Localic Katětov-Tong insertion theorem and localic Tietze extension theorem

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Abstract. In this paper, localic upper, respectively lower continuous chains over a locale are defined. A localic Katětov-Tong insertion theorem is given and proved in terms of a localic upper and lower continuous chain. Finally, the localic Urysohn lemma and the localic Tietze extension theorem are shown as applications of the localic insertion theorem.

Keywords: frame, locale, lower (upper) continuous chain, normal locale

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Introduction

Let (X, τ) be a topological space, $f, g : X \to R$ be an upper, respectively lower semicontinuous function such that $f \leq g$, whether we can insert a continuous function $h : X \to R$ such that $f \leq h \leq g$. We consider the famous classical problem. Katětov [5] and Tong [11] gave the insertion function h in the case of a normal space, for the further works see [5] and [8]. The solution of the classical insertion problem dependens seriously on the existence of a point of the topological space X and on the analytic properties of the real line. On the other hand, locales have longly been recognized as an important generalization of topological spaces, a notion which points the study of topological questions in context where intuitionistic logic rather Boolean logic prevails and in which spaces without points occur naturally. An excellent exposition of the history of this generalization can be found in Johnstone's "The point of pointless topology" [4]. In this work, it is emphasized that the locale theory is inherently constructive and methodologically algebraic, all independent of the properties of the "point". So it is natural to generalize the above classical insertion problem to the frames which are independent of the point. While, the first problem we encounter is that we must choose an appropriate real open locale such that we can define it in any topos with natural numbers as objects, so we drop out the choice of the open sets locale $\Omega(R)$ for real numbers R, but we prefer the constructive real open locale $L(R)$ defined by Fourman and Hyland in [2]. Corresponding to this, we use the universal algebra method to give the constructive real upper open locale $\mathcal{R}_u$ or the real lower open locale $\mathcal{R}_l$ respectively, which is not equivalent to the classical

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real upper or lower topology even in the Boolean logic. It is very surprising and interesting that the corresponding continuous morphism from a locale to $\mathcal{R}_l$ forms a locale. This points out that the theory of localic semicontinuous functions is more general and interesting than that in the classical case even when the locale is spatial. This also brings another direction to the study, to compare a locale with its corresponding semicontinuous functions locale. We shall do some work here, in particular, we will prove the localic Katětov-Tong insertion theorem, its proof is more constructive and direct and independent of the analytic methods in [5], [11]. This shows to be more transparent and informative. Moreover, this method can be applied to any topos with natural numbers as objects, and as it is well known, this degree of generality is quite relevant to "down to earth" mathematics in the usual set-theoretic universe.

The contents of this paper is as follows.

Section 1 gives some basic definitions of upper continuous chains, lower continuous chains and continuous chains. Using the universal algebra point of view, we introduce the locale $\mathcal{R}_u$, $\mathcal{R}_l$ and $\mathcal{R}$, and explain the formal forms of upper, lower and continuous chains. Finally, the locale of lower continuous chains over a locale is obtained.

Section 2 provides a localic Katětov-Tong insertion theorem, and also gives another characterization of the normal locale.

In Section 3, the weak and strong form for the localic Urysohn lemma and Tietze extension theorem are given as corollaries of the localic Katětov-Tong insertion theorem.

1. Preliminaries, localic semicontinuous functions

First, let us recall some facts about frame theory, all these have been described in [3].

A frame A is a complete lattice with a largest element $\top$ (or 1_A) and a least element $\perp$ (or 0_A) satisfying infinite distributivity laws $a \wedge \bigvee S = \bigvee_{s\in S}(a \wedge s)$ ($S \subseteq A$ and $a \in A$). A frame homomorphism is a mapping $f : A \to B$ between frames A and B which preserves joins and finite meets. Frames and frame homomorphisms form a category Frm, its dual category is the locale category Loc, the objects of Loc are called locales. As frames and locales are the same objects without referring to morphisms, we will use frames and locales equally in this paper. Let Sp denote the category of general topological spaces, then there is an adjoint between categories Sp and Loc by functor $\Omega : Sp \to Loc$ and $pt : Loc \to Sp$, where $\Omega(X)$ denotes the topology of X for every topological space X and $pt(A) = \{x : A \to 2 \,|\, x \text{ is a frame homomorphism}\}$ for every locale A such that $\Omega(pt(A)) = \{\phi(a) \,|\, a \in A\}$, where $\phi(a) = \{x \in pt(A) \,|\, x(a) = 1\}$. The corresponding actions on morphisms are natural.

For a locale A, a nucleus over A is just a closure operator $j : A \to A$ preserving finite meets. Then there is a one to one correspondence between the nuclei over A and the sublocales of A. For a nucleus $j : A \to A$, if there exists an $a \in A$ such that $j(b) = a \vee b$ for every $b \in A$, then the sublocale A_a corresponding to

j is the closed sublocale of A, and the corresponding frame onto morphism is denoted by $c_a : A \to A_a$. For every $a \in A$, we define the mapping $u_a : A \to A$ by $u_a(b) = a \to b = \bigvee\{c \in A : a \wedge c \leq b\}$ for a nucleus $b \in A$; it corresponds to an open sublocale $A_{u(a)}$ of A. For any locale A, let $N(A)$ denote all the nuclei on A with pointwise ordering, then $N(A)$ is a frame and $c : A \to N(A)$ (by $a \mapsto c_a$) is a frame monomorphism. This frame monomorphism has the following categorical property: for every frame homomorphism $f : A \to B$, there exists a frame homomorphism $N(f) : N(A) \to N(B)$ (N is a functor) such that the following diagram is commutative

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ c\big\downarrow & & \big\downarrow c \\ N(A) & \xrightarrow{N(f)} & N(B) \end{array}$$

and $N(f)(u_a) = u_{f(a)}$. In $N(A)$, c_a and u_a are complemented for any $a \in A$, that is, $c_a \wedge u_a = c_{0_A} = 0_{N(A)}$ and $c_a \vee u_a = c_{1_A} = 1_{N(A)}$.

Let $N^*(A)$ denote all the sublocales of A, then $N^*(A) = \{imj \,|\, j \in N(A)\}$. $N^*(A)$ is a co-frame, it is closed under set-intersection and it is anti-order-isomorphic to $N(A)$. Let $\mathcal{O}(A)$ denote all the open sublocales of A, $\mathcal{C}(A)$ denotes all the closed sublocales of A, then $A \cong \mathcal{O}(A)$ and for every $U \in \mathcal{O}(A)$, $F \in \mathcal{C}(A)$, let U^c, F^c denote the complement of U and F in $N^*(A)$, then $U^c \in \mathcal{C}(A)$ and $F^c \in \mathcal{O}(A)$. Each sublocale S of A has a closure $\overline{S}$, the least closed sublocale of A containing S. Each sublocale S of A has an interior S°, the largest open sublocale of A containing in S. For a sublocale S of A, let $\neg S = \bigcap\{E \in N^*(A) : E \vee S = \top\}$, then $\neg S$ is the least sublocale of A such that $\neg S \vee S = \top$.

