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Remarks on continuous images of Radon-Nikodým compacta

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Abstract. A family of compact spaces containing continuous images of Radon-Nikodým compacta is introduced and studied. A family of Banach spaces containing subspaces of Asplund generated (i.e., GSG) spaces is introduced and studied. Further, for a continuous image of a Radon-Nikodým compact K we prove: If K is totally disconnected, then it is Radon-Nikodým compact. If K is adequate, then it is even Eberlein compact.

Keywords: Asplund generated space, continuous image of Radon-Nikodým compact, totally disconnected compact, adequate compact, Eberlein compact

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Introduction

A Banach space X is called *Asplund* if every subspace of it has separable dual. X is called *Asplund generated* (or *GSG*) if it contains a linear continuous image of an Asplund space as a dense set. Thus, every weakly compactly generated space is Asplund generated.

All topological spaces in this note are assumed to be Hausdorff. A compact space is called *Radon-Nikodým* if it can be found, up to a homeomorphism, in the dual to an Asplund space, endowed with the weak* topology. Note that every Eberlein compact is Radon-Nikodým. We recall

Theorem 0 ([St1], [F, Theorem 1.5.4]). *For a compact space K the following assertions are equivalent:*

(i) *K is a Radon-Nikodým compact;*
(ii) *$C(K)$ is an Asplund generated space;*
(iii) *the dual unit ball $\left(B_{C(K)^*}, w^*\right)$ endowed with the weak* topology is a Radon-Nikodým compact.*

Corollary 1 ([F, Theorem 1.5.5]). *For a compact space K the following assertions are equivalent:*

(i) *K is a continuous image of a Radon-Nikodým compact;*
(ii) *$C(K)$ is a subspace of an Asplund generated space;*
(iii) *$\left(B_{C(K)^*}, w^*\right)$ is a continuous image of a Radon-Nikodým compact.*

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Corollary 2 ([F, Theorem 1.5.6]). *For a Banach space Z the following assertions are equivalent:*

(i) *Z is a subspace of an Asplund generated space;*
(ii) *the dual unit ball $\left(B_{Z^*}, w^*\right)$ is a continuous image of a Radon-Nikodým compact;*
(iii) *$C((B_{Z^*}, w^*))$ is a subspace of an Asplund generated space.*

A main open question raised by Namioka [N1], [N2], but going back to Grothendieck's memoir [Gr], sounds as:

($*$) *Is a continuous image of a Radon-Nikodým compact a Radon-Nikodým compact?*

A related question for Banach spaces has a negative answer: There exists an Asplund generated space and its subspace which is not Asplund generated. Indeed, Stegall [St1] observed that Rosenthal's counterexample [Ro] of a weakly compactly generated space (L_1 on a "big" measure space) and its non weakly compactly generated subspace fits the job. For another example (now of the form $C(K)$) see [A1]. However, according to [BRW], the dual unit ball of such subspaces is Eberlein, hence Radon-Nikodým compact.

Note also that if ($*$) has a positive answer then ($**$) below has a positive answer and vice versa.

($**$) *If Z is a subspace of X and (B_{X^*}, w^*) is a Radon-Nikodým compact, is such the compact (B_{Z^*}, w^*)?*

To see this, let φ be a continuous mapping of a Radon-Nikodým compact L onto a compact K. Then the assignment $f \mapsto f \circ \varphi$ maps the Banach space $C(K)$ onto a closed subspace of $C(L)$ isometrically. We observe that $(B_{C(L)^*}, w^*)$ is a Radon-Nikodým compact by Theorem 0. Now assume that ($**$) has a positive answer. Then $(B_{C(K)^*}, w^*)$ is a Radon-Nikodým compact and hence so is K.

Note that if (B_{Z^*}, w^*) is a Radon-Nikodým compact, then Z may not be Asplund generated, see [F, Theorem 1.6.3].

The aim of this note is twofold. First we define and study a class of compacta which is, at least formally, larger than that of continuous images of Radon-Nikodým compacta. We call them countably lower fragmentable compacta. Paralelly we do the same for a Banach space counterpart of such compacta. Namely, we consider Banach spaces whose dual unit ball with the weak* topology is countably lower fragmentable. This class extends, at least formally, the class of subspaces of Asplund generated spaces.

The second part studies our concepts in the framework of totally disconnected compacta. We prove, in a different way, a result of Argyros that *a totally disconnected compact, which is a continuous image of a Radon-Nikodým compact, is Radon-Nikodým compact* ([A2]). We also get that, *an adequate compact, which is a continuous image of a Radon-Nikodým compact is even Eberlein compact.*

We hope that this note will bring a bit of light to the open questions mentioned above.

