Finite spaces and the universal bundle of a group

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Abstract. We find sufficient conditions for a cotriad of which the objects are locally trivial fibrations, in order that the push-out be a locally trivial fibration. As an application, the universal G-bundle of a finite group G, and the classifying space is modeled by locally finite spaces. In particular, if G is finite, then the universal G-bundle is the limit of an ascending chain of finite spaces. The bundle projection is a covering projection.

Keywords: covering projection, fibration, finite space, push-out

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The work of Alexandroff [1] supplemented by that of McCord [6], describes the functorial relationship between finite topological spaces and finite posets. Finite posets arise in different mathematical problems, for example, certain subposets of the lattice of subgroups of a group was introduced by K.S. Brown in the study of Euler characteristic of groups with torsion. Such posets play an important role in the representation and cohomology of finite groups. In particular, the work of McCord shows that one can work with a finite T_0 space directly, instead of going via the polyhedron associated with it. In [3] it is shown how one can rectify the lack of morphisms in the category of finite T_0 spaces when studying homotopy properties of its objects. Different aspects of finite spaces are studied in the paper [8] of Stong. In particular, [8, Theorem 6] shows that a finite T_0 space F with base point admits an H-structure of type II (see [8, Section 5]) if and only if F is pointed homotopy equivalent to a discrete space. Thus the finite T_0 H-spaces of type II are essentially the finite (discrete) groups. In this article we study the action of a discrete group on a locally finite topological space. This leads to a model for the universal principal G-bundle projection of a discrete group G, as a covering projection between locally finite T_0 spaces. In our constructions the cylinder object $\mathbf{I}$ (the compact unit interval), is replaced by a finite space, so as to yield *non-Hausdorff* constructions as in [6]. We prove a basic theorem for push-outs of locally trivial fibrations. This theorem is then adapted to be utilized in the formation of non-Hausdorff homotopy push-outs.

Notation. By a *map* we shall mean a continuous function between topological spaces. The category of topological spaces and maps is denoted by **Top**. By $\mathbf{Top}^2$ we mean the category of which the objects are the morphisms of **Top** and a morphism $p_0 \to p_1$ in $\mathbf{Top}^2$ is a pair (g, f) of maps such that $f \circ p_0 = p_1 \circ g$, that is, diagram **A** is commutative.

$$\begin{tikzcd} E_0 \arrow[r, "g"] \arrow[d, "p_0"'] & E_1 \arrow[d, "p_1"] \\ B_0 \arrow[r, "f"'] & B_1 \end{tikzcd} \qquad \mathbf{A}$$

Such a $\mathbf{Top}^2$-morphism $(g, f) : p_0 \to p_1$ is said to be a *homeomorphism of fibres* if for each $b \in B_0$, the induced map $p_0^{-1}(b) \to p_1^{-1}[f(b)]$ is a homeomorphism.

We shall often work with a commutative diagram **B** in **Top**. This can be regarded as a cotriad in $\mathbf{Top}^2$. The spaces obtained as the push-outs of the **Top**-cotriads appearing as the upper and lower rows of diagram **B**, are denoted by E and B respectively.

$$\begin{tikzcd} E_1 \arrow[d, "p_1"] & E_0 \arrow[l, "g_1"'] \arrow[r, "g_2"] \arrow[d, "p_0"] & E_2 \arrow[d, "p_2"] \\ B_1 & B_0 \arrow[l, "f_1"] \arrow[r, "f_2"'] & B_2 \end{tikzcd} \qquad \mathbf{B}$$

The push-out of the $\mathbf{Top}^2$-cotriad coincides with the unique map $p : E \to B$ that exists since E and B are push-outs.

1. Locally trivial fibrations

We shall sometimes require a space to satisfy the following condition **Q**. This condition is satisfied by every locally compact Hausdorff space F. A proof can be found in [2, Problem 25, p. 330]. In Section 2 we show that the property also holds for locally finite spaces.

Condition Q (for a space F)**.** Whenever a map $q : X \to Y$ is a quotient map, then the map $q \times F : X \times F \to Y \times F$ is a quotient map. □

The concept of locally trivial fibration is well known. The other terminology defined in Definition 1.1 below is not so standard.

Definition 1.1. Let $p : E \to B$ be a map of topological spaces.

(a) For a subset U of B, U is said to be p-*projected* if there exists a space G and a homeomorphism $\phi : U \times G \to p^{-1}(U)$ such that for every $(x, g) \in U \times G$, we have $p\phi(x, g) = x$. The map ϕ is said to be a *trivialization* of p over U.

(b) The map p is a *locally trivial fibration* if B has an open cover of p-projected subsets.

