Finite row-column exchangeable arrays

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Abstract. We generalize well known results about the extendibility of finite exchangeable sequences and provide necessary conditions for finite and infinite extendibility of a finite row-column exchangeable array. These conditions depend in a simple way on the correlation matrix of the array.

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1. Introduction

A random vector, whose law is invariant under permutations, is called exchangeable. An exchangeable random vector of length n is called r-extendible if it is the initial segment of an exchangeable random vector of length r $(r > n)$. It is well known (see e.g. de Finetti [4]) that not every exchangeable random vector is extendible. We talk about infinite extendibility when r-extendibility holds for every $r \in \mathbb{N}$. By de Finetti's theorem, the law of infinitely extendible sequences can be represented as a mixture of independent laws and much stronger properties hold for infinitely extendible exchangeable sequences. Weaker forms of exchangeability have been studied by different authors, mainly in the infinite setting. Finite partially exchangeable sequences and the problem of their extendibility have been studied by von Plato [11] and Scarsini and Verdicchio [9]. The reader is referred to this last paper for references on the extendibility of exchangeable vectors.

Aldous [1] studied random arrays whose law is invariant with respect to permutations of rows and columns (row-column exchangeable arrays) and provided interesting representation results for them (see also Aldous [2], [3], Hoover [6], [7]).

In this paper we consider finite row-column exchangeable matrices and provide necessary conditions for their finite and infinite extendibility. These conditions, based on the analysis of the correlation matrix, are the natural generalization of the results obtained by de Finetti [4] for exchangeable sequences and by Scarsini and Verdicchio [9] for partially exchangeable sequences.

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2. Results

All the random quantities in the paper will be defined on the probability space $(\Omega, \mathcal{A}, P)$.

Definition 1. A random vector $\mathbf{Y} \equiv [Y_i]$, $i = i, \ldots, n$ is called exchangeable if

$$P(Y_1 \in A_1, Y_2 \in A_2, \ldots, Y_n \in A_n) = P(Y_{\pi(1)} \in A_1, Y_{\pi(2)} \in A_2, \ldots, Y_{\pi(n)} \in A_n)$$

for all n-permutations π, for all Borel sets $A_1, A_2, \ldots, A_n$. The vector $\mathbf{Y}$ is called r-extendible if there exists a vector $\mathbf{V}$ such that the vector $(\mathbf{Y}, \mathbf{V})$ has dimension r and is exchangeable.

The following well known necessary condition for r-extendibility can be found in de Finetti [4].

Theorem 2 (de Finetti [4])**.** *If* $\mathbf{Y}$ *is exchangeable and* r*-extendible, then* $\mathrm{Corr}(Y_1, Y_2) \geq -(r-1)^{-1}$.

Definition 3. A random matrix $\mathbf{X} \equiv [X_{ij}]$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, is called row-column exchangeable if

$$\begin{aligned} P(X_{11} &\in A_{11}, X_{12} \in A_{12}, \ldots, X_{1m} \in A_{1m}, X_{21} \in A_{21}, X_{22} \in A_{22}, \ldots, \\ &X_{2m} \in A_{2m}, \ldots, X_{n1} \in A_{n1}, X_{n2} \in A_{n2}, \ldots, X_{nm} \in A_{nm}) = \\ &P(X_{\sigma(1)\pi(1)} \in A_{11}, X_{\sigma(1)\pi(2)} \in A_{12}, \ldots, X_{\sigma(1)\pi(m)} \in A_{1m}, \\ &X_{\sigma(2)\pi(1)} \in A_{21}, X_{\sigma(2)\pi(2)} \in A_{22}, \ldots, X_{\sigma(2)\pi(m)} \in A_{2m}, \ldots, \\ &\quad X_{\sigma(n)\pi(1)} \in A_{n1}, X_{\sigma(n)\pi(2)} \in A_{n2}, \ldots, X_{\sigma(n)\pi(m)} \in A_{nm}), \end{aligned}$$

for all n-permutations σ and m-permutations π, for all Borel sets $A_{11}, A_{12}, \ldots, A_{1m}, A_{21}, A_{22}, \ldots, A_{2m}, \ldots, A_{n1}, A_{n2}, \ldots, A_{nm}$.