Countably lower fragmentable compacta

Let X be a Banach space, H a set in X^*, A a bounded set in X and $\Delta > 0$. We say that

(i) H is (A, Δ)-*fragmentable* if for every nonempty set $M \subset H$ there is a weak* open set $G \subset X^*$ such that $M \cap G \neq \emptyset$ and

$$\operatorname{diam}_A(M \cap G) := \sup\{\langle x_1^* - x_2^*, x\rangle : x_1^*, x_2^* \in M \cap G, x \in A\} \leq \Delta;$$

(ii) H is (A, Δ)-*dentable* if for every nonempty set $M \subset H$ there are $x \in X$ and $\alpha > 0$ such that $\operatorname{diam}_A S(M, x, \alpha) \leq \Delta$, where

$$S(M, x, \alpha) = \{x^* \in M : \langle x^*, x\rangle > \sup\langle M, x\rangle - \alpha\};$$

(iii) H is (A, Δ)-*separable* if there exists an at most countable set $C \subset H$ such that for every $x^* \in H$ there is $y^* \in C$ such that $\sup\{|\langle x^* - y^*, x\rangle| : x \in A\} \leq \Delta$;

(iv) the dual X^* is *countably weak* dentable* if there exist bounded sets $A_{n,p}$, $n, p \in \mathbb{N}$, in X such that $\bigcup_{n=1}^{\infty} A_{n,p} = X$ for every $p \in \mathbb{N}$ and the dual unit ball B_{X^*} is $(A_{n,p}, \frac{1}{p})$-dentable for every $n, p \in \mathbb{N}$. (Note that in [H1] such an X is called a countably dentable space.)

Let K be a compact space, A a bounded set in $C(K)$, and $\Delta > 0$. We say that

(i) K is (A, Δ)-*fragmentable* if for every nonempty set $M \subset K$ there is an open set $G \subset K$ such that $M \cap G \neq \emptyset$ and

$$\operatorname{diam}_A(M \cap G) := \sup\{f(k_1) - f(k_2) : \ k_1, k_2 \in M \cap G, \ f \in A\} \leq \Delta;$$

(ii) K is (A, Δ)-*separable* if there exists an at most countable set $C \subset K$ such that for every $k \in K$ there is $k' \in C$ such that $\sup\{|f(k) - f(k')| : \ f \in A\} \leq \Delta$;

(iii) K is *countably lower fragmentable* if there exist bounded sets $A_{n,p}$, $n, p \in \mathbb{N}$, in $C(K)$ such that $\bigcup_{n=1}^{\infty} A_{n,p} = C(K)$ and K is $(A_{n,p}, \frac{1}{p})$-fragmentable for every $p \in \mathbb{N}$.

Propositions below relate the above concepts mutually and to the known notions of Radon-Nikodým compacta and Asplund generated spaces.

Proposition 1. *Let K be a compact space, A a bounded subset of $C(K)$ and $\Delta > 0$. Let $\kappa : K \to C(K)^*$ be the canonical mapping sending $k \in K$ to the point mass δ_k. Then:*

(i) *K is (A, Δ)-fragmentable if and only if $(\kappa(K), w^*)$ is (A, Δ)-fragmentable if and only if $\kappa(K)$ is (A, Δ)-dentable;*

(ii) *K is (A, Δ)-separable if and only if $\kappa(K)$ is (A, Δ)-separable;*

(iii) *if $C(K)^*$ is countably weak* dentable, then K is countably lower fragmentable.*

Proof: Everything is trivial but the fact that (A,Δ)-fragmentability of K implies (A,Δ)-dentability of $\kappa(K)$. So assume that K is (A,Δ)-fragmentable and consider $\emptyset \neq \kappa(M) \subset \kappa(K)$. We find an open set $G \subset K$ such that $M \cap G \neq \emptyset$ and $\operatorname{diam}_A(M \cap G) \leq \Delta$. Pick $k \in M \cap G$ and find $f \in C(K)$ such that $f(k) = 1$ and $f(k') = 0$ for all $k' \in K \backslash G$. Then surely $S := S(\kappa(M), f, 1/2) \subset \kappa(M) \cap \kappa(G) = \kappa(M \cap G)$ and so $\operatorname{diam}_A S \leq \operatorname{diam}_A(M \cap G) \leq \Delta$. □

Proposition 2. (i) *If X is an Asplund space, then X^* is countably weak* dentable.*

(ii) *If Z is a subspace of a Banach space X and X^* is countably weak* dentable, then so is Z^*.*

(iii) *If $T : X \to Y$ is a linear continuous mapping, with $\overline{TX} = Y$, and X^* is countably weak* dentable, then so is Y^*.*

(iv) *If Z is a subspace of an Asplund generated space, then Z^* is countably weak* dentable.*

(v) *If Z^* is countably weak* dentable, then Z is a weak Asplund space, that is, that every continuous convex function on Z is Gâteaux differentiable at the points of a dense G_δ set.*

Proof: (i) the sets $A_{n,p} = nB_X$, $n, p \in \mathbb{N}$, fit the job. (ii), (iii), and (v) are proved in [H1]. (iv) follows by putting together (i), (ii) and (iii). □

Proposition 3. *Continuous images of Radon-Nikodým compacta are countably lower fragmentable.*

Proof: Put together Corollary 1, Proposition 2(iv), and Proposition 1(iii). □

We do not know if the converse to (iv) in Proposition 2 holds. Likewise, it is unclear if Proposition 3 can be reversed. Using Corollary 1, Theorem 1 and Theorem 2, we can check that these two questions are equivalent.

