# A new look at pointfree metrization theorems

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*In memory of Miroslav Katětov*

*Abstract.* We present a unified treatment of pointfree metrization theorems based on an analysis of special properties of bases. It essentially covers all the facts concerning metrization from Engelking [1] which make pointfree sense. With one exception, where the generalization is shown to be false, all the theorems extend to the general pointfree context.

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The recognition that the notion of metric diameters in frames provides a pointfree axiomatization of metrics in spaces ([6], [7]) naturally raised the question of pointfree metrization theorems. The first of these was the pointfree version of the classical uniform metrization theorem, saying that a frame is metrizable iff it has a uniformity with a countable basis ([6]). Note that the latter uniformities are exactly those called metric in Isbell [2] so that this theorem may be viewed as providing geometric justification for this terminology. Other known results in this area are the pointfree version of the Moore and the Bing metrization theorems in [7] and of the Nagata-Smirnov metrization theorem in Isbell [3].

The purpose of this paper is to present a unified treatment of these and one further result, the pointfree Archangelskij theorem, based on a detailed analysis of special properties of bases, which leads to entirely new, and we believe especially transparent and succinct, proofs in all cases. Furthermore, our treatment covers all metrization theorems presented in Engelking [1] which make pointfree sense except for the Jones theorem involving the stars of compact subspaces which we show does *not* hold in general.

As an additional feature, our treatment reveals two particular aspects of the Moore theorem which give it much greater significance here than it has in the classical context. First, while the notion of development may seem somewhat ad hoc for spaces, its counterpart for frames, that is: admissibility, is an absolutely essential concept that plays an important role in various different ways, quite apart from the question of metrizability. Secondly, the Moore theorem has a central position here in that all the other theorems are, in a sense, derived from

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it. We note that an important tool of recent origin to obtain the Moore theorem from the original uniform metrization theorem was provided by Kaiser [5].

As a general reference, we use the excellent survey of metrization theorems in IV.4 and V.4 of Engelking [1].

## 1. Preliminaries

**1.1.** A *frame* (see [4] and [11] for details) is a complete lattice $L$ satisfying the distribution law

$$a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}$$

for all $a \in L$ and $S \subseteq L$.

A *basis* of a frame $L$ is a subset $B \subseteq L$ such that $x = \bigvee\{b \in B \mid b \leq x\}$ for each $x \in L$.

The *pseudocomplement* of an $a \in L$ is $a^* = \bigvee\{x \mid x \wedge a = 0\}$, the largest element $b$ such that $b \wedge a = 0$. It is easy to see that $(\bigvee X)^* = \bigwedge\{x^* \mid x \in X\}$ for any $X \subseteq L$.

**1.2.** We write $a \prec b$ if $a^* \vee b = 1$. A frame $L$ is said to be *regular* if

$$x = \bigvee\{y \mid y \prec x\}$$

for all $x \in L$. As usual, $a \in L$ is called *minimal* if it is a minimal non-zero element, that is, an $a \neq 0$ such that $0 \neq b \leq a$ implies $b = a$. The following simple observation will be useful:

*If $L$ is regular and $0 \neq a \in L$ is not minimal then $a = \bigvee\{b \mid b < a\}$. Consequently, if $B$ is a basis of $L$, $a = \bigvee\{b \mid b \in B,\ b < a\}$.*

Proof: Let $b \neq 0$, $b < a$. By regularity, there is a $c \neq 0$ such that $b \vee c^* = 1$. Thus,

$$a = a \wedge (b \vee c^*) = b \vee (a \wedge c^*).$$

Now, if $a \wedge c^*$ were equal to $a$ we would have $a \leq c^*$ and $c = c \wedge a = 0$, a contradiction.

□

**1.3.** A *cover* of a frame $L$ is a subset $A \subseteq L$ such that $\bigvee A = 1$. If $A, B$ are covers we say that $A$ *refines* $B$, and write

$$A \leq B,$$

if, for each $a \in A$, there is a $b \in B$ such that $a \leq b$.

We set

$$A \wedge B = \{a \wedge b \mid a \in A,\ b \in B\}$$

which is obviously a common refinement of $A$ and $B$.

For a cover $A$ of $L$ and $x \in L$ we put

$$Ax = \bigvee\{a \mid a \in A,\ a \wedge x \neq 0\},$$

and for covers $A$ and $B$,

$$AB = \{Ab \mid b \in B\}.$$

We say that $A$ *star-refines* $B$, and write

$$A \leq^* B$$

if $AA \leq B$.