For $r > n$ and $q > m$ the matrix $\mathbf{X}$ is called (r, q)-extendible if there exist matrices $\mathbf{T}$, $\mathbf{Z}$, $\mathbf{W}$, with

$$\begin{aligned} \mathbf{T} &\equiv [X_{ij}] \quad i = 1, \ldots, n, \quad j = m+1, \ldots, q, \\ \mathbf{Z} &\equiv [X_{ij}] \quad i = n+1, \ldots, r, \quad j = 1, \ldots, m, \\ \mathbf{W} &\equiv [X_{ij}] \quad i = n+1, \ldots, r, \quad j = m+1, \ldots, q, \end{aligned}$$

such that

$$\mathbf{X}^* \equiv \begin{bmatrix} \mathbf{X} & \mathbf{T} \\ \mathbf{Z} & \mathbf{W} \end{bmatrix} \tag{1}$$

is row-column exchangeable. A matrix that is (r, q) extendible for all $r > n$ and $q > m$ is called infinitely extendible.

Representation results for infinitely extendible row-column exchangeable matrices can be found in Aldous [1], [2], [3], Hoover [6], [7].

Row-column exchangeability implies the following form for the covariance matrix of $\mathbf{X}$:

$$\text{(2)} \quad \operatorname{Var}(X_{ij}) = \sigma^2 \quad \forall\, i \in \{1,\dots,n\}, \quad \forall\, j \in \{1,\dots,m\},$$

$$\text{(3)} \quad \operatorname{Cov}(X_{ij}, X_{ik}) = \sigma^2\rho \quad \forall\, i \in \{1,\dots,n\}, \quad \forall\, j,k \in \{1,\dots,m\},\ j \neq k,$$

$$\text{(4)} \quad \operatorname{Cov}(X_{ij}, X_{hj}) = \sigma^2\beta \quad \forall\, j \in \{1,\dots,m\}, \quad \forall\, i,h \in \{1,\dots,n\},\ i \neq h,$$

$$\text{(5)} \quad \operatorname{Cov}(X_{ij}, X_{hk}) = \sigma^2\alpha \quad \forall\, i,h \in \{1,\dots,n\},\ \forall\, j,k \in \{1,\dots,m\},\ j \neq k,\ i \neq h.$$

The following theorem gives necessary conditions for the (r,q)-extendibility of a $(n \times m)$ row-column exchangeable matrix.

Theorem 4. *Let $r, q \geq 4$. If a row-column exchangeable random matrix $\mathbf{X}$ is (r,q)-extendible, then*

$$\text{(6)} \quad 1 + \alpha \geq \rho + \beta,$$

$$\text{(7)} \quad 1 + (k-1)\rho \geq |\beta + (k-1)\alpha|, \quad \forall\, k \in \{1,\dots,q\},$$

$$\text{(8)} \quad 1 + (h-1)\beta \geq |\rho + (h-1)\alpha|, \quad \forall\, h \in \{1,\dots,r\},$$

$$\text{(9)} \quad 1 + (k-1)\rho + (h-1)\beta + (h-1)(k-1)\alpha \geq 0, \ \forall\, k \in \{1,\dots,q\}, \ \forall\, h \in \{1,\dots,r\}.$$

Several lemmata will be needed in order to prove the above theorem.

Lemma 5. *If a row-column exchangeable random matrix $\mathbf{X}$ is (r,q)-extendible, then*

$$(1-\rho-\beta+\alpha)^{(h-1)(k-1)}(1+(h-1)\beta-\rho-(h-1)\alpha)^{k-1}\cdot$$
$$(1+(k-1)\rho-\beta-(k-1)\alpha)^{h-1}(1+(h-1)\beta+(k-1)\rho+(h-1)(k-1)\alpha) \geq 0,$$

for all $h \in \{1,\dots,r\}$, $k \in \{1,\dots,q\}$.

Proof: Let $\mathbf{X}^*$ be a row-column exchangeable matrix obtained by extending $\mathbf{X}$ as in (1). By definition of row-column extendibility, for $i \in \{1,\dots,r\}$, $X_{i1},\dots,X_{iq}$ are exchangeable, hence, by Theorem 2

$$\text{(10)} \quad \rho \geq -(q-1)^{-1},$$

and, for $j \in \{1,\dots,q\}$, $X_{1j},\dots,X_{rj}$ are exchangeable, hence,

$$\text{(11)} \quad \beta \geq -(r-1)^{-1}.$$

Consider the column vector $\operatorname{vec}(\mathbf{X})$ obtained by stacking the columns of $\mathbf{X}$. Conditions (2), (3), (4), (5) imply that the correlation matrix of $\operatorname{vec}(\mathbf{X})$ is

$$\operatorname{Corr}(\operatorname{vec}(\mathbf{X})) = \begin{bmatrix} \mathbf{B} & \mathbf{A} & \cdots & \mathbf{A} \\ \mathbf{A} & \mathbf{B} & \cdots & \mathbf{A} \\ \vdots & & \ddots & \\ \mathbf{A} & \mathbf{A} & \cdots & \mathbf{B} \end{bmatrix},$$

where

$$\mathbf{B} = \begin{bmatrix} 1 & \beta & \cdots & \beta \\ \beta & 1 & \cdots & \beta \\ \vdots & & \ddots & \\ \beta & \beta & \cdots & 1 \end{bmatrix},$$

and

$$\mathbf{A} = \begin{bmatrix} \rho & \alpha & \cdots & \alpha \\ \alpha & \rho & \cdots & \alpha \\ \vdots & & \ddots & \\ \alpha & \alpha & \cdots & \rho \end{bmatrix}.$$