The effort below is devoted to proving a converse to the implication (iii) in Proposition 1. Actually, we prove a complete analogue of Corollary 1:

Theorem 1. *For a compact space K the following assertions are equivalent:*

(i) *K is a countably lower fragmentable compact;*

(ii) *the dual $C(K)^*$ is countably weak* dentable;*

(iii) *the dual unit ball $(B_{C(K)^*}, w^*)$ is a countably lower fragmentable compact.*

The proof is a compilation of several lemmas listed (and proved) below.

Lemma 1. *Let K be a compact space, $A \subset C(K)$ a bounded set and $\Delta > 0$. Then K is (A,Δ)-fragmentable if and only if for every at most countable set $A_0 \subset A$ the space K is (A_0,Δ)-separable.*

Proof: Throughout the whole proof we use techniques known from the theory of Asplund spaces, see [Ph]. Assume that K is (A,Δ)-fragmentable and take a countable set $A_0 \subset A$. Let Z be the closed linear span of A_0; so Z is a separable Banach space. For $k \in K$ define $\phi(k)(z) = z(k)$, $z \in Z$. Then $\phi(k) \in Z^*$ and the mapping $\phi: K \to (Z^*, w^*)$ is continuous. Hence $(\phi(K), w^*)$ is a compact space.

We claim that $\phi(K)$ *is* (A_0, Δ)-*fragmentable.* Take $\emptyset \neq M \subset \phi(K)$. Using Zorn's lemma, we find a closed set $N \subset K$, minimal with respect to the inclusion, such that $\phi(N)$ is equal to the weak* closure $\overline{M}^{\,*}$ of M. Since K is (A_0, Δ)-fragmentable, there is an open set $U \subset K$ such that $N \cap U \neq \emptyset$ and $\operatorname{diam}_{A_0}(N \cap U) \leq \Delta$. The set $N \backslash U$ is closed so $\phi(K) \backslash \phi(N \backslash U)$ is an open set in $(\phi(K), w^*)$. We find a weak* open set $G \subset Z^*$ such that $\phi(K) \backslash \phi(N \backslash U) = G \cap \phi(K)$. Also $\overline{M}^{\,*} \cap G = \overline{M}^{\,*} \backslash \phi(N \backslash U)$ and this set is nonempty because of the minimality of N and since $N \cap U \neq \emptyset$. Further we observe that $M \cap G \subset \phi(N \cap U)$. Therefore

$$\begin{aligned}
\operatorname{diam}_{A_0}(M \cap G) &= \sup\{\langle z_1^* - z_2^*, z\rangle :\ z_i^* \in M \cap G,\ z \in A_0\} \\
&\leq \sup\{\langle z_1^* - z_2^*, z\rangle :\ z_i^* \in \phi(N \cap U),\ z \in A_0\} \\
&= \sup\{\langle \phi(k_1) - \phi(k_2), z\rangle :\ k_i \in N \cap U,\ z \in A_0\} \\
&= \sup\{z(k_1) - z(k_2) :\ k_i \in N \cap U,\ z \in A_0\} \\
&= \operatorname{diam}_{A_0}(N \cap U) \leq \Delta.
\end{aligned}$$

The claim is thus proved.

Now assume that K is not (A_0, Δ)-separable. Then there exists an uncountable set $C \subset K$ such that

$$\sup\{|f(k) - f(k')| :\ f \in A_0\} > \Delta \quad \text{whenever} \quad k, k' \in C \quad \text{and} \quad k \neq k'.$$

Then

$$\sup\{|\langle \phi(k) - \phi(k'), f\rangle| :\ f \in A_0\} > \Delta \quad \text{whenever} \quad k, k' \in C, \quad \text{and} \quad k \neq k'.$$

Since Z is a separable Banach space, there is a sequence $\{z_n :\ n \in \mathbb{N}\}$ contained in and dense in the unit ball of Z. Define a mapping $\psi :\ (Z^*, w^*) \to \mathbb{R}^{\mathbb{N}}$ by the formula $\psi(z^*)(n) = \langle z^*, z_n\rangle$, $z^* \in Z^*$, $n \in \mathbb{N}$; it is injective and continuous. It then follows that $\psi(\phi(K))$ is a metrizable compact. Hence $(\phi(K), w^*)$ is a metrizable compact and $L := (\phi(C), w^*)$ is a metrizable separable space. Thus, the topology on L has a countable basis, say $\mathcal{B}$. Put $\mathcal{B}_0 = \{U \in \mathcal{B} :\ L \cap U$ is at most countable$\}$. Then the set $L \cap \left(\bigcup \mathcal{B}_0\right)$ is at most countable and hence the set $\widetilde{L} = L \backslash \left(\bigcup \mathcal{B}_0\right)$ is nonempty. We shall check that $\widetilde{L}$ has no isolated point. So take any $z^* \in \widetilde{L}$. If $z^* \in U \in \mathcal{B}$, then $U \notin \mathcal{B}_0$ and hence the set $L \cap U$ is uncountable and so is the set $\widetilde{L} \cap U$. Therefore z^* is not an isolated point of $\widetilde{L}$.

