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Fixed point theorems for nonexpansive operators with dissipative perturbations in cones

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Abstract. Let P be a cone in a Hilbert space H, $A : P \to 2^P$ be an accretive mapping (equivalently, $-A$ be a dissipative mapping) and $T : P \to P$ be a nonexpansive mapping. In this paper, some fixed point theorems for mappings of the type $-A+T$ are established. As an application, we utilize the results presented in this paper to study the existence problem of solutions for some kind of nonlinear integral equations in $L^2(\Omega)$.

Keywords: nonexpansive mapping, accretive mapping, fixed point theorem, nonlinear integral equations

Classification: 45H10, 47H06, 47H09, 47H15

1. Introduction

The study of fixed point problem for nonexpansive mapping for a closed convex subset of a Banach space into itself was started by F.E. Browder, D. Göhde and W.A. Kirk. Since then, whether a nonexpansive mapping from a bounded closed convex subset of a Banach space into itself has a fixed point, has become an interesting problem. As we know, the answer to this problem is negative in general Banach spaces ([2]). But this problem is still open for reflexive Banach spaces. Recently, some existence theorems for fixed points of nonexpansive type mappings related to accretive mappings have been considered by some authors ([8], [10], [11], [12]).

The purpose of this paper is to establish some existence theorems for mappings of the type $-A+T$ in a cone P of a real Hilbert space H, where the mapping $A : D(A) \subseteq P \to 2^P$ is accretive and $T : P \to P$ is nonexpansive. As an application, we utilize the results presented in this paper to study the existence problem of solutions for some kind of nonlinear integral equations in $L^2(\Omega)$.

2. Main results

Throughout this paper, suppose that H is a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$, P is a cone in H and there is an order " $\leq$ " induced by P in H, i.e., for any given $x, y \in H$, define $x \leq y \Longleftrightarrow y - x \in P$.

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A mapping $A : D(A) \subseteq H \to 2^H$ is said to be *accretive* (or *monotone*) if

$$\langle u - v, x - y\rangle \geq 0$$

for all $x, y \in D(A)$, $u \in Ax$ and $v \in Ay$. If A is accretive, then $-A$ is said to be *dissipative*. In this paper, we always assume that

$$(A + I)(D(A)) = P. \tag{2.1}$$

For further details on accretive mappings which satisfy the condition (2.1), we may refer to [5], [6], [7].

Theorem 2.1. *Let $A : D(A) \subseteq P \to 2^P$ be an accretive mapping satisfying the condition* (2.1), *Ω be an open bounded subset of H with $\theta \in \Omega$ and $T : P \to P$ be a nonexpansive mapping. If the condition $Tx \not\geq x$ for all $x \in \partial\Omega \cap D(A)$ is satisfied, then $-A + T$ has a fixed point in $D(A)$.*

PROOF: For each $0 \leq k_n < 1$, $k_nT : \bar{\Omega} \cap P \to P$ is a k_n-set contraction. Since $Tx \not\geq x$ for all $x \in \partial\Omega \cap D(A)$, $k_nTx \not\geq x$ for all $x \in \partial\Omega \cap D(A)$. By Lemma 1 in [5], we have

$$i(-A + k_nT, \Omega \cap D(A)) = 1$$

and so $-A + k_nT$ has a fixed point $x_n \in \Omega \cap D(A)$, i.e.,

$$x_n = (I + A)^{-1}k_nTx_n.$$

Letting $k_n \to 1$, $\{x_n\}$ has a subsequence (for simplicity, we still denote it by $\{x_n\}$) such that $\{x_n\}$ converges weakly to x^*. Let $x_\lambda = (1 - \lambda)x^* + \lambda(I + A)^{-1}Tx^*$ for all $\lambda \in (0, 1)$. Since $(I + A)^{-1}$ is nonexpansive, we have

$$\begin{aligned}&\langle x_\lambda - (I + A)^{-1}Tx_\lambda - [x_n - (I + A)^{-1}Tx_n,\ x_\lambda - x_n\rangle\\ &\quad \geq \|x_\lambda - x_n\|^2 - \langle (I + A)^{-1}Tx_\lambda - (I + A)^{-1}Tx_n,\ x_\lambda - x_n\rangle\end{aligned}$$

and, as $n \to \infty$,

$$\begin{aligned}\|x_n - (I + A)^{-1}Tx_n\| &= \|(I + A)^{-1}k_nTx_n - (I + A)^{-1}Tx_n\|\\ &\leq (1 - k_n) \cdot \|Tx_n\| \longrightarrow 0.\end{aligned}$$