For any $n \times n$ matrix $\mathbf{C}$ of the form

$$\mathbf{C} = \begin{bmatrix} \gamma & \delta & \cdots & \delta \\ \delta & \gamma & \cdots & \delta \\ \vdots & & \ddots & \\ \delta & \delta & \cdots & \gamma \end{bmatrix},$$

we have

$$\det(\mathbf{C}) = (\gamma + (n-1)\delta)(\gamma - \delta)^{n-1}.$$

Repeated applications of the above result yield

$$\begin{aligned} \det(\operatorname{Corr}(\operatorname{vec}(\mathbf{X}))) = {} & \det(\mathbf{B} + (m-1)\mathbf{A}) \cdot \det(\mathbf{B}-\mathbf{A})^{m-1} \\ = {} & (1-\rho-\beta+\alpha)^{(n-1)(m-1)}(1+(n-1)\beta-\rho-(n-1)\alpha)^{m-1} \times \\ & (1+(m-1)\rho-\beta-(m-1)\alpha)^{n-1} \times \\ & (1+(n-1)\beta+(m-1)\rho+(n-1)(m-1)\alpha). \end{aligned}$$

The correlation matrix of $\operatorname{vec}(\mathbf{X}^*)$ will have the same structure and the same parameters (just a different number of lines) as the correlation matrix of $\operatorname{vec}(\mathbf{X})$. Any correlation matrix is positive semidefinite, and therefore its determinant and all its principal minors are nonnegative. Hence

$$(1-\rho-\beta+\alpha)^{(h-1)(k-1)}(1+(h-1)\beta-\rho-(h-1)\alpha)^{k-1} \times \tag{12}$$

$$(1+(k-1)\rho-\beta-(k-1)\alpha)^{h-1}(1+(h-1)\beta+(k-1)\rho+(h-1)(k-1)\alpha) \geq 0,$$

for all $h \in \{1, \dots, r\}$, $k \in \{1, \dots, q\}$. □

Define

$$\begin{aligned} A &= (1-\rho-\beta+\alpha), \\ B(h) &= (1+(h-1)\beta-\rho-(h-1)\alpha), \\ C(k) &= (1+(k-1)\rho-\beta-(k-1)\alpha), \\ D(h,k) &= (1+(h-1)\beta+(k-1)\rho+(h-1)(k-1)\alpha). \end{aligned}$$

Thus (12) reads

$$A^{(h-1)(k-1)}B(h)^{k-1}C(k)^{h-1}D(h,k) \geq 0.$$

The following result is obvious.

Lemma 6. *The functions $B(\cdot)$ and $C(\cdot)$ are linear on $\mathbb{N}$. The function $D(\cdot,\cdot)$ is linear in each of its two arguments. Since $B(1) = 1-\rho$, either $B(h)$ is always positive, or it is negative for all h larger than some value h_0. Analogously for $C(k)$.*

Lemma 7. $D(h,k) \geq 0 \ \ \forall\, h \in \{1,\dots,r\},\ \ \forall\, k \in \{1,\dots,q\}$.

Proof: We have

$$\begin{aligned}
0 &\leq \mathrm{Var}\,(\sum_{i=1}^{h}\sum_{j=1}^{k} X_{ij}) \\
&= \sum_{i=1}^{h}\sum_{j=1}^{k}\sum_{u=1}^{h}\sum_{v=1}^{k} \mathrm{Cov}\,(X_{ij}, X_{uv}) \\
&= hk \sum_{i=1}^{h}\sum_{j=1}^{k} \mathrm{Cov}\,(X_{ij}, X_{11}) \\
&= hk \Bigg[\mathrm{Var}\,(X_{11}) + \sum_{i=2}^{h} \mathrm{Cov}\,(X_{i1}, X_{11}) \\
&\qquad\qquad + \sum_{j=2}^{k} \mathrm{Cov}\,(X_{1j}, X_{11}) + \sum_{i=2}^{h}\sum_{j=2}^{k} \mathrm{Cov}\,(X_{ij}, X_{11}) \Bigg] \\
&= \sigma^2 hk[1 + (h-1)\beta + (k-1)\rho + (h-1)(k-1)\alpha] \\
&= \sigma^2 hk D(h,k).
\end{aligned}$$