Now we apply the claim to the (nonempty) set $M := \widetilde{L} \subset \phi(K)$. We get a weak* open set $G \subset Z^*$ such that $\widetilde{L} \cap G \neq \emptyset$ and $\operatorname{diam}_{A_0}(\widetilde{L} \cap G) \leq \Delta$. According to the property of the set C, we then conclude that $\widetilde{L} \cap G$ is a singleton. This means that $\widetilde{L} \cap G$ consists of an isolated point of $\widetilde{L}$. However $\widetilde{L}$ does not have isolated points. This is a contradiction and therefore K is (A_0, Δ)-separable. Second, assume K is not (A, Δ)-fragmentable. For $s \in \{0,1\} \cup \{0,1\}^2 \cup \dots$ we shall construct nonempty open sets $G_s \subset K$, functions $f_s \in A$, and numbers $\Delta_s > \Delta$ as follows: Put $G_0 = G_1 = K$. Assume that for some $s = (s_1, \dots, s_n)$ we already have nonempty open sets $G_s \subset K$ with $\operatorname{diam}_A G_s > \Delta$. We find $f_s \in A$ and $k_0,\ k_1 \in G_s$ such that $f_s(k_1) - f_s(k_0) > \Delta$. Then we find Δ_s satisfying $f_s(k_1) - f_s(k_0) > \Delta_s > \Delta$. Take $a > 0$ so that $f_s(k_1) > a > f_s(k_0) + \Delta_s$. Then put

$$\widetilde{G}_{s,0} = \{k \in G_s :\ f_s(k) < a - \Delta_s\}, \quad \widetilde{G}_{s,1} = \{k \in G_s :\ f_s(k) > a\}.$$

Thus $k_j \in \widetilde{G}_{s,j}$, $j = 0, 1$. Find open sets $G_{s,j}$ such that $k_j \in G_{s,j} \subset \overline{G_{s,j}} \subset \widetilde{G}_{s,j}$, $j = 0, 1$. This finishes the induction step. Now put $A_0 = \{f_s :\ s \in \{0,1\} \cup \{0,1\}^2 \cup \dots\}$; this is a countable subset of A. For every $\sigma = (s_1, s_2, \dots) \in \{0,1\}^{\mathbb{N}}$ choose $k_\sigma \in \bigcap_{n \in \mathbb{N}} G_{s_1, s_2, \dots, s_n}$; such a k_σ exists. Observe that

$$\sup\{|f(k_\sigma) - f(k_{\sigma'})| :\ f \in A_0\} > \Delta \quad \text{whenever} \quad \sigma,\ \sigma' \in \{0,1\}^{\mathbb{N}}, \quad \text{and} \quad \sigma \neq \sigma'.$$

Therefore the space K is not (A_0, Δ)-separable. $\square$

Lemma 2. *Let K be a compact space, $A \subset C(K)$ a bounded countable set and $\Delta > 0$. If K is (A, Δ)-separable, then the unit ball $B_{C(K)^*}$ in $C(K)^*$ is $(A, 2\Delta)$-separable.*

Proof: We shall immitate an argument due to W.B. Moors, see the proof of [F, Lemma 1.5.3]. Let $\{k_n :\ n \in \mathbb{N}\}$ be a sequence which is (A, Δ)-dense in K, i.e. for every $k \in K$ there exists $n \in \mathbb{N}$ such that $\sup\{|f(k) - f(k_n)| :\ f \in A\} \le \Delta$. Denote

$$H = \{\nu \in C(K)^* :\ \sup |\langle \nu, A \rangle| \le \Delta\},$$

and

$$K_n = \big\{k \in K :\ \delta_k \in \{\delta_{k_1}, \dots, \delta_{k_n}\} + H\big\}, \quad n \in \mathbb{N}.$$

(Here $\delta_k \in C(K)^*$ means the point mass measure at $k \in K$.) Clearly H is convex symmetric and weak* closed, K_n is closed, and $K = \bigcup_{n=1}^{\infty} K_n$. Take any $\mu \in B_{C(K)^*}$. (We use here and below F. Riesz' representation theorem.) We find $n \in \mathbb{N}$ so that $|\mu|(K \backslash K_n) < \Delta/(2c)$ where $c = \sup\{\|f\| :\ f \in A\}$. ($|\mu|$ means the total variation of μ.) Define