**1.4.** For a system $\mathfrak{A}$ of covers of a frame $L$ (always understood to be non-void) we write

$$x \lhd_{\mathfrak{A}} y \quad \text{or simply} \quad x \lhd y$$

if there is an $A \in \mathfrak{A}$ such that $Ax \leq y$, and $\mathfrak{A}$ is called *admissible* if

$$x = \bigvee\{y \mid y \lhd_{\mathfrak{A}} x\}$$

for each $x \in L$. Note that $x \lhd_{\mathfrak{A}} y$ implies $x \prec y$ for any $\mathfrak{A}$ and hence

*any frame with an admissible system of covers is regular.*

Obviously

(1.4.1) *if $\mathfrak{B}$ is a system of covers such that for each $A \in \mathfrak{A}$ there is a $B \in \mathfrak{B}$ with $B \leq A$ then $x \lhd_{\mathfrak{A}} y$ implies $x \lhd_{\mathfrak{B}} y$; consequently, if $\mathfrak{A}$ is admissible then so is $\mathfrak{B}$.*

Further we have

(1.4.2) *if $\mathfrak{B}$ is an admissible system of covers then $\bigcup \mathfrak{B}$ is a basis,*

since, for any $x \in L$,

$$x = \bigvee\{b \mid b \in B \in \mathfrak{B} \ \text{ such that } \ By \leq x \ \text{ and } \ b \wedge y \neq 0 \ \text{ for some } \ y\}.$$

An admissible system of covers is called a *uniformity* if

(1) for any $A, B \in \mathfrak{A}$ there is a common refinement $C \in \mathfrak{A}$,
(2) for each $A \in \mathfrak{A}$ there is a star refinement $B \in \mathfrak{A}$, and
(3) for any cover $B$, if $A \leq B$ for some $A \in \mathfrak{A}$ then $B \in \mathfrak{A}$.

Further, $\mathfrak{A}$ is called a *basis of uniformity* whenever (1) and (2) hold.

**1.5.** The set of all non-negative reals augmented by $+\infty$ will be denoted by

$$\mathbb{R}^+.$$

A *metric diameter* on a frame $L$ is a monotone zero-preserving map

$$d : L \to \mathbb{R}^+$$

such that

(1) for all $a, b$, $d(a \vee b) \leq d(a) + d(b)$ whenever $a \wedge b \neq 0$,
(2) for each $\varepsilon \geq 0$, $A(d, \varepsilon) = \{a \mid d(a) < \varepsilon\}$ is a cover,
(3) the system $\{A(d, \varepsilon) \mid \varepsilon > 0\}$ is admissible, and
(4) for all $a \in L$ and $\varepsilon > 0$, $d(a) = \sup\{d(x \vee y) \mid x, y \leq a,\ d(x), d(y) < \varepsilon\}$.

A frame that admits a metric diameter is said to be *metrizable*. It should be noted that this provides a pointfree expression of the metrizability of spaces: *a space is metrizable in the classical sense iff its frame of open sets is metrizable*, where the passage from a metric $\rho$ to a metric diameter $d$ is provided by the usual diameter

$$d(U) = \sup\{\rho(p, q) \mid p, q \in U\}.$$

(see, e.g., [7], [8]).

**1.6.** In [6] the following characterization of metrizability was proved:

**Theorem.** *A frame $L$ is metrizable iff it admits a countable basis of uniformity.*

Clearly, this extends the classical uniform metrization theorem for spaces to frames. It should be added that, by Isbell's definition in [2], a frame is called metrizable whenever it has a uniformity with a countable basis. Thus, this theorem confirms that this terminology indeed carries the correct geometric connotation.

## 2. Additional background

**2.1.** We say that elements $a, b$ of a frame $L$ *meet* if $a \wedge b \neq 0$. A subset $X$ of a frame $L$ is called *locally finite* respectively *discrete* if there is a cover $W$ of $L$ such that

$$\text{each } w \in W \text{ meets only finitely many } x \in X,$$

respectively

$$\text{each } w \in W \text{ meets at most one } x \in X.$$

The cover $W$ is said to *witness* the local finiteness respectively discreteness of $X$.

A countable union of locally finite (discrete) subsets of $L$ is called *$\sigma$-locally finite* (*$\sigma$-discrete*).

