mathbf{R} \times \mathbf{R}^n \to P_{cp}(\mathbf{R}^n)$, with $\Omega = [t_0 - a, t_0 + a] \times \tilde{B}_{\mathbf{R}^n}(x^0, b)$, $(a, b > 0)$.

The main result of this note is the following existence theorem:

Theorem 3.1. *Consider the multivalued Cauchy problem* (2). *We suppose that:*

(i) $F : \Omega \to P_{cp}(\mathbf{R}^n)$ *is u.s.c. and integrably bounded,*
(ii) $|t - t_0| H(F(t,u), F(t,v)) \leq k\|u - v\|$, *for every* $(t,u), (t,v) \in \Omega$,
(iii) $|t - t_0|^\beta H(F(t,u), F(t,v)) \leq A\|u - v\|^\alpha$, *for every* $(t,u), (t,v) \in \Omega$,
(iv) $A, k > 0$, $0 < \alpha < 1$, $\beta < \alpha$ *and* $k(1 - \alpha) < 1 - \beta$.

Then, the multivalued Cauchy problem (2) *has a solution.*

PROOF: From Proposition 2.7 it follows the existence of a real constant $M > 0$ such that $\|F(t,x)\| \leq M$ on Ω.

We denote by I the interval $I = [t_0 - h, t_0 + h]$, where $h = \min\{a, \frac{b}{M}\}$. We shall prove, by an application of Theorem 2.9, the existence of a solution of problem (2) on this interval I.

For this purpose we shall consider a space X with a generalized metric d, as follows:

$$X = \{\varphi \in C(I, \mathbf{R}^n) \mid \|\varphi(t) - x_0\| \leq b,\ \forall\, t \in I,\ \varphi(t_0) = x^0\}$$

$$d : X \times X \to \mathbf{R}_+ \cup \{+\infty\}$$

$$d(\varphi_1, \varphi_2) := \sup\left\{ \frac{\|\varphi_1(t) - \varphi_2(t)\|}{|t - t_0|^{pk}} \mid t \in I \right\},$$

where $p > 1$, $pk(1-\alpha) < 1 - \beta$.

From [9] we have that (X, d) is a complete generalized metric space.

Finally, we choose the multivalued operator $T : X \multimap X$,

$$T(x) := \left\{ v \in X \mid v(t) \in x^0 + \int_{t_0}^{t} F(s, x(s))\, ds \ \text{ a.e. } \ I \right\},$$

(where $\int_{t_0}^{t} F(s, x(s))\, ds$ denotes the multivalued integral of Aumann).

It is easy to see that a function φ^* is a fixed point of T if and only if φ^* is a solution of problem (2).

We shall prove now that T satisfies all the hypotheses of Theorem 2.9.

(a) $T(x) \neq \emptyset$ for each $x \in X$.

Consider the multivalued operator F_x, given by $F_x(t) = F(t, x(t))$. By the Kuratowski-Ryll-Nardzewski selection theorem, F_x has a measurable selection $w(t) \in F_x(t)$, for all $t \in I$.

Define $v(t) = x^0 + \int_{t_0}^{t} w(s)\, ds$, $t \in I$. We obtain $v \in T(x)$ and so $T(x) \neq \emptyset$.

(b) $T(x)$ is closed for each $x \in X$.

Suppose (x_n) is a sequence in $T(x)$ which converges to $y \in X$. But $x_n(t) \in x^0 + \int_{t_0}^{t} F(t, x(t))$ a.e. and $x^0 + \int_{t_0}^{t} F(t, x(t))$ is closed (see [7]). Hence $y(t) \in x^0 + \int_{t_0}^{t} F(t, x(t))$ a.e.

(c) T is a multivalued contraction.

We shall prove that there exists $L \in (0,1)$ such that for each $x, y \in X$ with $d(x,y) < \infty$ one obtains $H(T(x), T(y)) \leq L d(x,y)$.