A locale A is called normal if, for each pair $a, b \in A$ with $a \vee b = 1_A$, there exist $c, d \in A$ such that $c \wedge d = 0_A$ and $c \vee a = d \vee b = 1_A$.

A compact locale, σ-compact locale and connected locale are defined classically. A locally compact locale is just a locale which is also a continuous lattice. A locally connected locale is just a locale in which every element is the join of some connected elements ([7]).

Write Q for the totally ordered set of rational numbers throughout the paper.

Definition 1.1. For every locale A, a chain in A indexed by rational numbers, $C = \{x_\alpha\}_{\alpha \in Q}$, is called an *increasing chain* in A, if $x_\alpha \in A$ for every $\alpha \in Q$ and $x_\alpha \leq x_\beta$ whenever $\alpha < \beta$.

An increasing chain $C = \{x_\alpha\}_{\alpha \in Q}$ is called an *upper continuous chain*, if $\bigvee_{\beta < \alpha} x_\beta = x_\alpha$ is true for any $\alpha \in Q$.

Furthermore, an upper continuous chain $C = \{x_\alpha\}_{\alpha \in Q}$ in A is called a *proper upper continuous chain*, if $\bigvee_{\alpha \in Q} x_\alpha = \top$, that is, C is a cover of A.

Dually, we can define descending chain, lower continuous chain and proper lower continuous chain for a locale in the following.

Definition 1.2. For a locale A, a chain $D = \{y_\alpha\}_{\alpha\in Q}$ in A indexed by rational numbers is called a *descending chain* in A, if $y_\alpha \in A$ for every $\alpha \in Q$ and $y_\alpha \geq y_\beta$ whenever $\alpha < \beta$.

A descending chain $D = \{y_\alpha\}_{\alpha\in Q}$ in A is called a *lower continuous chain*, if $\bigvee_{\beta>\alpha} y_\beta = y_\alpha$ is true for any $\alpha \in Q$.

Furthermore, a lower continuous chain $D = \{y_\alpha\}_{\alpha\in Q}$ in A is called a *proper lower continuous chain*, if $\bigvee_{\alpha\in Q} y_\alpha = \top$, that is, D is a cover of A.

Write $\mathcal{R}_u(A)$ for the set of all the upper continuous chains, $\overline{\mathcal{R}}_u(A)$ for the set of all the proper upper continuous chains, $\mathcal{R}_l(A)$ for the set of all the lower continuous chains, $\overline{\mathcal{R}}_l(A)$ for the set of all the proper lower continuous chains.

We give an order over $\mathcal{R}_l(A)$ and $\mathcal{R}_u(A)$, respectively, in the following.

Definition 1.3. *For* $D^1 = \{y^1_\alpha\}_{\alpha\in Q}$, $D^2 = \{y^2_\alpha\}_{\alpha\in Q} \in \mathcal{R}_l(A)$, $D^1 \leq D^2$ *iff* $y^1_\alpha \leq y^2_\alpha$ *for any* $\alpha \in Q$.

For $C^1 = \{x^1_\alpha\}_{\alpha\in Q}$, $C^2 = \{x^2_\alpha\}_{\alpha\in Q} \in \mathcal{R}_u(A)$, $C^1 \leq C^2$ *iff* $x^1_\alpha \geq x^2_\alpha$ *for any* $\alpha \in Q$.

According to the order just defined, $\mathcal{R}_u(A)$ and $\mathcal{R}_l(A)$ become partially order sets. Let $C \in \mathcal{R}_u(A)$, for any $\alpha \in Q$, let $y_\alpha = x_{-\alpha}$, then $D = \{y_\alpha\}_{\alpha\in Q}$ is a lower continuous chain, and this correspondence defines an anti-isomorphism of $\mathcal{R}_l(A)$ and $\mathcal{R}_u(A)$. So $\mathcal{R}_l(A)$ and $\mathcal{R}_u(A)$ have the dual order properties.

First, it is easy to verify that $\mathcal{R}_l(A)$ is a locale, for $D^i = \{y^i_\alpha\}_{\alpha\in Q}$ in $\mathcal{R}_l(A)$, $\bigvee D^i = \{\bigvee_i y^i_\alpha\}$ and $D^1 \wedge D^2 = \{y^1_\alpha \wedge y^2_\alpha\}$. The constant chain $\{0\}$ and $\{1\}$ is respectively the least and the largest element of $\mathcal{R}_l(A)$. For $a \in A$, denote $[a] = \{a\}$ the constant chain, then $[a] \in \mathcal{R}_u(A)$ and $[a] \in \mathcal{R}_l(A)$. So A is a subframe of $\mathcal{R}_l(A)$, in fact, the inclusion map $m : A \to \mathcal{R}_l(A)$ is an equationally closed frame embedding in [10]. Therefore, we can consider the following problem:

> *What are the relations between the properties of a locale A, such as separateness, countableness, covering properties or connectedness, and those of the corresponding locale $\mathcal{R}_l(A)$ or $\overline{\mathcal{R}}_l(A)$.*

This is an interesting topic for which, we think, it will be useful to study locales or frames. We will do some work in this direction.

Definition 1.4. *Let* $C = \{x_\alpha\}_{\alpha\in Q} \in \mathcal{R}_u(A)$, $D = \{y_\alpha\}_{\alpha\in Q} \in \mathcal{R}_l(A)$, *define* $C \leq D$ *iff* $x_\alpha \vee y_\beta = \top$ *whenever* $\beta < \alpha$ *in* Q, $D \leq C$ *iff* $x_\alpha \wedge y_\alpha = \bot$ *for any* $\alpha \in Q$. *A pair* $E = (E_1, E_2) \in \mathcal{R}_u(A) \times \mathcal{R}_l(A)$ *is called a continuous chain if* $E_1 \leq E_2$ *and* $E_2 \leq E_1$. *Moreover, if* $E_1 \in \overline{\mathcal{R}}_u(A)$ *and* $E_2 \in \overline{\mathcal{R}}_l(A)$, *we call* $E = (E_1, E_2)$ *a proper continuous chain. In the following, we will see that proper continuous chains just correspond to continuous functions over a locale.*

Write $\mathcal{R}(A)$, $\overline{\mathcal{R}}(A)$ for the set of all continuous chains and proper continuous chains, respectively. For $E = (E_1, E_2)$, $F = (F_1, F_2) \in \mathcal{R}(A)$, we define $E \leq F$ iff $E_1 \geq F_1$, or equivalently, $E_2 \leq F_2$ or $E_1 \leq F_2$ or $E_2 \leq F_1$. Under this order, $\mathcal{R}(A)$ and $\overline{\mathcal{R}}(A)$ become a poset. In fact, $\mathcal{R}(A)$ forms a lattice.

In [3], Johnstone described and justified a method of specifying locales by giving generators and relations, and Vickers [12] developed this method by giving a representation of a frame as same particular universal algebra. Since we shall use this repeatedly below, let us briefly review some facts.