**2.2 Lemma.** *If $X \subseteq L$ is locally finite and $x \prec a$ for each $x \in X$, then $\bigvee X \prec a$.*

Proof: Let $W$ be a witnessing cover. For any $w \in W$ let $x_1, \cdots, x_n$ be all the elements of $X$ met by $w$, and $Y = X \smallsetminus \{x_1, \cdots, x_n\}$. Then $w \wedge \bigvee Y = 0$, hence $w \leq (\bigvee Y)^*$ and we have

$$(\bigvee X)^* \vee a = ((\bigvee Y)^* \wedge (\bigwedge_{j=1}^{n} x_j)^*) \vee a \geq$$
$$\geq (w \vee a) \wedge (\bigwedge_{j=1}^{n} (x_j^* \vee a)) = w \vee a \geq w.$$

As $W$ is a cover this shows that $(\bigvee X)^* \vee a = 1$, that is, $\bigvee X \prec a$. $\square$

**2.3.** A regular frame $L$ is said to be *paracompact* if each cover has a locally finite refinement. As in the classical case of spaces,

*the following are equivalent for a regular frame $L$:*

(P1) *$L$ is paracompact,*
(P2) *each cover of $L$ has a $\sigma$-discrete refinement,*
(P3) *each cover of $L$ has a star-refinement.*

(See [10], [9].)

**2.4.** In [7] was proved that

- *a frame $L$ is metrizable iff it has a countable admissible system of covers*, and
- *any frame with a countable admissible system of covers is paracompact.*

Recently T. Kaiser ([5]) found a remarkable formula from which these two facts follow immediately:

*Denote by $y \mapsto y/C$ the right adjoint to $x \mapsto Cx$, that is, $Cx \leq y$ iff $x \leq y/C$. If $\mathfrak{C}$ is a countable admissible system of covers and $A$ any cover then*

$$B = \{c \mid \exists C \in \mathfrak{C}\ \exists a \in A \ \text{ such that } \ c \in C \ \text{ and } \ c \wedge (a/(C(CC))) \neq 0\}$$

*is a cover which star-refines $A$.*

## 3. Some properties of bases

**3.1.** Besides the standard notions of a *$\sigma$-locally finite* and *$\sigma$-discrete basis* we will use a straightforward pointfree counterpart of a *regular basis*, and two others, connected with admissibility.

A basis of a frame $L$ is said to be *regular* if there are, for each $0 \neq a \in L$, subsets $C(a) \subseteq L$ such that

(R1) $\bigvee C(a) = a$, and
(R2) for each $c \in C(a)$, the set $\{b \in B \mid b \wedge c \neq 0 \text{ and } b \nleq a\}$ is finite.

A basis of a frame $L$ is said to be *$\sigma$-admissible* if it can be written as $\bigcup_{n=1}^{\infty} B_n$ where $\{B_n \mid n = 1, 2, \cdots\}$ is an admissible system of covers. It is said to be *$\sigma$-stratified* if, moreover,

(S1) each $B_n$ is locally finite, and
(S2) $B_n \leq^* B_{n-1}$ for each $n \geq 2$.

**3.2 Observation.** *If $L$ has a $\sigma$-admissible basis then*

(1) *each basis of $L$ is $\sigma$-admissible,*
(2) *$L$ has a $\sigma$-discrete basis, and*
(3) *$L$ has a $\sigma$-stratified basis.*

Indeed, let $\{A_n\}$ be admissible. If $B$ is any basis we can put $B_1 = B$, $B_{n+1} = \{b \in B \mid b \leq a \text{ for an } a \in A_n\}$ and since $\{B_n\}$ is admissible by (1.4.1), we have (1). Now as $L$ is paracompact by 2.4, there first are $\sigma$-discrete refinements $B_n$ of $A_n$ and we have a $\sigma$-discrete basis $B = \bigcup B_n$ by (1.4.1) and (1.4.2). Secondly, each cover $C$ has a locally finite star-refinement $C^s$; putting, inductively, $B_1 = A_1^s$, $B_{n+1} = (A_n \wedge B_n)^s$ we obtain a $\sigma$-stratified $B = \bigcup B_n$.