To see this, let $v_1 \in T(x)$. Then $v_1 \in X$ and $v_1(t) \in x^0 + \int_{t_0}^{t} F(s, x(s))\, ds$, a.e. on I. It follows that there is a mapping $f_x \in S^1_{F(\cdot, x(\cdot))}$ such that $v_1(t) = x^0 +$

$\int_{t_0}^t f_x(s)\,ds$ a.e. on I. Since $H(F(t,x(t)),F(t,y(t))) \le k\frac{\|x(t)-y(t)\|}{|t-t_0|}$ one obtains that there exists $w \in F(t,y(t))$ such that $\|f_x(t)-w\| \le k\frac{\|x(t)-y(t)\|}{|t-t_0|}$ on I. Thus the multivalued operator G defined by $G(t)=F_y(t)\cap K(t)$, $t\in I$ (where $F_y(t)=F(t,y(t))$ and $K(t)=\{w\in F(t,y(t))\mid \|f_x(t)-w\|\le k\frac{\|x(t)-y(t)\|}{|t-t_0|}\}$) has nonempty values.

F_y and K are measurable and hence G is also measurable. Let f_y be a measurable selection for G (which exists by the Kuratowski-Ryll-Nardzewski selection theorem). Then $f_y(t)\in F(t,y(t))$ a.e. on I and $\|f_x(t)-f_y(t)\|\le k\frac{\|x(t)-y(t)\|}{|t-t_0|}$ on I.

Define $v_2(t)=x^0+\int_{t_0}^t f_y(s)\,ds$, $t\in I$. It follows that $v_2\in T(y)$ and

$$\begin{aligned}
&\|v_1(t)-v_2(t)\| = \|x^0+\int_{t_0}^t f_x(s)\,ds - x^0 - \int_{t_0}^t f_y(s)\,ds\|\\
&\le \int_{t_0}^t \|f_x(s)-f_y(s)\|\,ds \le k\int_{t_0}^t \frac{\|x(s)-y(s)\|}{|s-t_0|}\,ds\\
&= k\int_{t_0}^t \frac{\|x(s)-y(s)\|}{|s-t_0|^{pk}}|s-t_0|^{pk-1}\,ds \le kd(x,y)\int_{t_0}^t |s-t_0|^{pk-1}\,ds\\
&= kd(x,y)\frac{|t-t_0|^{pk}}{pk}.
\end{aligned}$$

Finally, one obtains:

$$\frac{\|v_1(t)-v_2(t)\|}{|t-t_0|^{pk}} \le \frac{1}{p}d(x,y) \quad \text{a.e.}$$

Hence $d(v_1,v_2)\le \frac{1}{p}d(x,y)$.

From this and the analogous inequality obtained by interchanging the roles of x and y, we get

$$H(T(x),T(y)) \le \frac{1}{p}d(x,y), \quad \text{for each } \ x,y\in X \ \text{ with } \ d(x,y)<\infty.$$

(d) T admits a sequence of successive approximations $(\varphi_n)_{n\in\mathbf{N}}$ with the property that there exists an index $N\in\mathbf{N}$ such that $d(\varphi_N,\varphi_{N+l})<\infty$, for all $l\in\mathbf{N}^*$.

To see this, let $(\varphi_n)_{n\in\mathbf{N}}$ a sequence of successive approximations for T (where $\varphi_0\in X$ is arbitrary). Let $\varphi_1\in T(\varphi_0)$. It follows that there exists $f_0\in L^1(I,\mathbf{R}^n)$, $f_0(s)\in F(s,\varphi_0(s))$ a.e. such that

$$\varphi_1(t) = x^0 + \int_{t_0}^t f_0(s)\,ds \text{ a.e.}$$

Let $\varphi_2 \in T(\varphi_1)$. By the definition of T, one obtains again that there exists $f_1 \in L^1(I, \mathbf{R}^n)$, $f_1(s) \in F(s, \varphi(s))$ a.e. such that

$$\varphi_2(t) = x_0 + \int_{t_0}^{t} f_1(s)\, ds \text{ a.e.}$$

By the boundedness of F we have

$$\|\varphi_2(t) - \varphi_1(t)\| = \| \int_{t_0}^{t} (f_1(s) - f_0(s))\, ds\| \le \int_{t_0}^{t} \|f_1(s) - f_0(s)\|\, ds \le 2M|t - t_0|.$$