□

Lemma 8. *Let $r, q \geq 2$. Then $D(h,k) > 0\ \forall\, h \in \{1,\dots,r-1\}$, $\forall\, k \in \{1,\dots,q-1\}$.*

Proof: Observe that $D(1,1) = 1$ and, by Lemma 7, $D(1,q) \geq 0$. For any $k \in \{1,\dots,q-1\}$, we have $D(1,k) > 0$, since D is linear in the second argument (Lemma 6). Furthermore, again by Lemma 7, we have that $D(r,k) \geq 0$, and hence $D(h,k) > 0\,\forall\, h \in \{1,\dots,r-1\}$, since D is linear in its first argument. □

Lemma 9. *Let $r, q \geq 3$. Then $A \geq 0$, $B(2) \geq 0$, $C(2) \geq 0$.*

Proof: We have

$$\begin{aligned}
A &= (1-\rho-\beta+\alpha), \\
B(2) &= (1-\rho+\beta-\alpha), \\
C(2) &= (1+\rho-\beta-\alpha).
\end{aligned}$$

Thus

$$A + B(2) = 2 - 2\rho \geq 0, \tag{13}$$
$$A + C(2) = 2 - 2\beta \geq 0, \tag{14}$$
$$B(2) + C(2) = 2 - 2\alpha \geq 0. \tag{15}$$

Hence, the family $\{A, B(2), C(2)\}$ cannot contain two (or more) strictly negative terms.

By Lemma 5 we know that

$$A \cdot B(2) \cdot C(2) \cdot D(2,2) \geq 0.$$

By Lemma 8, we have $D(2,2) > 0$, and hence

$$A \cdot B(2) \cdot C(2) \geq 0. \tag{16}$$

Suppose that there exists a strictly negative term. Then (16) can be satisfied only if there is another negative (possibly zero) term. But this is impossible, since the sum of these two terms would be strictly negative. Thus, the claim follows. □

Lemma 10. *If $A = 0$, then $B(h) \geq 0$ for all $h \in \mathbb{N}$ and $C(k) \geq 0$ for all $k \in \mathbb{N}$.*

PROOF: If $A = 0$, then $1 - \rho = \beta - \alpha$, hence

$$B(h) = (1 - \rho + (h-1)(\beta - \alpha)) = h(1 - \rho) \geq 0, \ \forall\, h \in \mathbb{N},$$

and $1 - \beta = \rho - \alpha$, hence

$$C(k) = (1 - \beta + (k-1)(\rho - \alpha)) = k(1 - \beta) \geq 0, \ \forall\, k \in \mathbb{N}.$$

□

Lemma 11. *If either $B(h) \equiv 0$, or $C(k) \equiv 0$, then $A = 0$.*

PROOF: If $B(h) \equiv 0$, then $\alpha = \beta$, and $\rho = 1$, hence $A = 0$. If $C(k) \equiv 0$, then $\alpha = \rho$, and $\beta = 1$, hence $A = 0$. □

Lemma 12. *Let $r, q \geq 2$. The following are equivalent:*

(a) $D(1, q) = 0$,
(b) $\rho = -(q-1)^{-1}$,
(c) *there exists $c \in \mathbb{R}$ such that $\sum_{j=1}^{q} X_{1j} = c$, a.s.*

Any of the above implies:

(d) $\beta + (q-1)\alpha = 0$,
(e) $C(q) = 0$.

Proof: First recall that

$$D(1,q) = 1 + (q-1)\rho = \frac{1}{\sigma^2 q}\operatorname{Var}\Big(\sum_{j=1}^{q} X_{1j}\Big) \tag{17}$$

(see also the proof of Lemma 7). Conditions (a) and (b) are obviously equivalent. Formula (17) implies that $D(1,q) = 0$ iff $\sum_{j=1}^{q} X_{1j}$ is a degenerate random variable, i.e. iff (c) holds.