(c) Let F be any topological space. The map p is a *locally trivial fibration with fibre F* if p is a locally trivial fibration and $p^{-1}(b)$ is homeomorphic to F for every $b \in B$.

Examples of locally trivial fibrations are abundant, see for example [9]. In particular, every covering projection is a locally trivial fibration.

The proof of the following proposition is elementary and we omit it. We refer to diagram **C** below.

$$\begin{tikzcd} & X \arrow[r, "g"] \arrow[dl, "f"'] & Z \arrow[dl, "f'"] \\ Y \arrow[r, "g'"'] & P & \end{tikzcd} \qquad \mathbf{C}$$

Proposition 1.2. *Suppose that the commutative diagram* **C** *is a push-out square in* **Top**. *Suppose further that* F *is a space satisfying Condition* **Q**, *and in diagram* **D** *the three unlabeled vertical arrows are projection maps.*

$$\begin{tikzcd} & X \times F \arrow[rr, "g \times F"] \arrow[dl, "f \times F"'] \arrow[dd] & & Z \times F \arrow[dl, "f' \times F"] \arrow[dd] \\ Y \times F \arrow[rr, "g' \times F"] \arrow[dd] & & P \times F \arrow[dd, dashed, "v"] & \\ & X \arrow[rr] \arrow[dl, "f"'] & & Z \arrow[dl, "f'"] \\ Y \arrow[rr, "g'"] & & P & \end{tikzcd} \qquad \mathbf{D}$$

Then the rhombus in the top of diagram **D** *is a push-out square, and the unique map* $v : P \times F \to P$, *guaranteed by push-out properties to make diagram* **D** *commutative, is the projection map.* $\square$

Theorem 1.3. *Suppose that in diagram* **B** *every map* p_i *is a locally trivial fibration with fibre* F, *where* F *is a space satisfying Condition* **Q**. *Suppose further that there exists an index set* S *which determines for each* $s \in S$ *and each* $j \in \{1, 2\}$ *an open subset* U_j^s *of* B_j, *such that the following conditions hold.*

(1) *For each* $j \in \{1, 2\}$, *the collection* $\{U_j^s : s \in S\}$ *covers* B_j.
(2) *For each* $s \in S$, $f_1^{-1}(U_1^s) = f_2^{-1}(U_2^s)$. *This subset of* B_0 *will be denoted by* U_0^s.
(3) *For each* $s \in S$ *with* U_0^s *nonempty, we require the following (and now we drop the superscript* s *of the sets* U_i^s*):*
There exist homeomorphisms $h_i : U_i \times F \to p_i^{-1}(U_i)$ *such that diagram* **E** *is commutative.*

$$\begin{tikzcd}
U_1 \times F \arrow[dr, "h_1"] \arrow[ddr, "q_1"'] & & U_0 \times F \arrow[ll, "k_1 \times F"'] \arrow[rr, "k_2 \times F"] \arrow[d, "h_0"] & & U_2 \times F \arrow[d, "h_2"] \\
& V_1 \arrow[d, "r_1"] & V_1 \arrow[l, "l_1"'] \arrow[rr, "l_2"] \arrow[d, "r_0"] & & V_2 \arrow[d, "r_2"] \\
& U_1 & U_0 \arrow[l, "k_1"'] \arrow[rr, "k_2"] & & U_2
\end{tikzcd} \qquad \mathbf{E}$$

In this diagram, $V_i = p_i^{-1}(U_i)$ and the vertical arrow r_i is induced by p_i. The maps k_j and l_j are induced by f_j and g_j respectively. The maps $r_i \circ h_i : U_i \times F \to U_i$, which we shall denote by q_i, are required to be projections.
Then the push-out $p : E \to B$ of the $\mathbf{Top}^2$-cotriad of diagram $\mathbf{B}$ is a locally trivial fibration with fibre F.

Proof: For each $s \in S$, there are $\mathbf{Top}^2$-cotriads in diagram $\mathbf{E}$, formed by the triples h_i, r_i and q_i. In view of Condition $\mathbf{Q}$ and (1), Proposition 1.2 applies, ensuring that the push-out of the $\mathbf{Top}^2$-cotriad formed by the maps q_i is precisely the projection map $q : U \times F \to U$. The map q together with the push-outs of the triples h_i and r_i, yield a commutative triangle as in diagram $\mathbf{F}$.

$$\begin{tikzcd}
U \times F \arrow[r, "h"] \arrow[dr, "q"'] & V \arrow[d, "r"] \\
& U
\end{tikzcd} \qquad \mathbf{F}$$