**3.3 Lemma.** *Every $\sigma$-stratified basis is regular.*

Proof: Let $B = \bigcup B_n$ satisfy (S1) and (S2). Let the local finiteness of $B_n$ be witnessed by $W_n$. Obviously, $W_n$ can be chosen so that

$$W_n \leq B_n \quad \text{and} \quad W_1 \geq W_2 \geq W_3 \geq \cdots$$

and then the $W_n$ witness the local finiteness for each $B_k$ with $k \leq n$. For $a \in L$ put

$$C(a) = \{w \mid \exists n,\ w \in W_n \ \text{ and } \ B_n w \leq a\}.$$

Since $\{B_n \mid n = 1, 2, \cdots\}$ is admissible, to prove that $\bigvee C(a) = a$ it suffices to show that $\bigvee C(a) \geq x$ for any $x \lhd a$. Let $B_{n-1}x \leq a$ and let $w \in W_n$ meet $x$. Choose $b \in B_n$ such that $w \leq b$ (recall that $W_n \leq B_n$), and a $b' \in B_{n-1}$ such that $B_n b \leq b'$. Then $b' \wedge x \neq 0$, hence $B_n w \leq B_n b \leq b' \leq a$, therefore $w \in C(a)$ and we see that $x \leq W_n x \leq a$.

Take any $w \in C(a)$, say, $w \in W_n$ with $B_n w \leq a$, and put

$$F = \{b \mid \exists k < n \ \text{ such that } \ b \in B_k \ \text{ and } \ b \wedge w \neq 0\}.$$

By (S1) and the choice of $W_j$, $F$ is finite. Now let $b \in B$ be such that $b \wedge w \neq 0$ and $b \not\leq a$. Let $b \in B_k$. For $c \in B_j$ with $j \geq n$ and $c \wedge w \neq 0$ one has $c \leq B_j w \leq B_n w \leq a$; thus, $k < n$. As $b \wedge w \neq 0$ we have $b \in F$ and hence the set from (R2) is a subset of $F$. $\square$

**3.4 Lemma.** *In a regular frame $L$, every regular basis is $\sigma$-locally finite.*

Proof: Let $B$ be a regular basis of $L$ with associated subsets $C(a) \subseteq L$ as in 3.1. Then $B \cap \{x \mid x \geq a\}$ is finite for any $a \neq 0$ since each $c \in C(a)$ meets every member of this set, and we let $\nu(a)$ be the length of the largest chain in this. Obviously $\nu(c) < \nu(a)$ whenever $0 < a < c$.

Now define

$$B_n = \{b \in B \mid \nu(b) = n, \quad \text{or} \quad b \ \text{ is minimal and } \ \nu(b) \leq n\}$$

and

$$W_n = \bigcup \{C(b) \mid b \in B_n\}.$$

We first show by induction that each $B_n$ is a cover. This is obvious for $n = 1$ since $\bigvee B = 1$ and for any $b \in B$ there exist $c \geq b$ in $B$ such that $\nu(c) = 1$. Now consider any $B_n$ which is a cover, and a $b \in B_n$. By 1.2,

$$b = \bigvee \{c \in B \mid c < b\}$$

for each non-minimal $b \in B$, and hence $\nu(c) > \nu(b) = n$ for all the $c$ that occur. Further, replacing these $c$ by $\overline{c} \in B$ such that $c \le \overline{c}$ and $\nu(\overline{c}) = n+1$ we obtain

$$b \le \bigvee\{\overline{c} \mid c \in B,\ c < b\}$$

and hence $B_{n+1}$ is a cover.

As a result the same holds for $W_n$. We now show that $B_n$ is locally finite witnessed by $W_n$. For any $c \in W_n$ let $b \in B_n$ be such that $c \wedge b \neq 0$. Now, $c \in C(d)$ for some $d \in B_n$. We cannot have $b < d$; this is excluded in the case of minimal $d$, and if $d$ is not minimal, $\nu(d) = n$ and $\nu(b)$ for $b < d$ is greater than $n$. Thus, $b = d$ or $b \nleq d$, and as $c \wedge b \neq 0$, there are only finitely many such $b \in B$. $\square$

**3.5 Lemma.** *In a regular frame $L$, any $\sigma$-locally finite basis is $\sigma$-admissible.*

Proof: Let $B = \bigcup_{n=1}^{\infty} B_n$ be a basis with locally finite $B_n$ witnessed by $W_n$. Then, for each $x \in L$ and $n$, put

$$x_n = \bigvee\{b \in B_n \mid b \prec x\} \quad (\prec x \ \text{ by 2.2})$$

and for each $w \in W_n$ and $k$,

$$\{b(w,1), b(w,2) \cdots, b(w,l_w)\} = \{b \in B_n \mid w \wedge b \neq 0\}$$

and

$$S_k(w) = \{b(w,1), (b(w,1)_k)^*\} \wedge \cdots \wedge \{b(w,l_w), (b(w,l_w)_k)^*\}.$$