Next, observe that

$$1 + (k-1)\rho - [\beta + (k-1)\alpha] = D(1,k) - [\beta + (k-1)\alpha] = C(k), \tag{18}$$

$$1 + (k-1)\rho + [\beta + (k-1)\alpha] = D(1,k) + [\beta + (k-1)\alpha] = D(2,k). \tag{19}$$

If $D(1,q) = 0$, then $D(2,q) = 0$. In fact

$$D(2,q) = \operatorname{Var}\Big(\sum_{j=1}^{q}(X_{1j} + X_{2j})\Big) = \operatorname{Var}(2c) = 0.$$

Thus, if $D(1,q) = 0$, taking $k = q$ in (19) yields (d). Condition (e) follows immediately from (d) and (18). $\square$

Lemma 13. *Let $r, q \geq 2$. The following are equivalent:*

(a) $D(r,1) = 0$,
(b) $\beta = -(r-1)^{-1}$,
(c) *there exists $c \in \mathbb{R}$ such that $\sum_{i=1}^{r} X_{i1} = c$, a.s.*

Any of the above implies:

(d) $\rho + (r-1)\alpha = 0$,
(e) $B(r) = 0$.

Proof: The proof is similar to the previous one. $\square$

Lemma 14. *Let $r, q \geq 4$. Then $B(r) \geq 0$ and $C(q) \geq 0$.*

Proof: Let us prove $C(q) \geq 0$. The proof of $B(r) \geq 0$ is similar.

If $D(1,q) = 0$, the claim follows from Lemma 12. Let $D(1,q) > 0$. If $A = 0$ the result follows from Lemma 10. Assume $A > 0$. Since $D(1,q) > 0$ and $D(r,q) \geq 0$ (Lemma 7), we have

$$D(2,q) > 0 \quad \text{and} \quad D(3,q) > 0. \tag{20}$$

By Lemma 5, we have

$$A^{(h-1)(k-1)}(B(h))^{k-1}(C(k))^{h-1}D(h,k) \geq 0, \ \forall\, h \in \{1,\dots,r\}, \ \forall\, k \in \{1,\dots,q\},$$

which implies

$$(B(2))^{k-1}C(k) \geq 0, \ \forall\, k \in \{1, \dots, q\}, \tag{21}$$

$$(B(3))^{k-1}(C(k))^2 \geq 0, \ \forall\, k \in \{1, \dots, q\}. \tag{22}$$

The above inequalities stem from Lemma 8 for $k \in \{1, \dots, q-1\}$ and from (20) for $k = q$.

Suppose that $C(q) < 0$. Then (21) (with $k = q$) and Lemma 9 imply that $B(2) = 0$. We cannot have $B(1) = B(2) = B(3) = \dots = 0$, in view of Lemma 11. Hence, by the linearity of B and by the relation $B(1) = 1-\rho \geq 0$, we get $B(3) < 0$. Then

$$[B(3)(C(2))^2 \leq 0.$$

This, together with (22), yields $C(2) = 0$. Thus, we have: $A > 0$, $B(2) = C(2) = 0$. From equations (13), (14) and (15) we obtain: $\rho < 1$, $\beta < 1$, $\alpha = 1$. Hence,

$$C(4) = 1 - \beta + 3(\rho - \alpha) < 1 - \beta + \rho - \alpha = C(2) = 0.$$

Therefore

$$(B(3))^3(C(4))^2 < 0,$$

which contradicts (22). Hence $C(q) \geq 0$. □

Proof of Theorem 4: Condition (6) has been proved in Lemma 9. Next, observe that, in view of (18) and (19), in order to prove (7) it is enough to check that $C(k) \geq 0$ and $D(2,k) \geq 0$ for $k \in \{1, \dots, q\}$. Lemma 7 ensures that $D(2,k) \geq 0$. The relation $C(k) \geq 0$ stems from Lemma 10 when $A = 0$, and from Lemma 14 and the linearity of C, when $A > 0$. The proof of (8) is similar. Condition (9) was proved in Lemma 7. □

Corollary 15. *If* $\mathbf{X}$ *is infinitely extendible, then*

$$\max(0, \rho + \beta - 1) \leq \alpha \leq \min(\rho, \beta).$$

Proof: Conditions (7) and (8) and a passage to the limit yield $\alpha \leq \rho$ and $\alpha \leq \beta$. The condition $\alpha \geq \rho + \beta - 1$ is just $A \geq 0$ (proved in Lemma 9). The condition $\alpha \geq 0$ stems from (9) and a passage to the limit. □

Remark 16. In complete analogy with the case of exchangeable and partially exchangeable sequences, the extendibility conditions depend only on the parameters of the correlation matrix.

Remark 17. A weaker exchangeability property was proposed by Silverman [10], Gaifman [5], Krauss [8]. It makes sense only for square (or infinite) matrices and it coincides with the property of Definition 3, except that $\sigma = \pi$, namely, the two permutations that operate on the rows and columns are the same. If we assume this weaker property for $\mathbf{X}$, its implications for the correlation matrix are the same as before, therefore the necessary extendibility conditions do not change.

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