$$\nu(M) = \mu(M \cap K_n), \quad M \subset K \quad \text{Borel}.$$

Then $\nu \in B_{C(K)^*}$ and $\|\mu - \nu\| \le |\mu|(K \backslash K_n) < \Delta/(2c)$; so $\mu \in \nu + \frac{1}{2}H$. Now we observe, using the separation theorem, that ν belongs to the weak* closed convex hull of $\{\pm\delta_k :\ k \in K_n\}$. Thus, denoting $S = \operatorname{co}\{\pm\delta_{k_1}, \pm\delta_{k_2}, \dots\}$, we have

$$\begin{aligned} \mu \in \nu + \tfrac{1}{2}H &\subset \overline{\operatorname{co}}^*\{\pm\delta_k :\ k \in K_n\} + \tfrac{1}{2}H \\ &\subset \operatorname{co}\{\pm\delta_{k_1}, \dots, \pm\delta_{k_n}\} + H + \tfrac{1}{2}H \subset S + \tfrac{3}{2}H. \end{aligned}$$

Therefore $B_{C(K)^*} \subset S + \frac{3}{2}H$. Now we observe that S is a norm separable set, hence S is also $(A, \Delta/2)$-separable. Thus $B_{C(K)^*}$ is $(A, 2\Delta)$-separable. $\square$

Lemma 3. *Let X be a Banach space, A a bounded set in X, $\Delta > 0$ and assume that B_{X^*} is (A,Δ)-fragmentable. Then B_{X^*} is $(A,2\Delta)$-dentable.*

Proof: We just copy the proof of (iv) $\Rightarrow$ (i) in [NP, Lemma 3]. $\square$

Lemma 4. *Let L be a compact space, X a linear subset of $C(L)$ which separates the points of L, and $\Delta > 0$. Suppose that there exist bounded sets $A_n \subset X$, $n \in \mathbb{N}$, such that $\bigcup_{n=1}^{\infty} A_n = X$ and L is (A_n,Δ)-fragmentable for each $n \in \mathbb{N}$. Then there exist bounded sets $F_1 \subset F_2 \subset \cdots \subset C(L)$ such that $\bigcup_{n=1}^{\infty} F_n = C(L)$, and L is $(F_n, 2\Delta)$-fragmentable for each $n \in \mathbb{N}$.*

Proof: Denote by 1 the function on L which is identically equal to one. We observe that L is (A,Δ)-fragmentable where A is the convex symmetric hull of the set $A_n \cup \{n \cdot 1\}$. Therefore, by considering the linear span of $\{1\} \cup X$ instead of X and the convex symmetric hull of $A_n \cup \{n \cdot 1\}$ instead of A_n for each $n \in \mathbb{N}$, we may, and do assume that X contains the constants and the sets A_n are bounded, convex, and symmetric.

Put $E_1 = A_1$. If E_n was already defined for some $n \in \mathbb{N}$, let E_{n+1} be the convex symmetric hull of the set

$$\{f_1 \vee f_2 :\ f_1, f_2 \in A_{n+1} \cup E_n\},$$

where $f \vee g$ denotes the pointwise maximum of the functions f and g. Clearly L is (E_1,Δ)-fragmentable. Suppose L is (E_n,Δ)-fragmentable for some $n \in \mathbb{N}$. Then L is also $(A_{n+1} \cup E_n, \Delta)$-fragmentable. Take $f, g \in E_n \cup A_{n+1}$ and $l_1, l_2 \in L$. Then

$$(f \vee g)(l_1) - (f \vee g)(l_2) \leq \max\{f(l_1) - f(l_2)\ ,\ g(l_1) - g(l_2)\} \leq \operatorname{diam}_{E_n \cup A_{n+1}}\{l_1, l_2\}.$$

Hence L is also (B,Δ)-fragmentable where $B = \{f \vee g :\ f, g \in E_n \cup A_{n+1}\}$, and finally, L is (E_{n+1},Δ)-fragmentable.

Clearly, $E_1 \subset E_2 \subset \ldots$, and the set $Y = \bigcup_{n=1}^{\infty} E_n$ is closed under the operation of taking pointwise maximum and minimum of two functions, separates the points of L, and contains the constant functions. We shall show that Y is a linear subset of $C(L)$. Then, by the Stone-Weierstrass theorem (see the proof of [DS, Theorem IV.6.16]), Y is dense in $C(L)$. Therefore the sets

$$F_n = E_n + \tfrac{\Delta}{2} B_{C(L)}, \quad n \in \mathbb{N},$$

have the required properties.