Let G be a set and R be a set of equations (or inequalities) between frame words (that is, infinite joins of G or finite meets of G) in the symbols of G. Then a model of this set subject to the relation R is just a frame A equipped with a mapping $[\,] : G \to A$ such that for any relation $e_1 = e_2$ in R, $[e_1] = [e_2]$, where $[e]$ denotes the evaluation of expression e in A by mapping $[\,]$. In particular, the frame freely generated by G subject to the relation R, denoted by $Frm\,\langle G \,|\, R\rangle$ in the sequel, is a model A of the set G subject to the relation R, such that, for any other model B, there is a unique frame homomorphism $\theta : A \to B$ such that $\theta(g_A) = g_B$ for every generator $g \in G$. In [3] or [12], they give a procedure to describe $Frm\,\langle G \,|\, R\rangle$. In [9], Madden uses the method of universal algebra to give another procedure to describe $Frm\,\langle G \,|\, R\rangle$. For details, see [3], [9], [12]. We use this method to define some frames needed in this paper. These frames have tight connections with real numbers.

$$\mathcal{R}_u = Frm\,\langle \overline{\alpha}(\alpha \in Q) \,|$$

$$\begin{aligned} &(1) \quad \alpha \le \beta \Rightarrow \overline{\alpha} \le \overline{\beta} \\ &(2) \quad \bigvee_{\beta<\alpha} \overline{\beta} = \overline{\alpha}\rangle, \end{aligned}$$

$$\overline{\mathcal{R}}_u = Frm\,\langle \overline{\alpha}(\alpha \in Q) \,|\, (1)(2) \text{ in } \mathcal{R}_u \text{ and}$$

$$(3) \quad \bigvee_{\alpha\in Q} \overline{\alpha} = \top\rangle,$$

$$\mathcal{R}_l = Frm\,\langle \underline{\alpha}(\alpha \in Q) \,|$$

$$\begin{aligned} &(1) \quad \alpha \le \beta \Rightarrow \underline{\alpha} \ge \underline{\beta} \\ &(2) \quad \bigvee_{\alpha<\beta} \underline{\beta} = \underline{\alpha}\rangle, \end{aligned}$$

$$\overline{\mathcal{R}}_l = Frm\,\langle \underline{\alpha}(\alpha \in Q) \,|\; (1), (2) \text{ and}$$

$$(3) \quad \bigvee_{\alpha\in Q} \underline{\alpha} = \top\rangle,$$

$\mathcal{R} = Frm\,\langle \overline{\alpha}, \underline{\alpha}(\alpha \in Q) \mid$

$$\begin{array}{ll} (1) & \bigvee\limits_{\beta<\alpha} \overline{\beta} = \overline{\alpha} \\ (2) & \bigvee\limits_{\alpha<\beta} \underline{\beta} = \underline{\alpha} \\ (3) & \beta < \alpha \Rightarrow \overline{\alpha} \vee \underline{\beta} = \top \\ (4) & \alpha \le \beta \Rightarrow \overline{\alpha} \wedge \underline{\beta} = \bot \rangle, \end{array}$$

$\overline{\mathcal{R}} = Frm\,\langle \langle \alpha, \beta \rangle (\alpha, \beta \in Q) \mid$

$$\begin{array}{ll} (1) & \beta \le \alpha \Rightarrow \langle \alpha, \beta \rangle = \bot \\ (2) & \langle \alpha, \beta \rangle \wedge \langle \gamma, \delta \rangle = \langle \max\{\alpha, \gamma\}, \min\{\beta, \delta\} \rangle \\ (3) & \alpha \le \gamma < \beta \le \delta \Rightarrow \langle \alpha, \beta \rangle \vee \langle \gamma, \delta \rangle = \langle \alpha, \delta \rangle \\ (4) & \bigvee\limits_{\alpha<\gamma<\delta<\beta} \langle \gamma, \delta \rangle = \langle \alpha, \beta \rangle \\ (5) & \bigvee\limits_{\alpha,\beta\in Q} \langle \alpha, \beta \rangle = \top \rangle. \end{array}$$

$\overline{\mathcal{R}}$ as a real locale was described by Fourmann and Hyland, Johnstone and Madden. Classically, it was just the interval topology of real line, but in the point of view of constructiveness, these two notions are not the same, see [2]. For the same reason, $\overline{\mathcal{R}}_u$, $\overline{\mathcal{R}}_l$ is the upper topology and lower topology, respectively, of the real line in the classical sense, but not in the sense of the constructive view. We choose $\overline{\mathcal{R}}_u$, $\overline{\mathcal{R}}_l$ and $\overline{\mathcal{R}}$ in this paper, as the classical upper topology, lower topology and interval topology of the real line is not very useful to us.

We can give another equivalent representation of the real locale $\overline{\mathcal{R}}$ in the following, and we use these two representations arbitrarily in our paper.

$\overline{\mathcal{R}} = Frm\,\langle \overline{\alpha}, \underline{\alpha}(\alpha \in Q) \mid$ the relation (1), (2), (3), (4) in $\mathcal{R}$ and

$$\begin{array}{ll} (5) & \bigvee\limits_{\alpha\in Q} \overline{\alpha} = \top \\ (6) & \bigvee\limits_{\alpha\in Q} \underline{\alpha} = \top \rangle, \end{array}$$

where $\overline{\alpha} = \bigvee_{\beta\in Q} \langle \beta, \alpha \rangle$, $\underline{\alpha} = \bigvee_{\beta\in Q} \langle \alpha, \beta \rangle$ and $\langle \alpha, \beta \rangle = \overline{\beta} \wedge \underline{\alpha}$.

The following lemma is very useful, its proof is routine and we omit it.

Lemma 1.1. *Let A be a locale, then*

(1) $C = \{x_\alpha\}_{\alpha\in Q} \in \mathcal{R}_u(A)$ *iff there exists a frame homomorphism* $f : \mathcal{R}_u \to A$ *such that* $f(\overline{\alpha}) = x_\alpha$ *for every* $\alpha \in Q$;

(1)′ $C = \{x_\alpha\}_{\alpha\in Q} \in \overline{\mathcal{R}}_u(A)$ *iff there exists a frame homomorphism* $f : \overline{\mathcal{R}}_u \to A$ *such that* $f(\overline{\alpha}) = x_\alpha$ *for every* $\alpha \in Q$;

(2) $D = \{y_\alpha\}_{\alpha\in Q} \in \mathcal{R}_l(A)$ *iff there exists a frame homomorphism* $g : \mathcal{R}_l \to A$ *such that* $g(\underline{\alpha}) = y_\alpha$ *for every* $\alpha \in Q$;

(2)′ $D = \{y_\alpha\}_{\alpha\in Q} \in \overline{\mathcal{R}}_l(A)$ *iff there exists a frame homomorphism* $g : \overline{\mathcal{R}}_l \to A$ *such that* $g(\underline{\alpha}) = y_\alpha$ *for every* $\alpha \in Q$;