Since each of the maps h_i is a homeomorphism, h is a homeomorphism. U can be considered to be a subset of B while r is the pull-back of p over the inclusion $U \subset B$. By commutativity of diagram $\mathbf{F}$, it immediately follows that U is p-projected. By condition (2), the open subset $U_1 + U_0 + U_2$ of $B_1 + B_0 + B_2$ is saturated with respect to the quotient map $\eta : B_1 + B_0 + B_2 \to B$. Hence U is open in B. Thus U is a p-projected open subset of B. By condition (1) it follows that these sets U^s, for the different $s \in S$, form a cover of B. This completes the proof. $\square$

2. Locally finite spaces

A locally finite space is a topological space in which every point has a neighbourhood which is a finite set. Note that for every point x in a locally finite space X, X has a smallest (finite) neighbourhood for x. In this section we look at two properties of such spaces which make them particularly useful when studying locally trivial fibrations.

Definition 2.1. Let F and A be topological spaces. We say that F is a *rigid fibre over* A if for every open subset B of A, with $p : F \times B \to B$ denoting the projection map, the following condition is satisfied:

Whenever $f : F \times B \to F \times B$ is a map such that $p \circ f = p$ and for every $b \in B$ the map $F \times b \to F \times b$ induced by f is a homeomorphism, then f is itself a homeomorphism.

$$\begin{tikzcd} F \times B \arrow[rr, "f"] \arrow[dr, "p"'] & & F \times B \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

The rigidity property has the following significance. Suppose that in diagram **A**, (g, f) is a homeomorphism of fibres. Then if B_1 is p_1-projected and p_0 is a locally trivial fibration with fibre F which is a rigid fibre over B_0, then B_0 is p_0-projected.

It is known that a Hausdorff space F is a rigid fibre over any space B provided that F is either compact or locally compact and locally connected. This rigidity property is studied in greater generality in [10]. Returning to locally finite spaces, we have the following result.

Proposition 2.2. *If F and A are locally finite spaces, then F is a rigid fibre over A.*

PROOF: Let B be any open subspace of A, and let $p : F \times B \to B$ be the projection map. Suppose that $f : F \times B \to F \times B$ is a map such that $p \circ f = p$, and for every $b \in B$, the map $f_b : F \times b \to F \times b$ induced by f is a homeomorphism. We show that f is a homeomorphism by exhibiting an open cover $\mathcal{U}$ of its codomain such that for every $U \in \mathcal{U}$, the induced map $f^{-1}(U) \to U$ is a homeomorphism.

Let $z = (e, b)$ be any point of $F \times B$. Let U_z be the subset $V \times W$, where V is the smallest open subset of F containing e and W is the smallest open subset of B containing b. From the definition of product topology, it turns out that U_z is the smallest neighbourhood of z. Let $V' = \{y \in F : (y, b) = f(x, b) \text{ for some } x \in V\}$. Then since f_b is a homeomorphism, it follows that V' is an open subset of F. Thus $V' \times W$ is open in $F \times B$. Since f is continuous, the set $f^{-1}(V' \times W)$ is open. Therefore, $f^{-1}(V' \times W)$ is a neighbourhood of z. By minimality of U_z, it follows that $U_z \subset f^{-1}(V' \times W)$. However the latter two sets are cardinally equivalent and are finite, and so $U_z = f^{-1}(V' \times W)$. Thus $f(U_z)$ is open. Moreover U_z is homeomorphic to $f(U_z)$, and a continuous bijection between finite spaces which are known to be homeomorphic, is necessarily a homeomorphism. We can choose $\mathcal{U}$ to be the collection, $\mathcal{U} = \{f(U_z) : z \in F \times B\}$. $\square$

Proposition 2.3. *If F is a locally finite space, then F satisfies Condition* **Q**.

PROOF: Let U be any open subset of $F \times X$ which is saturated with respect to the map $F \times q$. We show that $U' = (F \times q)(U)$ is open in $F \times Y$.

Let z be an arbitrary element of U'. Then z is of the form $(e, y) \in F \times Y$. Let V be the minimal neighbourhood of e in F. Let $W = \{x \in X : (e, x) \in U\}$. For each $x \in W$, by minimality of V and since U is open, there exists T_x open in X such that $(e, x) \in V \times T_x \subset U$. Thus W is open in X. Note also that $V \times W \subset U$.

Since U is saturated with respect to $F \times q$, W is saturated with respect to q. Consequently $W' = q(W)$ is open in Y. Since $V \times W \subset U$, $V \times W' \subset U'$. So $V \times W'$ is a neighbourhood of z that lies entirely in U'. Thus U' is open in $F \times Y$. □

3. The non-Hausdorff double mapping cylinder

In order to construct suitable adjunction spaces, we require a version of mapping cylinders in the context of locally finite spaces.