Since $f_1(s) \in F(s, \varphi(s))$ a.e. and F has compact values, we obtain that there exists $w \in F(s, \varphi_2(s))$, for each $s \in I$ such that

$$\|w - f_1(s)\| \le H(F(s, \varphi_2(s)), F(s, \varphi_1(s))).$$

Consider the multivalued operator G defined by $G(s) = F_{\varphi_2}(s) \cap H^*(s)$ (where $F_{\varphi_2}(s) := F(s, \varphi_2(s))$ and

$$H^*(s) := \{w \in X \mid \|w - f_1(s)\| \le H(F(s, \varphi_2(s)), F(s, \varphi_1(s))) \text{ a.e.}\}.)$$

Clearly G is measurable and by the Kuratowski-Ryll-Nardzewski selection theorem it admits a measurable selection $f_2(s) \in G(s)$ a.e. on I. Thus $f_2(s) \in F(s, \varphi_2(s))$ a.e. and

$$\|f_2(s) - f_1(s)\| \le H(F(s, \varphi_2(s)), F(s, \varphi_1(s))).$$

Let $\varphi_3(t) := x_0 + \int_{t_0}^{t} f_2(s)\, ds$. We have:

$$\begin{aligned}\|\varphi_3(t) - \varphi_2(t)\| &\le \int_{t_0}^{t} \|f_2(s) - f_1(s)\| \le \int_{t_0}^{t} H(F(s, \varphi_2(s)), F(s, \varphi_1(s)))\, ds \\ &\le A \int_{t_0}^{t} \frac{\|\varphi_2(s) - \varphi_1(s)\|}{|s - t_0|^{\beta}}\, ds \le A(2M)^{\alpha} \int_{t_0}^{t} |s - t_0|^{\alpha - \beta}\, ds \\ &= A(2M)^{\alpha} \frac{|t - t_0|^{1+\alpha-\beta}}{1 + \alpha - \beta} \le A(2M)^{\alpha} |t - t_0|^{1+\alpha-\beta}.\end{aligned}$$

Generally

$$\begin{aligned}\|\varphi_{n+1}(t) - \varphi_n(t)\| &\le A^{1+\alpha+\cdots+\alpha^{n-2}} (2M)^{\alpha^{n-1}} |t - t_0|^{(1-\beta)(1+\cdots+\alpha^{n-2})+\alpha^{n+1}} \\ &< B|t - t_0|^{(1-\beta)(1+\alpha+\cdots+\alpha^{n-2})+\alpha^{n-1}},\end{aligned}$$

where $B = A^{\frac{1}{1-\alpha}} \max\{2M, 1\}$.

In view of $pk(1 - \alpha) < 1 - \beta$ there exists an index $N \in \mathbf{N}$ such that $(1 - \beta)(1 + \alpha + \cdots + \alpha^{n-2}) + \alpha^{n-1} > pk$, for each $n \ge N$. Hence for $n \ge N$, we have:

$$\frac{\|\varphi_{n+1}(t) - \varphi_n(t)\|}{|t - t_0|^{pk}} \le B|t - t_0|^{\gamma_n},$$

where $\gamma_n = (1 - \beta)(1 + \cdots + \alpha^{n-2}) + \alpha^{n-1} - pk > 0$.

This shows that $d(\varphi_{n+1}, \varphi_n) < \infty$, for all $n \ge N$, which completes the proof.

After these verifications, an application of Theorem 2.9 in the preceding section gives the desired conclusion. $\square$

Remark 3.2. For $\beta = 0$, we get an existence result which is an improvement of Theorem 2 from [5].

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"Babeş-Bolyai" University Cluj-Napoca, Faculty of Mathematics and Computer Science, Str. Kogălniceanu 1, 3400 Cluj-Napoca, Romania

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