(3) $E \in \mathcal{R}(A)$ *iff there exists a frame homomorphism* $h : \mathcal{R} \to A$ *such that* $h(\overline{\alpha}) = z^1_\alpha$ *and* $h(\underline{\alpha}) = z^2_\alpha$ *for every* $\alpha \in Q$, *where* $E = (E_1, E_2)$, $E_i = \{z^i_\alpha\}_{\alpha\in Q}$ *for* $i = 1, 2$ *and* $E_1 \in \mathcal{R}_u(A)$, $E_2 \in \mathcal{R}_l(A)$;

(3)′ $E \in \overline{\mathcal{R}}(A)$ *iff there exists a frame homomorphism* $h : \overline{\mathcal{R}} \to A$ *such that* $h(\overline{\alpha}) = z^1_\alpha$ *and* $h(\underline{\alpha}) = z^2_\alpha$ *for every* $\alpha \in Q$, *where* $E = (E_1, E_2)$, $E_i = \{z^i_\alpha\}_{\alpha\in Q}$ *for* $i = 1, 2$ *and* $E_1 \in \overline{\mathcal{R}}_u(A)$, $E_2 \in \overline{\mathcal{R}}_l(A)$;

Consequently, (proper) upper continuous chains, (proper) lower continuous chains and (proper) continuous chains are just generalizations of upper semicontinuous functions, lower semicontinuous functions and continuous functions in the general sense. In the sequel, we also use function forms of upper or lower continuous chains.

We give some general results on the comparison of a locale A and its corresponding locale $\mathcal{R}_l(A)$ without proof.

Proposition 1.1. *A locale A is σ-compact iff for every proper lower continuous chain $D = \{y_\alpha\}_{\alpha\in Q}$ there exists an $\alpha \in Q$ such that $y_\alpha = \top$.*

Proposition 1.2. *A locale A is locally compact iff $\mathcal{R}_l(A)$ is locally compact.*

Proposition 1.3. *Let A be a locale, then*

(1) *A is connected iff $\mathcal{R}_l(A)$ is connected;*

(2) *A is locally connected iff $\mathcal{R}_l(A)$ is locally connected.*

Proposition 1.4. *Let A be a locale, then A is a spatial locale iff $\mathcal{R}_l(A)$ is a spatial locale.*

In fact, the points of $\mathcal{R}_l(A)$ have the following structure.

Proposition 1.5. *Let A be a locale, $x \in pt(A)$ and D a nonempty upper subset of Q. Define $P^D_x : \mathcal{R}_l(A) \to \{\perp, \top\}$ by*

$$P^D_x(\{y_\alpha\}_{\alpha\in Q}) = \bigvee\{x(y_\alpha) \mid \alpha \in D\}$$

for any $\{y_\alpha\}_{\alpha\in Q} \in \mathcal{R}_l(A)$, then $P^D_x \in pt(\mathcal{R}_l(A))$ and the points of $\mathcal{R}_l(A)$ are just these forms.

2. Localic Katětov-Tong insertion theorem

First, we prove the localic Katětov-Tong insertion theorem (abbreviated to insertion theorem) related to $\mathcal{R}$.

Theorem 2.1. *A locale A is normal iff for every upper continuous chain $f : \mathcal{R}_u \to A$ and lower continuous chain $g : \mathcal{R}_l \to A$ with $f \le g$, there exists a continuous chain $h : \mathcal{R} \to A$ such that $f \le h \le g$.*

Proof: Sufficiency. Let $a, b \in A$ with $a \vee b = 1_A$, then $[a] \in \mathcal{R}_u(A)$ and $[b] \in \mathcal{R}_l(A)$ such that $[a] \le [b]$, then from the condition of this theorem there exists a continuous chain $E = (E_1, E_2)$ such that $[a] \le E \le [b]$, hence $[a] \le E_2$ and $E_1 \le [b]$. Let $c = E_1 = (\overline{1})$, $d = E_2(\underline{1})$, then $[a] \le E_2 \Rightarrow [a](\overline{2}) \vee E_2(\underline{1}) = \top$, that is, $a \vee d = \top$, $E_1 \le [b] \Rightarrow E_1(\overline{1}) \vee [b](\underline{0}) = \top$, that is, $b \vee c = \top$; $E_1 \le E_2 \Rightarrow E_1(\overline{1}) \wedge E_2(\underline{1}) = \perp$, that is, $c \wedge d = \perp$. So A is a normal locale.

For the necessity, we need some auxiliary lemmas which themselves are interesting for the locale theory.

Lemma 2.1. *A locale A is normal iff for any pair of closed sublocales E, F, if $E \cap F = \perp$, then there exists an open sublocale U such that $E \subseteq U$ and $F \cap \overline{U} = \perp$.*

For $S, T \in N^*(A)$, if $\overline{S} \wedge T = S \wedge \overline{T} = \perp$, then we call S and T *separated* in A.

Lemma 2.2. *Let A be a normal locale, $D = \bigwedge_{n \in N} U_n$ and $H = \bigvee_{n \in N} F_n$, where $U_n \in \mathcal{O}(A)$ and $F_n \in \mathcal{C}(A)$ for any $n \in N$, such that $\overline{\neg D} \wedge H = \perp = \neg D \wedge \overline{H}$, then there exists an open sublocale V such that $H \le V \le \overline{V} \le D$.*

Proof: As $\overline{\neg D} \wedge H = \perp$ and $H = \bigvee_{n \in N} F_n$, we have $\overline{\neg D} \wedge F_n = \perp$ for any $n \in N$. By Lemma 2.1, there exists an open sublocale V_n such that $F_n \le V_n$ and $\overline{V_n} \cap \overline{\neg D} = \perp$ for any $n \in N$.

Similarly, as $\neg D \wedge \overline{H} = \perp$ and $D = \bigwedge_{n \in N} U_n$, we have $U_n^c \wedge \overline{H} = \perp$ for any $n \in N$ and by Lemma 2.1 there exists an open sublocale W_n such that $U_n^c \le W_n$ and $\overline{W_n} \cap \overline{H} = \perp$.

Let $V_n' = V_n - \bigvee_{i=1}^n \overline{W_i}$, $W_n' = W_n - \bigvee_{i=1}^n \overline{V_n}$, $W = \bigvee_{n \in N} W_n'$, $V = \bigvee_{n \in N} V_n'$, then we have: $U_n^c \le W_n$ and $U_n^c \wedge \bigvee_{i=1}^n \overline{V_i} \le \overline{\neg D} \wedge \bigvee_{i=1}^n \overline{V_i} = \perp$, that is, $U_n^c \le W_n' = W_n - \bigvee_{i=1}^n \overline{V_i}$, thus $U_n^c \le W$, which is equivalent to $W^c \le U_n$. Hence, $W^c \le D$.

Similarly, since $F_n \le V_n$ and $F_n \wedge \bigvee_{i=1}^n \overline{W_i} \le \overline{H} \wedge \bigvee_{i=1}^n \overline{W_i} = \perp$, we have $F_n \le V_n' \le V$. We deduce that $H \le V$.