Let R_d be the set of integers, topologized by taking as a subbase the collection of all subsets of the form $\{2n-1; 2n; 2n+1\}$, $n \in \mathbf{Z}$. The space R_d will be referred to as the digital line. This topological space was introduced by Khalimsky [4]. Let I_d be the subspace $\{0, 1, 2\}$ of R_d. As is pointed out in [3], the subspace I_d together with its barycentric subdivisions constitute the appropriate cylinder object for locally finite spaces, to fulfill the role of the compact unit interval in the category of Hausdorff spaces. Without further subdivision, I_d will suffice for the purposes of this paper. Every compact interval in R_d is contractible onto every point which is closed as a one-point set. The following argument shows a particular instance of this phenomenon.

Example 3.1. Let $[0, 1]$ denote the unit interval of real numbers, and let I_d be our conventional subset of the digital line. We define a map $h : I_d \times [0, 1] \to I_d$ as follows:

$$h(x,t) = \begin{cases} x & \text{if } \ t = 0; \\ 1 & \text{if } \ t \in (0,1) \text{ and } \ x = 1, 2; \\ 0 & \text{otherwise.} \end{cases}$$

Then h is a homotopy from the identity map of I_d to the constant map $I_d \to \{0\}$. □

We now describe the version of the double mapping cylinder that arises from this cylinder object I_d, for a cotriad,

$$E_1 \xleftarrow{\quad g_1 \quad} E_0 \xrightarrow{\quad g_2 \quad} E_2 \ .$$

Let E be the disjoint union $E_1 + E_0 \times I_d + E_2$. We form the quotient space E' by identifying a point $(x, 0) \in E_0 \times I_d$ with the point $g_1(x) \in E_1$, and a point $(x, 2) \in E_0 \times I_d$ is identified with the point $g_2(x) \in E_2$.

Let D_1 and D_2 respectively, be the images under the identification map $E \to E'$, of the subsets $E_1 + E_0 \times \{0, 1\}$ and $E_2 + E_0 \times \{1, 2\}$. The sets D_1 and D_2 are open. Let $D_0 = D_1 \cap D_2$. The resulting double mapping cylinder construction, to which we shall refer as the *non-Hausdorff double mapping cylinder*, is functorial. We further note that the non-Hausdorff double mapping cylinder construction is possible even if the spaces in the cotriad are not locally finite, just as the standard double mapping cylinder construction is possible regardless of the nature of the spaces in the cotriad. Note furthermore that by taking special cotriads, we obtain *non-Hausdorff mapping cylinders* and *non-Hausdorff mapping cones.*

In Proposition 3.2 below, we compare the non-Hausdorff double mapping cylinder with the standard double mapping cylinder. A proof of the proposition can be found in [6].

Proposition 3.2. *Let* $[0,1]$ *be the compact unit interval in* $\mathbf{R}$*. Let* $h : [0,1] \to I_d$ *be the map defined by:*

$$0 \mapsto 0, \quad (0,1) \mapsto 1 \quad \textit{and} \quad 1 \mapsto 2.$$

Then for every **Top***-cotriad, the map* h *induces a weak equivalence from the ordinary double mapping cylinder to the non-Hausdorff double mapping cylinder.*

For the purposes of the following adjunction theorem, we refer to the notation agreed upon just prior to Section 1. With reference to diagram **B**, we note that there is an obvious map $p : E' \to B'$ where E' and B' are the non-Hausdorff double mapping cylinders of the **Top**-cotriads constituted by the rows in (respectively) the top and the bottom.

Theorem 3.3. *Suppose that in the commutative diagram* **B***, every map* p_i *is a locally trivial fibration with fibre* F*, and that* F *and every space shown in diagram* **B** *is a locally finite space. Suppose further that each of the* $\mathbf{Top}^2$*-morphisms* $p_0 \to p_j$ *is a homeomorphism of fibres,* $j = 1; 2$*.*
Then the map $p' : E' \to B'$ *of non-Hausdorff double mapping cylinders determined by the* $\mathbf{Top}^2$*-cotriad in diagram* **B***, is a locally trivial fibration with fibre* F*.*

Proof: The commutative square in the right hand part of diagram **C** induces a map $\zeta : Z' \to Z$ from the non-Hausdorff mapping cylinder of g_1 to the non-Hausdorff mapping cylinder of f_1.