Since each of the sets $E_1 \subset E_2 \subset \ldots$ is convex and symmetric, in order to show that Y is linear it is enough to find for each $n \in \mathbb{N}$, each $f \in E_n$, and each $c > 0$ some $i \in \mathbb{N}$ so that $cf \in E_i$. We shall show this by induction on n. If $f \in E_1$ $(= A_1)$ and $c > 0$ then $cf \in X$ and hence there exists $i \in \mathbb{N}$ so that $cf \in A_i \subset E_i$. Suppose that the statement holds for some $n \in \mathbb{N}$, and let $f \in E_{n+1}$

and $c > 0$ be given. Then there exist $m \in \mathbb{N}$, $f_1, f_2, \ldots, f_{2m} \in A_{n+1} \cup E_n$, and $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ such that $\sum_{j=1}^{m} |\lambda_j| \leq 1$ and

$$f = \sum_{j=1}^{m} \lambda_j (f_{2j-1} \vee f_{2j}).$$

For each $j \in \{1, \ldots, 2m\}$ we find $i_j \in \mathbb{N}$ so that $cf_j \in E_{i_j}$, and put $i = 1 + \max\{i_1, i_2, \ldots, i_{2m}\}$. Then

$$cf = \sum_{j=1}^{m} \lambda_j ((cf_{2j-1}) \vee (cf_{2j})) \in E_i.$$

□

Proof of Theorem 1: : (i)⇒(ii). Assume that K is a countably lower fragmentable compact. Let $A_{n,p}$ be the sets from the definition of countable lower fragmentability. Fix any $n, p \in \mathbb{N}$. We know that K is $(A_{n,p}, \frac{1}{p})$-fragmentable. By Lemma 1, for every countable set $A_0 \subset A_{n,p}$, the space K is $(A_0, \frac{1}{p})$-separable. Then, by Lemma 2, $B_{C(K)^*}$ is $(A_0, \frac{2}{p})$-separable for every such A_0. Again, by Lemma 1, $B_{C(K)^*}$ is $(A_{n,p}, \frac{2}{p})$-fragmentable. (Here, in order to be able to use Lemma 1, we consider A_0 and $A_{n,p}$ as if they were subsets of $C((B_{C(K)^*}, w^*))$ — this can be ensured by a canonical embedding.) Finally, by Lemma 3, $B_{C(K)^*}$ is $(A_{n,p}, \frac{4}{p})$-dentable. This means nothing else than that (ii) holds.

(ii)⇒(iii). Let K satisfy (ii) in our theorem. We observe that the Banach space $C(K)$ embeds isometrically as a closed subspace into $C((B_{C(K)^*}, w^*))$ and this subspace separates the points of $B_{C(K)^*}$. By (ii), there exist sets $A_{n,p} \subset C(K)$, $n, p \in \mathbb{N}$, so that $\bigcup_{n=1}^{\infty} A_{n,p} = C(K)$ for each $p \in \mathbb{N}$ and that $B_{C(K)^*}$ is $(A_{n,p}, \frac{1}{p})$-fragmentable for each $n, p \in \mathbb{N}$. Fix any $p \in \mathbb{N}$ and apply Lemma 4 for $L := (B_{C(K)^*}, w^*)$, $X := C(K) \ \big(\subset C((B_{C(K)^*}, w^*))\big)$, and $A_n := A_{n,p} \ \big(\subset C((B_{C(K)^*}, w^*))\big)$, $n \in \mathbb{N}$. In this way, we get bounded sets $F_{n,p} \subset C((B_{C(K)^*}, w^*))$, $n, p \in \mathbb{N}$, so that $\bigcup_{n=1}^{\infty} F_{n,p} = C((B_{C(K)^*}, w^*))$ for each $p \in \mathbb{N}$, and the compact space $(B_{C(K)^*}, w^*)$ is $(F_{n,p}, \frac{2}{p})$-fragmentable for each $n, p \in \mathbb{N}$.

(iii)⇒(i). Let $A_{n,p} \subset C((B_{C(K)^*}, w^*))$, $n, p \in \mathbb{N}$, do the job in (iii). Then the sets $A'_{n,p} = \{f_{|K} : \ f \in A_{n,p}\}$ do the job for (i). (We assumed that K is a subspace of $(B_{C(K)^*}, w^*)$.) □

Theorem 2. *For a Banach space X the following assertions are equivalent:*

(i) *the dual space X^* is countably weak* dentable;*
(ii) *the compact space (B_{X^*}, w^*) is countably lower fragmentable;*
(iii) *the dual $C((B_{X^*}, w^*))^*$ is countably weak* dentable.*

PROOF: (i)⇒(ii) follows from Lemma 4. (ii)⇒(iii) follows from (i)⇒(ii) in Theorem 1. (iii)⇒(i) follows from Proposition 2(ii). □

Countably lower fragmentable compacta which are totally disconnected

A topological space is called *totally disconnected* if its topology has a basis consisting from sets which are both open and closed. Thus, a compact space is totally disconnected if and only if it is homeomorphic to a closed subspace of the product $\{0,1\}^\Gamma$ for some set Γ. For $A \subset \Gamma$ we define $\chi_A : \Gamma \to \{0,1\}$ as $\chi_A(\gamma) = 1$ if $\gamma \in A$ and $\chi_A(\gamma) = 0$ if $\gamma \in \Gamma \backslash A$. If $k \in \{0,1\}^\Gamma$, we put $\operatorname{supp} k = \{\gamma \in \Gamma : \ k(\gamma) = 1\}$. A compact space K is called *adequate* if it is a closed subspace of $\{0,1\}^\Gamma$ for some set Γ,

(i) $\chi_{\{\gamma\}} \in K$ for every $\gamma \in \Gamma$, and
(ii) if $B \subset A \subset \Gamma$ and $\chi_A \in K$, then $\chi_B \in K$.