We have $W \cap V = \bigvee_{n \in N} \bigvee_{m \in N} W_n' \wedge V_m'$ and $W_n' \cap V_m' = W_n \cap V_m - \bigvee_{i=1}^n \overline{V_i} \vee \bigvee_{j=1}^m \overline{W_j} = \perp$, that is, $W \cap V = \perp$. Hence $V \le W^c$.

Thus $H \le V \le \overline{V} \le W^c \le D$. □

Lemma 2.3. *For a locale A and $a, b \in A$,*
(1) $\neg(a \vee b) = \neg a \wedge \neg b$, and
(2) if b has a complement b^c in A, then $\neg(a \wedge b) = \neg a \vee b^c$.

The proof is trivial, we omit it.

Lemma 2.4. *Let $f \in \mathcal{R}_u(A)$, $g \in \mathcal{R}_l(A)$. If $f \leq g$, then for any $\alpha \in Q$, $\neg g[\alpha]$ and $\bigvee\{f[\beta] : \alpha < \beta\}$ are separated, where $g[\alpha] = \bigwedge\{g(\beta) : \beta < \alpha\}$ in $N^*(A)$, $f[\beta] = \neg f(\beta)$.*

Proof: Let us first show a general inequality, that is, $f[\alpha] \leq g[\alpha]$ for any $\alpha \in Q$. For any $\gamma < \alpha$, since $f \leq g$, we have $f(\alpha) \vee g(\gamma) = \top$. It follows that $f[\alpha] = f(\alpha)^c \leq g(\gamma)$. Since this inequality is true for any $\gamma < \alpha$ in Q, we have $f[\alpha] \leq g[\alpha] = \bigwedge\{g(\gamma) : \gamma < \alpha\}$.

Write $E = \neg g[\alpha]$, $F = \bigvee\{f[\beta] : \alpha < \beta\}$. For any $\beta > \alpha$ in Q, we have $f(\alpha) \leq f(\beta)$ and thus $f[\beta] \leq f[\alpha]$, therefore, $\overline{F} \leq \overline{f[\alpha]} = f[\alpha] \leq g[\alpha]$, and then $\overline{F}^c \vee g[a] = \top$. Since $\overline{F}^c$ is the complement of $\overline{F}$ in $N^*(A)$ and $N^*(A)$ is a coframe, from the dual forms of Lemma 2.3(2) we have $\bot = \neg\top = \neg(\overline{F}^c \vee g[\alpha]) = \overline{F}^{cc} \wedge \neg g[\alpha]$, and since $\overline{F}^{cc} = \overline{F}$, we have $\overline{F} \wedge \neg g[\alpha] = \bot$.

On the other hand, for any $\beta > \alpha$, $f[\beta] \leq g[\beta] \leq g(\alpha) \leq g[\alpha]$, hence $F \leq g(\alpha) \leq g[\alpha]$. Since $\neg g[\alpha] \leq g(\alpha)^c$, we have $\overline{\neg g[\alpha]} \leq \overline{g(\alpha)^c} = g(\alpha)^c$, hence $\overline{\neg g[\alpha]} \wedge F \leq g(\alpha) \wedge g(\alpha)^c = \bot$, that is, $\overline{\neg g[\alpha]} \wedge F = \bot$. Therefore, $\neg g[\alpha]$ and $\bigvee\{f[\beta] : \beta > \alpha\}$ are separated. $\square$

Proof of necessity in Theorem 2.1: Since Q is countable, we can index it by natural numbers, say $Q = \{\alpha_i : i \in N\}$. For any $\alpha \in Q$, let $g[\alpha] = \bigwedge\{g(\beta) : \beta < \alpha\}$, $f[\alpha] = f(\alpha)^c$, $G_i = \bigvee\{f[\alpha] : \alpha_i < \alpha\}$.

First, let us define $U_{\alpha_i} \in \mathcal{O}(A)$ inductively so that

(a) $G_i \leq U_{\alpha_i} \leq \overline{U_{\alpha_i}} \leq g[\alpha_i]$,
(b) $\alpha_i < \alpha_j \Rightarrow \overline{U_{\alpha_j}} \leq \overline{U_{\alpha_i}}$.

For $i = 1$, since $g[\alpha_1] = \bigwedge\{g(\beta) : \beta < \alpha_1\}$, $G_1 = \bigvee\{f[\alpha] : \alpha_1 < \alpha\}$ and $\neg g[\alpha_1]$ and G_1 are separated by Lemma 2.4. Furthermore, as Q is countable and due to Lemma 3.2, there exists a $U_{\alpha_1} \in \mathcal{O}(A)$ such that $G_1 \leq U_{\alpha_1} \leq \overline{U_{\alpha_1}} \leq g[\alpha_1]$.

Now suppose that, for any $i < k \in N$, U_{α_i} is well-defined and satisfies the conditions (a) and (b). We choose $U_{\alpha_k} \in \mathcal{O}(A)$ as follows.

Let $J_0 = \{i \in N : i < k \text{ and } \alpha_k < \alpha_i\}$, $J_1 = \{j \in N : j < k \text{ and } \alpha_j < \alpha_k\}$, $E = G_k \vee (\bigvee\{\overline{U_{\alpha_i}} : i \in J_0\})$, $H = g[\alpha_k] \wedge (\bigwedge\{U_{a_j} : j \in J_1\})$. Then E and $\neg H = \neg g[\alpha_k] \vee (\bigvee\{U^c_{\alpha_j} : j \in J_1\})$ are separated. This is because

(1) by Lemma 2.4, G_k and $\neg g[\alpha_k]$ are separated;
(2) by condition (b), $\bigvee\{\overline{U_{\alpha_i}} : i \in J_0\}$ and $\bigvee\{U^c_{\alpha_j} : j \in J_1\}$ are disjoint closed elements;
(3) by condition (a), if $j \in J_1$, then $\overline{G_k} \leq f[\alpha_k] \leq G_j \leq U_{\alpha_j}$, hence $\overline{G_k} \leq \bigwedge\{U_{\alpha_j} : j \in J_1\}$ and thus $\overline{G_k}$ and $\bigvee\{U^c_{\alpha_j} : j \in J_1\}$ are separated;
(4) if $j \in J_0$, then $\overline{U_{\alpha_i}} \leq g[\alpha_i] \leq (g[\alpha_k])^\circ \leq g[\alpha_k]$, hence $\neg g[\alpha_k]$ and $\bigvee\{\overline{U_{\alpha_i}} : i \in J_0\}$ are separated.

Thus, we have shown that E and $\neg H$ are separated. Since Q is countable, E and H satisfy the condition of Lemma 2.2 and there exists a $U_{\alpha_k} \in \mathcal{O}(A)$ such that $E \leq U_{\alpha_k} \leq \overline{U_{\alpha_k}} \leq g[\alpha_k]$. Observe that $\alpha_i < \alpha_k$ implies $\overline{U_{\alpha_k}} \leq U_{\alpha_j}$ and $\alpha_k < \alpha_i$ implies $\overline{U_{\alpha_i}} \leq U_{\alpha_k}$.

We thus complete the inductive procedure.