Hence we have

$$\langle x_\lambda - (I + A)^{-1}Tx_\lambda, \quad x_\lambda - x^*\rangle \geq 0$$

and so

$$\begin{aligned}&\langle (1 - \lambda)x^* + \lambda(I{+}A)^{-1}Tx^* - (I + A)^{-1}T((1 - \lambda)x^* + \lambda(I + A)^{-1}Tx^*),\\ &\qquad \lambda(I + A)^{-1}Tx^* - \lambda x^*\rangle \geq 0.\end{aligned} \tag{2.2}$$

Dividing (2.2) by λ and letting $\lambda \to 0^+$, we have

$$\langle x^* - (I + A)^{-1}Tx^*, (I + A)^{-1}Tx^* - x^*\rangle \geq 0.$$

This implies that $x^* = (I + A)^{-1}Tx^*$, i.e., x^* is a fixed point of $-A + T$ in $D(A)$. This completes the proof. □

Corollary 2.2. *Let A, Ω and T be the same as in Theorem* 2.1*. If the condition $Tx < x$ for all $x \in \partial\Omega \cap D(A)$ is satisfied, then $-A+T$ has a fixed point in $D(A)$.*

Proof: It is obvious that $Tx \ngeq x$ for all $x \in \partial\Omega \cap D(A)$. Hence the conclusion can be obtained from Theorem 2.1 immediately. □

Theorem 2.3. *Let $A : D(A) \subseteq P \to 2^P$ be an accretive mapping satisfying the condition* (2.1)*, Ω be an open bounded subset of H with $\theta \in \Omega$ and $T : P \to P$ be a nonexpansive mapping. If $\|Tx\| \le \|x\|$ for all $x \in \partial\Omega \cap D(A)$, then $-A+T$ has a fixed point in $D(A)$.*

Proof: For each $0 \le k_n < 1$, k_nT is a k_n-set contraction and we have

$$\|k_nTx\| \le \|Tx\| \le \|x\|$$

for all $x \in D(A) \cap \partial\Omega$. By Theorem 3 in [5], $-A+k_nT$ has a fixed point x_n in $\Omega \cap D(A)$. Without loss of generality, we may assume that $\{x_n\}$ converges weakly to x^* as $k_n \to 1$. Thus, by the same proof as given in Theorem 2.1, we can prove that x^* is a fixed point of $-A+T$ in $D(A)$. This completes the proof. □

Theorem 2.4. *Let $A : D(A) \subseteq P \to 2^P$ be an accretive mapping satisfying the condition* (2.1)*, Ω be an open bounded subset of H with $\theta \in \Omega$ and $T : P \to P$ be a nonexpansive mapping. If $\langle u - Tx, x\rangle \ge 0$ for all $x \in \partial\Omega \cap D(A)$ and $u \in Ax$, then $-A+T$ has a fixed point in $D(A)$.*

Proof: For each $0 \le k_n < 1$, k_nT is a k_n-set contraction. Let

$$H(t,x) = tk_nTx$$

for all $(t,x) \in [0,1] \times (\bar{\Omega} \cap P)$.

Now, we prove that

$$x \notin -Ax + H(t,x)$$

for all $(t,x) \in [0,1] \times (\partial\Omega \cap D(A))$. Suppose that this is not true. Then there exist $t_0 \in [0,1]$, $x_0 \in \partial\Omega \cap D(A)$ and $u_0 \in Ax_0$ such that $x_0 = -u_0 + t_0k_nTx_0$. Thus we have

$$\|x_0\|^2 = -t_0k_n\langle u_0 - Tx_0, x_0\rangle + (-1 + t_0k_n) \cdot \langle u_0, x_0\rangle. \tag{2.3}$$

Since $(I+A)(D(A)) = P$, we have $\theta \in A\theta$. In view of the assumption, it follows from (2.3) that $\|x_0\|^2 \le 0$, which implies that $x_0 = \theta$. This contradicts $\theta \in \Omega$. Thus, by using Theorem 1(c) in [5], we have

$$i(-A + k_nT, \Omega \cap D(A)) = i(-A + 0, \Omega \cap D(A)).$$

By virtue of Theorem 1(a) in [5] again, we have $i(-A+0, \Omega \cap D(A)) = 1$ and so $-A+k_nT$ has a fixed point x_n in $\Omega \cap D(A)$. Without loss of generality, we may assume that $\{x_n\}$ converges weakly to x^* as $k_n \to 1$. Therefore, by the same proof as given in Theorem 2.1, we can prove that x^* is a fixed point of $-A+T$ in $D(A)$. This completes the proof. □

3. Application

In this section, we shall use the results presented in Section 2 to study the existence problem of solutions for some kind of nonlinear integral equations.