Theorem 3. *For a totally disconnected compact space K the following assertions are equivalent:*

(i) *K is a Radon-Nikodým compact (i.e., $C(K)$ is Asplund generated);*
(ii) *K is a continuous image of a Radon-Nikodým compact (i.e., $C(K)$ is a subspace of an Asplund generated space);*
(iii) *K is a countably lower fragmentable compact (i.e., $C(K)^*$ is countably weak* dentable);*
(iv) *if φ is a homeomorphism of K into $\{0,1\}^\Gamma$, then there exist sets $\Gamma_n \subset \Gamma$, $n \in \mathbb{N}$, with $\bigcup_{n=1}^\infty \Gamma_n = \Gamma$, such that for every $n \in \mathbb{N}$ and every $\emptyset \neq M \subset K$ there are an open set $G \subset M$ and $\Gamma' \subset \Gamma_n$ such that $\operatorname{supp} \varphi(k) \cap \Gamma_n = \Gamma'$ whenever $k \in M \cap G$.*

If K is an adequate compact, then the above conditions are equivalent with:

(v) *K is an Eberlein compact (i.e., $C(K)$ is weakly compactly generated).*

PROOF: (i)⇒(ii) is trivial. (ii)⇒(iii) is Proposition 3.

(iii)⇒(iv) Let $A_{n,p}$, $n,p \in \mathbb{N}$, be the sets guaranteeing the countable lower fragmentability of K and let $\varphi : K \to \{0,1\}^\Gamma$ be a continuous injection. For $\gamma \in \Gamma$ put $\pi_\gamma(k) = \varphi(k)(\gamma)$, $k \in K$; thus $\pi_\gamma \in C(K)$. Define

$$\Gamma_n = \{\gamma \in \Gamma : \ \pi_\gamma \in A_{n,2}\}, \quad n \in \mathbb{N}.$$

Then $\bigcup_{n=1}^\infty \Gamma_n = \Gamma$. Now take $\emptyset \neq M \subset K$ and fix $n \in \mathbb{N}$. We find an open set $G \subset K$ such that $M \cap G \neq \emptyset$ and $\operatorname{diam}_{A_{n,2}}(M \cap G) \leq \frac{1}{2}$. Take $k, k' \in M \cap G$. Then for every $\gamma \in \Gamma_n$ we have $\pi_\gamma \in A_{n,2}$ and so

$$|\pi_\gamma(k) - \pi_\gamma(k')| \leq \frac{1}{2}.$$

This means that $\operatorname{supp}\varphi(k)\cap\Gamma_n=\operatorname{supp}\varphi(k')\cap\Gamma_n$ and so (iv) is satisfied.

(iv)⇒(i) Let $\varphi: K\to\{0,1\}^\Gamma$ be a continuous injection and let $\Gamma_n,\ n\in\mathbb{N}$, be found in (iv). Put

$$A_n=\{\pi_\gamma:\ \gamma\in\Gamma_n\},\quad n\in\mathbb{N},$$

and define

$$\rho(k,k')=\sup\Big\{|f(k)-f(k')|:\ f\in A_1\cup\tfrac12A_2\cup\tfrac13A_3\cup\cdots\Big\},\quad k,k'\in K.$$

Clearly, ρ is a lower semicontinuous metric on K. We shall show that ρ fragments K, that is, that every nonempty subset of K contains a nonempty relatively open subset whose ρ-diameter is less than an a priori given arbitrary positive number. Then, by [N2], we can conclude that K is a Radon-Nikodým compact. So let $\emptyset\neq M\subset K$ and $\epsilon>0$ be given. Take $n\in\mathbb{N}$ so that $1/n<\epsilon$. By (iv), there is an open set $G_1\subset K$, with $M\cap G_1\neq\emptyset$, and such that the set $\operatorname{supp}\varphi(k)\cap\Gamma_1$ does not depend upon $k\in M\cap G_1$. Then, again by (iv), there is an open set $G_2\subset K$, with $(M\cap G_1)\cap G_2\neq\emptyset$, and such that the set $\operatorname{supp}\varphi(k)\cap\Gamma_2$ does not depend upon $k\in(M\cap G_1)\cap G_2$. Continuing in this way, we finally find an open set $G_n\subset K$, with $M\cap G_1\cap\cdots\cap G_n\neq\emptyset$, and such that $\operatorname{supp}\varphi(k)\cap\Gamma_n$ does not depend upon $k\in M\cap G_1\cap\cdots\cap G_n$. Put $G=G_1\cap\cdots\cap G_n$. Then $M\cap G\neq\emptyset$ and the set $\operatorname{supp}\varphi(k)\cap(\Gamma_1\cup\cdots\cup\Gamma_n)$ does not depend upon $k\in M\cap G$. Thus, for $k,k'\in M\cap G$ we have

$$\rho(k,k')=\sup\Big\{|f(k)-f(k')|:\ f\in\tfrac{1}{n+1}A_{n+1}\cup\tfrac{1}{n+2}A_{n+2}\cup\cdots\Big\}\le\frac{1}{n+1}<\epsilon.$$

This means that K is fragmented by ρ and (i) is proved.