Let $C_{\alpha_k} = \bigwedge\{\overline{U_{\alpha_j}} : \alpha_j < \alpha_k\}$, $D_{\alpha_k} = \bigvee\{U_{\alpha_j} : \alpha_j > \alpha_k\}$, then it is obvious that $C_{\alpha_k} = \bigwedge\{C_{a_j} : \alpha_j < \alpha_k\}$ and $D_{\alpha_k} = \bigvee\{D_{\alpha_j} : \alpha_j > \alpha_k\}$. Furthermore, we have:

(A1) $f[\alpha_k] \leq C_{\alpha_k}$ and $D_{\alpha_k \leq g(\alpha_k)}$ for any $k \in N$. This is because, for any $\alpha_j < \alpha_k$, $f[\alpha_k] \leq G_k \leq \overline{U_{\alpha_j}}$, hence $f[\alpha_k] \leq C_{\alpha_k}$, and similarly, for any $\alpha_j > \alpha_k$, $U_{\alpha_j} \leq g(\alpha_k)$, and hence $D_{\alpha_k} \leq g(\alpha_k)$.

(A2) $D_{\alpha_k} \leq C_{\alpha_k}$ for any $k \in N$. This is because, if $\alpha_i > \alpha_k > \alpha_j$, then $U_{\alpha_i} \leq U_{\alpha_j}$ and thus $U_{\alpha_i} \leq \overline{U_{\alpha_j}}$, hence $D_{\alpha_k} \leq C_{\alpha_k}$.

(A3) If $\alpha_j < \alpha_k$, then $C_{\alpha_k} \leq D_{\alpha_j}$, that is, $\bigwedge\{\overline{U_{\alpha_i}} : \alpha_i < \alpha_k\} \leq \bigvee\{U_{\alpha_i} : \alpha_i > \alpha_j\}$. This is because, if $\alpha_j < \alpha_k$, then there exists $\alpha_l, \alpha_d \in Q$ such that $\alpha_j < \alpha_l < \alpha_d < \alpha_k$, hence $\bigwedge\{\overline{U_{\alpha_i}} : \alpha_i < \alpha_k\} \leq \overline{U_{\alpha_d}} \leq U_{\alpha_l} \leq \bigvee\{U_{\alpha_i} : \alpha_j < \alpha_i\}$, that is, $C_{\alpha_k} \leq D_{\alpha_j}$.

Let $z^1_{\alpha_k} = C^c_{\alpha_k}$, $z^2_{\alpha_k} = D_{\alpha_k}$, $E_1 = \{z^1_{\alpha_k}\}_{\alpha_k \in Q}$, $E_2 = \{z^2_{\alpha_k}\}_{\alpha_k \in Q}$. Then, as just proved, $E_1 \in \mathcal{R}_u(A)$ and $E_2 \in \mathcal{R}_l(A)$, $E_1 \leq E_2$, $E_2 \leq E_1$, $f \leq E_1$ and $E_2 \leq g$. Let $h = (E_1, E_2)$, then $h \in \mathcal{R}(A)$ and $f \leq h \leq g$.

The proof of necessity is thus complete. □

Remark 2.1. For a normal locale A, if $f \in \overline{\mathcal{R}_u}(A)$ and $g \in \overline{\mathcal{R}_l}(A)$ such that $f \leq g$, according to Theorem 2.1 there exists an $h \in \mathcal{R}(A)$ such that $f \leq h \leq g$. We can find h even as a proper continuous chain, but the proof of Theorem 2.1 is not applicable. We give a proof in the following.

In order to avoid confusion, we write $\overline{\alpha}$, α^*, $\underline{\alpha}$, α_* for the generator of $\mathcal{R}_u(A)$, $\overline{\mathcal{R}_u}(A)$, $\mathcal{R}_l(A)$, $\overline{\mathcal{R}_l}(A)$, respectively, for any $\alpha \in Q$, in the sequel.

We need some mappings, $e_u : \mathcal{R}_u \to \overline{\mathcal{R}_u}$, $e_l : \mathcal{R}_l \to \overline{\mathcal{R}_l}$, $d_u : \overline{\mathcal{R}_u} \to \mathcal{R}_u$, $d_l : \overline{\mathcal{R}_l} \to \mathcal{R}_l$ and $d : \overline{\mathcal{R}} \to \mathcal{R}$ in the sequel, they are defined on their generators by, for any $\alpha \in Q$, $e_u(\overline{\alpha}) = \alpha^*$; $e_l(\underline{\alpha}) = \alpha_*$; $d_u(\alpha^*) = \top$ if $\alpha > 1$, $\perp$ if $\alpha < 0$ and $\underline{\alpha}$ if $0 < \alpha \leq 1$; $d_l(\alpha_*) = \top$ if $\alpha < 0$, $\perp$ if $\alpha \geq 1$ and $\underline{\alpha}$ if $0 \leq \alpha < 1$; $d = (d_u, d_l)$. Then it is routine to verify that these mappings satisfy all the relations corresponding to their domains, so e_u, e_l, d_u, d_l and d are well-defined frame homomorphisms. We will fix these symbols to denote these mappings in the following.

We need the following trivial result.

Lemma 2.5. (1) *Let $f_1, f_2 \in \mathcal{R}_u(A)$, then $f_1 \leq f_2$ iff $N(f_1)(u_{\overline{\alpha}}) \leq N(f_2)(u_{\overline{\alpha}})$ for every $\alpha \in Q$.*

(2) *Let $g_1, g_2 \in \mathcal{R}_l(A)$, then $g_1 \leq g_2$ iff $N(g_1)(u_{\underline{\alpha}}) \geq N(g_2)(u_{\underline{\alpha}})$ for every $\alpha \in Q$.*

Theorem 2.2. *For any locale A, A is normal iff for every proper upper continuous chain $f : \overline{\mathcal{R}_u} \to A$ and for every proper lower continuous chain $g : \overline{\mathcal{R}_l} \to A$ with $f \leq g$ there exists a proper continuous chain $h : \overline{\mathcal{R}} \to A$ such that $f \leq h \leq g$.*

Proof: Sufficiency. Let $a, b \in A$ with $a \vee b = 1_A$, then $[a] \in \mathcal{R}_u(A)$ and $[b] \in \mathcal{R}_l(A)$ such that $[a] \leq [b]$, then $[a] \circ d_u \in \overline{\mathcal{R}_u}(A)$ and $[b] \circ d_l \in \overline{\mathcal{R}_l}(A)$, and from the definition of d_u and d_l, it is obvious that $[a] \circ d_u \leq [b] \circ d_l$, that is, $[a] \circ d_u(\beta^*) \vee [b] \circ d_l(\alpha_*) = \top$ provided that $\alpha < \beta$ in Q. From the condition of the theorem, there exists a proper continuous chain $E = (E_1, E_2)$ such that $[a] \circ d_u \leq E \leq [b] \circ d_l$, hence $[a] \circ d_u \leq E_2$ and $E_1 \leq [b] \circ d_l$. Let $c = E_1(1/2^*)$, $d = E_2(1/2_*)$, then $[a] \circ d_u \leq E_2 \Rightarrow [a] \circ d_u(1^*) \vee E_2(1/2_*) = \top$, that is, $a \vee d = \top$; $E_1 \leq [b] \circ d_l \Rightarrow E_1(1/2^*) \vee [b] \circ d_l(0_*) = \top$, that is, $b \vee c = \top$; $E_1 \leq E_2 \Rightarrow E_1(1/2^*) \wedge E_2(1/2_*) = \perp$, that is, $c \wedge d = \perp$. So A is a normal locale.