Let $\Omega \subset \mathbb{R}^n$ be a nonempty measurable subset with $m(\Omega) = 1$, where $m(\Omega)$ denotes the measure of Ω, and let $f(x, y) : \Omega \times [0, +\infty) \to [0, +\infty)$ be a function satisfying the Carathéodory condition, i.e.,

(i) for each $y \in [0, +\infty)$, $f(x, y)$ is measurable in x,

(ii) for almost all $x \in \Omega$, $f(x, y)$ is continuous in y.

In addition, if f satisfies the following conditions:

(a) $(f(x, y) - f(x, z))(y - z) \geq 0$ for all $x \in \Omega$ and $y, z \in [0, +\infty)$,

(b) for each $x \in \Omega$, there exists $N(y) > 0$ such that

$$f(x, y) \leq N(y) \cdot y$$

for all $y \in [0, +\infty)$, where $N(y)$ depends on y, then, by the condition (a), for $x \in \Omega$ such that $f(x, y)$ is continuous in y, $f(x, y) : [0, +\infty) \to [0, +\infty)$ is accretive. Therefore, it follows from the assumption (b) and Proposition 1 in [5] that

$$(I + f(x, \cdot))([0, +\infty)) = [0, +\infty)$$

for almost all $x \in \Omega$. Now let $P = \{u(\cdot) \in L^2(\Omega) \mid u(x) \geq 0 \text{ for almost all } x \in \Omega\}$. Then P is a cone in $L^2(\Omega)$. Denote

$$D(A) = \{u(\cdot) \in P | f(x, u(x)) \in L^2(\Omega)\}$$

and define the mapping $A : D(A) \subseteq P \to P$ by

$$Au(x) = f(x, u(x))$$

for all $x \in \Omega$ and $u(\cdot) \in D(A)$. By the condition (a), we know that $A : D(A) \subseteq P \to P$ is accretive. For $v(\cdot) \in P$, we have $(I + f(x, \cdot))^{-1} v(x) \in L^2(\Omega)$ since $(I + f(x, \cdot))[0, +\infty) = [0, +\infty)$ and $(I + f(x, \cdot))^{-1}$ is nonexpansive for almost all $x \in \Omega$. Therefore, we have

$$(I + A)(D(A)) = P.$$

Suppose that $k(x, y) : \Omega \times [0, +\infty) \to [0, +\infty)$ is a function satisfying the following conditions:

$$k(x, 0) \in L^2(\Omega)$$

and

$$|k(x, y) - k(x, z)| \leq |y - z|$$

for all $x \in \Omega$ and $y, z \in [0, +\infty)$. Let $T : P \to P$ be a mapping defined by

$$Tu(x) = \int_\Omega k(x, u(y))\, dy$$

for all $u(\cdot) \in L^2(\Omega)$ and $x \in \Omega$. Since $k(x,y) \leq y + k(x,0)$, T is well defined on P and it is easy to see that T is nonexpansive.

Now, we consider the following nonlinear integral equation:

$$u(x) = -f(x, u(x)) + \int_\Omega k(x, u(y))\, dy \tag{E3.1}$$

for all $x \in \Omega$. We make a further assumption on $k(\cdot,\cdot)$ as follows:

$$k(x,y) \leq M \cdot y + g(x)$$

for all $(x,y) \in \Omega \times [0,+\infty)$, where $0 < M < 1$ and $g(\cdot) \in P$. Choose $r > 0$ such that

$$M + r^{-1} \left(\int_\Omega g^2(x)\, dx \right)^{\frac{1}{2}} \leq 1.$$

Then, for each $u(\cdot) \in P$ with $(\int_\Omega u^2(x)\, dx)^{\frac{1}{2}} = r$, we have

$$\|Tu(x)\|_{L^2(\Omega)} \leq \|u(x)\|_{L^2(\Omega)}.$$

By Theorem 2.3, (E3.1) has a solution $u(\cdot) \in P$.

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