Finally, assume that K is an adequate compact. We find a set Γ so that K is a subspace of $\{0,1\}^\Gamma$. Assume that K satisfies (iv). Let Γ_n, $n\in\mathbb{N}$, be the sets from (iv). By replacing the set Γ_2 by $\Gamma_2\backslash\Gamma_1$, the set Γ_3 by $\Gamma_3\backslash(\Gamma_1\cup\Gamma_2)$ and so on, we may, and do assume that $\Gamma_i\cap\Gamma_j=\emptyset$ whenever $i\neq j$. We claim that for every $k\in K$ and every $n\in\mathbb{N}$ the set $\operatorname{supp}k\cap\Gamma_n$ is finite. Then the assignment $k\mapsto\{\frac1n k(\gamma):\ \text{if}\ \ \gamma\in\Gamma_n,\ n\in\mathbb{N}\}$ sends K into $(c_0(\Gamma),w)$ continuously and injectively and hence K is an Eberlein compact.

Assume that the claim is false. Then there exist $n\in\mathbb{N}$ and $k\in K$ such that $\operatorname{supp}k\cap\Gamma_n$ is an infinite set. Let $A\subset\Gamma$ be such that $k=\chi_A$; then $A\cap\Gamma_n$ is an infinite set. Denote

$$M=\{\chi_B:\ B\subset A\cap\Gamma_n\}.$$

Since K is adequate compact, M is a (nonempty) subset of K. By (iv), there exists an open set $G\subset K$ such that $M\cap G\neq\emptyset$ and $(B=)\ B\cap\Gamma_n=\text{const.}$ for every $\chi_B\in M\cap G$. Hence $M\cap G$ is a singleton. However, this contradicts to the definition of the set M and to the definition of the topology on $\{0,1\}^\Gamma$. □

In [T], Talagrand constructed an adequate compact K which is not Eberlein. In [OSV], it is shown that this K is not a Radon-Nikodým compact. Stegall showed in [St2] that this K is not a continuous image of a Radon-Nikodým compact, see also [F, Theorem 8.3.6]. From Theorem 3 we get the same fact in an easier way. Hence $C(K)$ is not a subspace of an Asplund generated space.

References

[A1] Argyros S., *Weakly Lindelöf determined Banach spaces not containing* ℓ_1, preprint.

[A2] Argyros S., *Private communication.*

[BRW] Benyamini Y., Rudin M.E., Wage M., *Continuous images of weakly compact subsets of Banach spaces*, Pacific J. Math. **70** (1977), 309–324.

[DS] Dunford N., Schwartz J.T., *Linear Operators, Part I*, Interscience Publ., New York, 1958.

[F] Fabian M., *Gâteaux Differentiability of Convex Functions and Topology – Weak Asplund Spaces*, John Wiley & Sons, Interscience, New York, 1997.

[Gr] Grothendieck A., *Produits tensoriels et espaces nucleaires*, Memoirs Amer. Math. Soc., No. **16**, 1955.

[H1] Heisler M., *Singlevaluedness of monotone operators on subspaces of GSG spaces*, Comment. Math. Univ. Carolinae **37** (1996), 255–261.

[H2] Heisler M., *Some aspects of differentiability and geometry on Banach spaces*, PhD. Thesis, Prague, 1996.

[N1] Namioka I., *Eberlein and Radon-Nikodým compact spaces*, Lecture Notes at University College, London, 1985.

[N2] Namioka I., *Radon-Nikodým compact spaces and fragmentability*, Mathematika **34** (1987), 258–281.

[NP] Namioka I., Phelps R.R., *Banach spaces which are Asplund spaces*, Duke Math. J. **42** (1975), 735–750.

[OSV] Orihuela J., Schachermayer W., Valdivia M., *Every Radon-Nikodým Corson compact is Eberlein compact*, Studia Math. **98** (1991), 157–174.

[Ph] Phelps R.R., *Convex functions, monotone operators and differentiability*, Lect. Notes Math., No. 1364, 2nd Edition, Springer Verlag, Berlin, 1993.

[Ro] Rosenthal H., *The heredity problem for weakly compactly generated Banach spaces*, Comp. Math. **28** (1974), 83–111.

[St1] Stegall Ch., *The Radon-Nikodým property in conjugate Banach spaces II*, Trans. Amer. Math. Soc. **264** (1981), 507–519.

[St2] Stegall Ch., *More facts about conjugate Banach spaces with the Radon-Nikodým property II*, Acta Univ. Carolinae – Math. et Phys. **32** (1991), 47–54.

[T] Talagrand M., *Espaces de Banach faiblement K-analytiques*, Annals of Math. **110** (1978), 407–438.

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