Necessity. Let A be a normal locale, and $f \in \overline{\mathcal{R}_u}(A)$, $g \in \overline{\mathcal{R}_l}(A)$ with $f \leq g$, then from Theorem 2.1 there exists a continuous chain $\widetilde{h} \in \mathcal{R}(A)$ such that $f \circ e_u \leq \widetilde{h} \leq g \circ e_l$. Let us define $h : \overline{\mathcal{R}} \to A$ by $h(\langle\alpha, \beta\rangle) = \widetilde{h}(\underline{\alpha}) \wedge \widetilde{h}(\overline{\beta})$ on the generator of $\overline{\mathcal{R}}$. Then h satisfies the relations (1)–(4) in $\overline{\mathcal{R}}$, we shall check that h satisfies also the relation (5) in $\overline{\mathcal{R}}$ i.e. that $h(\bigvee_{\alpha,\beta \in Q} \langle\alpha, \beta\rangle) = \bigvee_{\alpha,\beta \in Q} h(\langle\alpha, \beta\rangle) = \top$.

First, we check the following facts:

(a) $u_{\overline{\alpha}} \leq \bigvee_{\beta \in Q} c_{\underline{\beta}}$ for every $\alpha \in Q$:

Since $c_{\overline{\alpha}} \vee \bigvee_{\beta \in Q} c_{\underline{\beta}} = \bigvee_{\beta \in Q} c_{\overline{\alpha}} \vee c_{\underline{\beta}} = \bigvee_{\beta \in Q} c_{\overline{\alpha} \vee \underline{\beta}} \geq \bigvee_{\beta < \alpha} c_{\overline{\alpha} \vee \underline{\beta}} = \bigvee_{\beta \leq \alpha} c_\top = 1_{N(\mathcal{R})}$, that is, $c_{\overline{\alpha}} \vee \bigvee_{\beta \in Q} c_{\underline{\beta}} = \top$, we have $u_{\overline{\alpha}} = u_{\overline{\alpha}} \wedge 1_{N(\mathcal{R})} = u_{\overline{\alpha}} \wedge (c_{\overline{\alpha}} \vee \bigvee_{\beta \in Q} c_{\underline{\beta}}) = (u_{\overline{\alpha}} \wedge c_{\overline{\alpha}}) \vee (u_{\overline{\alpha}} \wedge \bigvee_{\beta \in Q} c_{\underline{\beta}}) = u_{\overline{\alpha}} \wedge \bigvee_{\beta \in Q} c_{\underline{\beta}}$. Hence $u_{\overline{\alpha}} \leq \bigvee_{\beta \in Q} c_{\underline{\beta}}$.

(b) $c_{\underline{\alpha}} \leq u_{\overline{\alpha}}$ for every $\alpha \in Q$:

Since $c_{\underline{\alpha}} \wedge c_{\overline{\alpha}} = c_{(\underline{\alpha} \wedge \overline{\alpha})} = c_\perp = 0_{N(\mathcal{R})}$, we have $u_{\overline{\alpha}} = u_{\overline{\alpha}} \vee 0_{N(\mathcal{R})} = (u_{\overline{\alpha}} \vee c_{\underline{\alpha}}) \wedge (u_{\overline{\alpha}} \vee c_{\overline{\alpha}}) = u_{\overline{\alpha}} \vee c_{\underline{\alpha}}$. Thus $c_{\underline{\alpha}} \leq u_{\overline{\alpha}}$.

From (a), (b) we have $\bigvee_{\alpha \in Q} c_{\underline{\alpha}} \leq \bigvee_{\alpha \in Q} u_{\overline{\alpha}} \leq \bigvee_{\alpha \in Q} c_{\underline{\alpha}}$, so $\bigvee_{\alpha \in Q} u_{\overline{\alpha}} = \bigvee_{\alpha \in Q} c_{\underline{\alpha}}$. For the same reason, we can deduce that $\bigvee_{\alpha \in Q} u_{\alpha^*} = \bigvee_{\alpha \in Q} c_{\alpha^*}$.

Thus

$$N(\widetilde{h})(\bigvee_{\alpha \in Q} c_{\underline{\alpha}}) = N(\widetilde{h})(\bigvee_{\alpha \in Q} u_{\overline{\alpha}}) = \bigvee_{\alpha \in Q} N(\widetilde{h})(u_{\overline{\alpha}})$$

(from $f \circ e_u \leq \widetilde{h}$ and Lemma 2.5, we thus have)

$$\begin{aligned} &\geq \bigvee_{\alpha \in Q} N(f \circ e_u)(u_{\overline{\alpha}}) = \bigvee_{\alpha \in Q} N(f)(N(e_u)(u_{\overline{\alpha}})) = \bigvee_{\alpha \in Q} N(f)(u_{e_u(\overline{\alpha})}) \\ &= \bigvee_{\alpha \in Q} N(f)(u_{\alpha^*}) = N(f)(\bigvee_{\alpha \in Q} u_{\alpha^*}) = N(f)(\bigvee_{\alpha \in Q} c_{\alpha^*}) = N(f)(c_{\bigvee_{\alpha \in Q} \alpha^*}) \\ &= N(f)(c_\top) = c_{f(\top)} = c_{1_a}. \end{aligned}$$

Hence $c_{\widetilde{h}(\bigvee_{\alpha \in Q} \underline{\alpha})} = N(\widetilde{h})(c_{\bigvee_{\alpha \in Q} \underline{\alpha}}) = N(\widetilde{h})(\bigvee_{\alpha \in Q} c_{\underline{\alpha}}) = c_{1_A}$, as c is an order-embedding, so $\widetilde{h}(\bigvee_{\alpha \in Q} \underline{\alpha}) = 1_A$.

Similarly, we can show that $\widetilde{h}(\bigvee_{\alpha \in Q} \overline{\alpha}) = 1_A$.

We thus deduce that $h(\bigvee_{\alpha,\beta\in Q}\langle\alpha,\beta\rangle) = \bigvee_{\alpha,\beta\in Q} h(\langle\alpha,\beta\rangle) = \bigvee_{\alpha,\beta\in Q}\widetilde{h}(\underline{\alpha}) \wedge \widetilde{h}(\overline{\beta}) = \bigvee_{\alpha\in Q}\widetilde{h}(\underline{\alpha}) \wedge \bigvee_{\alpha\in Q}\widetilde{h}(\overline{\alpha}) = \widetilde{h}(\bigvee_{\alpha\in Q}\underline{\alpha}) \wedge \widetilde{h}(\bigvee_{\alpha\in Q}\overline{\alpha}) = \top \wedge \top = 1_A$.