There is an open subset Y of Z which admits retractions $h : \zeta^{-1}(Y) \to E_0$ and $k : Y \to B_0$, such that $k\zeta(x) = \zeta h(x)$ for every $x \in \zeta^{-1}(Y)$, and the $\mathbf{Top}^2$-morphism $(h,k) : \eta \to p_1$ is a homeomorphism of fibres, where η is the pull-back of ζ over the inclusion $Y \subset Z$.

The space B_2 has an open cover consisting of p_2-projected subsets. Let U_2 be such an open subset, let $U_0 = f_2^{-1}(U_2)$ and let $U_1 = k^{-1}(U_0)$. Then U_0 is p_0-projected. Since (h,k) is a homeomorphism of fibres it follows that U_1 is p_1-projected. In fact, for the cotriad $U_1 \leftarrow U_0 \to U_2$, we can find trivializations satisfying the hypotheses as spelt out in 1.3(3). We further note that the image of B_0 in Z is a closed subset. Thus by Theorem 1.3, it follows that $p' : E' \to B'$ is a locally trivial fibration. $\square$

4. The universal bundle of a discrete group

We motivate this section by means of the following example, showing two actions of discrete groups on topological spaces. The examples show the importance of having totally discontinuous group action. In 4.2 we point out (omitting the straightforward proof) this significance more precisely. Theorem 4.3 is the main result of this article.

Example 4.1.

(a) For R_d the digital line, let S_d be the quotient space obtained from R_d by identifying elements which are congruent modulo 4. Notice that the additive group $\mathbf{Z}$ of integers acts on the topological space R_d, $(z : n \mapsto n + 4z)$, and in fact the orbits of this action are precisely our equivalence classes. The action is totally discontinuous and by a well-known theorem, see [5] for example, it follows that the quotient map is a covering projection. In fact we have found a locally finite space model for the classical exponential map $\mathbf{R} \to \mathbf{S}$, $\mathbf{R}$ being the real line and $\mathbf{S}$ the set of complex numbers with modulus 1.

(b) Taking S_d as in (a) above, we have a (unique) non-trivial topological action of the group $\mathbf{Z}/2$ with 2 elements on S. The action is however not totally discontinuous, and the orbit projection is very far from being a fibration. $\square$

Proposition 4.2. *Let G be a discrete group acting freely on a topological space E. Then the following conditions are equivalent:*

(1) *the action is totally discontinuous,*
(2) *the orbit projection $p : E \to E/G$ is a locally trivial fibration,*
(3) *the orbit projection is a covering projection.* $\square$

For a space A and a group G, we can define a (topological) action of G on $G \times A$ by the rule: ${}^{g}(g', a) = (gg', a)$. Let CA be the non-Hausdorff cone on A. Equip the spaces $G \times A$ and $G \times CA$ with the action defined above. Suppose now that A is a G-space. Then we have the cotriad of G-maps below, where α is the action on A. The space obtained as the push-out of the cotriad is a G-space, and we denote it by $G * A$.

$$\begin{tikzcd} G \times CA & G \times A \arrow[l, hook'] \arrow[r, "\alpha"] & A \end{tikzcd},$$

Theorem 4.3. *Let G be a discrete topological group, and E a G-space. If the action of G on E is totally discontinuous, then the action of G on $G * E$ is totally discontinuous.*

Proof: In view of 4.2, it suffices to show that the orbit projection $p : G*E \to (G*E)/G$ is a locally trivial fibration with fibre G. The latter follows by Theorem 3.3 applied to diagram $\mathbf{G}$.

$$\begin{tikzcd} G \times CE \arrow[d, "p_1"] & G \times E \arrow[l, hook'] \arrow[r, "\alpha"] \arrow[d] & E \arrow[d, "q"] \\ CE & E \arrow[l, hook'] \arrow[r, "q"] & E/G \end{tikzcd} \qquad \mathbf{G}$$

Here q is the orbit projection, and the other vertical arrows are product space projections. The space obtained in the push-out of the cotriad in the bottom, can be seen to coincide (up to homeomorphism) with $(G * E)/G$. $\square$

Theorem 4.3 gives a method for determining, for a given discrete group G, a sequence, $p_n : E_n \to B_n$ of locally trivial fibrations with fibre G. In view of 3.2 our construction is (weakly homotopy) equivalent to that of Milnor [7]. In our sequence, $E_0 = G$, and otherwise, $E_{n+1} = G * E_n$. For every non-negative integer n, $B_n = E_n/G$ and p_n is the orbit projection. The spaces E_n and B_n are locally finite. In particular, if G is a finite group of order k, then E_n has $(k+1)^{n+1} - 1$ elements. This follows by induction, using the relations $|B_{n+1}| = |B_n| + |E_n| + 1$ and $|E_n| = k|B_n|$.

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