Let $\mathfrak{A}$ be the set of covers

$$A_{nk} = \{w \wedge s \mid w \in W_n,\ s \in S_k(w)\}, \quad n,k = 1,2,\ldots\ .$$

We claim that $A_{nk} b_k \le b$ for any $b \in B_n$: if $w \wedge s \wedge b_k \neq 0$ for $w \in W_n$ and $s \in S_k(w)$ then also $w \wedge b \neq 0$, hence $b = b(w,i)$ for some $i$, and since $s \wedge b_k \neq 0$ this implies $s \le b$, showing that $w \wedge s \le b$. Now, $x = \bigvee x_k$ by regularity, and since $b_k \lhd_{\mathfrak{A}} b$ as shown, $\mathfrak{A}$ is admissible. Use 3.2. $\square$

**3.6.** Combining Lemmas 3.3, 3.4 and 3.5 with the observation 3.2 we obtain

**Theorem.** *The following statements are equivalent for a regular frame $L$:*

(1) *$L$ has a $\sigma$-discrete basis,*
(2) *$L$ has a $\sigma$-locally finite basis,*
(3) *$L$ has a regular basis,*
(4) *$L$ has a $\sigma$-admissible basis,*
(5) *$L$ has a $\sigma$-stratified basis.*

**Note.** The relations between these conditions are, of course, not quite symmetric: $\sigma$-admissibility is shared by all bases, if one is such; in some cases, a property $P$ of a basis implies $Q$ for the same basis; in others, the existence of a basis with property $P$ implies the existence of another basis with property $Q$.

## 4. Metrization theorems

**4.1.** By 1.6, any result which asserts the existence of a countable basis of uniformity for a certain class of frames is rightly considered a *metrization theorem*. In this section we describe a number of these, indicating how they follow from the basic Theorem 3.6. We note that, whenever the result in question is already known, the proof presented here is fundamentally different from the original one.

**4.2.** We identify metrization theorems by the author(s) of the corresponding classical result. Thus we have:

*A regular frame is metrizable iff it has*

(M1) *a countable admissible system of covers* (*Moore*),
(M2) *a $\sigma$-locally finite basis* (*Nagata-Smirnov*),
(M3) *a $\sigma$-discrete basis* (*Bing*),
(M4) *a regular basis* (*Archangelskij*).

The way these result from 3.6 is clear, given that the existence of a countable basis of uniformity is obviously equivalent to the existence of a $\sigma$-stratified basis. (M1) and (M3) were originally proved in [7], but with substantially more involved and unnecessarily circuitous proofs, and (M2) was established in Isbell [3] by arguing that, after some preparatory observations, the classical proofs generalize because they really deal with bases rather than with neighbourhoods of points. (M4) appears here for the first time.

**4.3.** It may be worth adding that (M3) immediately yields the generalization of Urysohn's original metrization theorem that

*any regular frame with a countable basis is metrizable*

and consequently its compact variant that

*a compact regular frame is metrizable iff it has a countable basis.*

The latter, of course, is not a proper extension of the classical result if the Axiom of Choice is assumed because then all compact regular frames are spatial.

On the other hand, (M1) also yields metrization theorems between Moore and the uniform one, notably the early result of Alexandroff-Urysohn (1923) concerning, in our terminology, countable admissible systems $\{A_n\}$ such that for any $a, b \in A_n$, if $a \wedge b \neq 0$ then there exist $c \geq a \vee b$ in $A_{n-1}$.

**4.4.** When considering the excellent survey in Engelking's book [1], we can observe that the theorems in 4.2 cover all except one of the cases mentioned there that are relevant in the pointfree context. The others are either point modifications of the Moore metrization theorem (Bing 1951) and the Archangelskij metrization theorem (Alexandroff 1960), or criteria concerning neighbourhood systems of points (Frink 1937, Morita 1955, Nagata 1957) or cases (regular compact, Čech complete) which do not allow for a pointfree extension since the frames concerned are necessarily spatial (Šnejder 1945, Nagata 1950, Katětov 1948).

The exceptional case is the Jones metrization theorem (1958) stating that

> *a Hausdorff space is metrizable iff there is a sequence $\mathcal{V}_1, \mathcal{V}_2, \cdots$ of open covers such that for each compact $Z \subseteq X$ and each open $U \supseteq Z$ one has $\mathrm{St}(Z, \mathcal{V}_k) \subseteq U$ for some $k$,*

which fails in the general case even though it makes perfectly good pointfree sense: any non-metrizable Boolean frame without atoms does not have any compact sublocales and hence satisfies the condition vacuously.

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