From the proof above, we know that h is a proper continuous chain. Furthermore, from $f\circ e \le \widetilde{h} \le g\circ e$, we can easily deduce that $f \le h \le g$, and thus the proof of necessity is complete. □

Remark 2.2. For f, g and h in Theorem 2.2, even if $pt(f)$, $pt(g)$ can attain the value $-\infty$ or $+\infty$, $pt(h)$ gets values only in R since h is continuous. In particular, for a spatial locale A (the definition can be found in [3]), if f is not the least element in $\overline{\mathcal{R}_u}(A)$, g is not the largest element in $\overline{\mathcal{R}_l}(A)$, then the above localic insertion theorem just corresponds to the classical Katětov-Tong insertion theorem.

3. Localic Urysohn lemma and Tietze extension theorem

As applications of Theorems 2.1 and 2.2., we give the localic Urysohn lemma and Tietze extension theorem. Their proofs are very easy.

Theorem 3.1 (Localic Urysohn lemma).

(Weak form) *A locale A is normal iff $a\vee b = \top$ implies that there exists a continuous chain $E = (E_1, E_2)$ such that $E_1(\overline{\alpha}) \le a$ and $E_2(\underline{\alpha}) \le b$ for any $\alpha \in Q$.*

(Strong form) *A locale A is normal iff $a\vee b = \top$ implies that there exists a continuous chain $E = (E_1, E_2)$ such that $E_1(\overline{\alpha}) \le a$ and $E_2(\underline{\alpha}) \le b$ for any $0 < \alpha < 1$ in Q.*

Proof: Necessity of the weak form. $a\vee b = \top$ implies $[a] \le [b]$, where we regard $[a]$ as an upper continuous chain and $[b]$ as a lower continuous chain. Then from Theorem 2.1 there exists a continuous chain $E = (E_1, E_2)$ such that $[a] \le E \le [b]$, hence $[a] \le E_1$ and $E_2 \le [b]$, that is, $E_1(\underline{\alpha}) \le a$ and $E_2(\overline{\alpha}) \le b$ for any $\alpha \in Q$.

Sufficiency of the weak form. Supposing that $a, b \in A$ such that $a\vee b = \top$, then as assumed, there exists a continuous chain $E = (E_1, E_2)$ such that $E_1(\underline{\alpha}) \le a$ and $E_2(\overline{\alpha}) \le b$ for any $\alpha \in Q$. Fix $\alpha < \delta < \beta$ in Q, and let $c = E_1(\overline{\delta})$, $d = E_2(\underline{\delta})$. Then $c\wedge d = \perp$ and $c \vee b \ge E_1(\overline{\delta}) \vee E_2(\underline{\alpha}) = \top$, $d\vee a = E_1(\overline{\beta}) \vee E_2(\underline{\delta}) = \top$, hence A is a normal locale.

The proof of sufficiency of the weak form is also applicable to that of strong form, we only restrict $0 < \alpha < \gamma < \beta < 1$. For the necessity of the strong form, we only replace $[a]$ by $[a]\circ d_u$ and $[b]$ by $[b]\circ d_l$ in that of the weak form. □

Theorem 3.2 (Localic Tietze extension theorem).

(Weak form) *For any locale A, A is normal iff for each closed sublocale A_a of A which corresponds to $c_a : A \to A_a$ by $c_a(b) = a \vee b$ for every $b \in A$, and for every continuous chain $f : \mathcal{R} \to A_a$, there always exits a continuous extension $g : \mathcal{R} \to A$ of f, that is, $f = c_a \circ g$.*

(Strong form) *For any locale A, A is normal iff for each closed sublocale A_a of A which corresponds to $c_a : A \to A_a$, and for every proper continuous chain*

$f : \overline{\mathcal{R}} \to A_a$, there always exits a proper continuous extension $g : \overline{\mathcal{R}} \to A$ of f, that is, $f = c_a \circ g$.

Proof: Sufficiency of the weak form. Supposing that $a \vee b = \top$, considering a closed sublocale $A_{a\wedge b}$ of A, we can define $f_1 = [a]$, $f_2 = [b]$. Then $f_1 \in \mathcal{R}_u(A_{a\wedge b})$, $f_2 \in \mathcal{R}_l(A_{a\wedge b})$ and $f_1 \leq f_2$, $f_2 \leq f_1$, so $f = (f_1, f_2)$ is a continuous chain in $A_{a\wedge b}$. As assumed, there exits a continuous chain $g = (g_1, g_2)$ in A such that $f = c_a \circ g$. Fix an $\alpha \in Q$, let $c = g(\overline{\alpha})$, $d = g(\underline{\alpha})$, then $a \vee g(\underline{\alpha}) = a \vee (a \wedge b) \vee g(\underline{\alpha}) = a \vee (c_{a\wedge b} \circ g)(\underline{\alpha}) = a \vee f(\underline{\alpha}) = a \vee b = \top$. Similarly, $b \vee c = \top$, so A is a normal locale.

Necessity. Assuming $f : \mathcal{R} \to A_a$ is a continuous chain, let us define $f_u : \mathcal{R}_u \to A$ by $f_u(\overline{\alpha}) = f(\overline{\alpha})$ for every $\alpha \in Q$. Then, obviously, $f_u \in \mathcal{R}_u(A)$ and the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{R} & \xrightarrow{f} & A_a \\ {\scriptstyle i_u}\uparrow & & \uparrow{\scriptstyle c_a} \\ \mathcal{R}_u & \xrightarrow{f_u} & A \end{array}$$

where $i_u : \mathcal{R}_u \to \mathcal{R}$ is defined by $i_u(\overline{\alpha}) = \overline{\alpha}$.

We can similarly define another frame homomorphism $f_l : \mathcal{R}_l \to A$ by $f_l(\underline{\alpha}) = f(\underline{\alpha})$ for every $\alpha \in Q$, then $f_l \in \mathcal{R}_l(A)$ and the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{R} & \xrightarrow{f} & A_a \\ {\scriptstyle i_l}\uparrow & & \uparrow{\scriptstyle c_a} \\ \mathcal{R}_l & \xrightarrow{f_l} & A \end{array}$$

where $i_l : \mathcal{R}_l \to \mathcal{R}$ is $i_l(\underline{\alpha}) = \underline{\alpha}$.

Then for $\alpha < \beta$ in Q, $f_u(\overline{\beta}) \vee f_l(\underline{\alpha}) = f(\overline{\beta}) \vee f(\underline{\alpha}) = f(\overline{\beta} \vee \underline{\alpha}) = f(\top) = 1_A$, so $f_u \leq f_l$. From Theorem 2.1, there exists a continuous chain $g : \mathcal{R} \to A$ such that $f_u \leq g \leq f_l$. The verification of $c_a \circ g = f$ is routine and we omit it.

Similarly, we can prove the strong form analogously as the strong form of the Urysohn lemma. □

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