The Banach-Saks property and Haar null sets

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Abstract. A characterization of Haar null sets in the sense of Christensen is given. Using it, we show that if the dual of a Banach space X has the Banach-Saks property, then closed and convex subsets of X with empty interior are Haar null.

Keywords: Banach space, convexity, Haar null set

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Introduction

A Borel subset A of a separable Banach space X is said to be Haar null if there is a Borel probability measure μ on X so that $\mu(A+x)=0$ for each $x \in X$ [C]. Haar null sets provide a substitute for Lebesgue null sets in "almost everywhere" type theorems in infinite dimensional Banach spaces ([C], [H], [HSY], [My]). As in the case of Lebesgue null sets, the Haar null sets are closed under countable unions [C], and isomorphisms (see e.g. Lemma 2.4). Unlike the Lebesgue null sets, in every infinite dimensional separable Banach space there exist uncountably many pairwise disjoint sets none of which is Haar null ([S]).

In Section 1 we give a characterization of Haar null sets. We show that a Borel subset A of a separable Banach space is not Haar null if and only if there exist $\delta > 0$ and $r > 0$ so that for every probability measure μ with spt $\mu \subset B(0,r)$ there exists $x \in X$ so that $\mu(A+x) \geq \delta$. We show that for a weakly compact set, δ can be (by shrinking r) chosen arbitrarily in $(0,1)$; also, for the "right-left" implication it is enough to check only those μ's which are arithmetic means of Dirac measures.

It was shown in [MS], that every separable nonreflexive Banach space X contains a closed convex set C with empty interior which is not Haar null; more specifically C contains a translate of each compact subset of X. On the other hand, in [M] it was shown that each closed, convex and nowhere dense set in a superreflexive Banach space is Haar null. In Section 2, we strengthen this result by showing that if the dual of a Banach space X has the Banach-Saks property then each closed, convex and nowhere dense subset of X is Haar null. We do not know whether or not there is a reflexive Banach space without this property;

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however, as we observe at the end of this paper, each weakly compact subset of a nonreflexive Banach space is Haar null.

We consider only real Banach spaces. In the following $B_X(x,r)$ denotes the open ball with center x and diameter r in a Banach space X. The support of a measure μ is denoted by spt μ; $\boldsymbol{\delta}_x$ is the Dirac measure supported at the point x.

For notation and concepts not introduced in the text we refer to the books [P] and [BL].

1. A characterization of Haar null sets

Let X be a separable Banach space (with the norm topology), $\mathcal{M}$ the set of Borel probability measures on X. The collection of sets

$$V_\mu(f,\varepsilon) = \{\nu \in \mathcal{M} : |\int f\,d\nu - \int f\,d\mu| < \varepsilon\},$$

where $\mu \in \mathcal{M}$, $\varepsilon > 0$, and f is a bounded real valued continuous function on X forms a subbasis of the so-called weak topology on $\mathcal{M}$. We will use the fact that $\mathcal{M}$ endowed with the weak topology is metrizable by a complete metric (see *e.g.* [P, p. 44]); we denote by d a complete metric on $\mathcal{M}$ generating the weak topology. Also, $\mathcal{M}$ with convolution (denoted later in the text by $*$) is an abelian topological semigroup; the Dirac measure $\boldsymbol{\delta}_o$ at the origin is its neutral element ([P]).

The proof of the following theorem is a modification of Christensen's proof ([C, p. 116]) of the fact that a countable union of Haar null sets is Haar null. The corresponding statement for Haar null sets in abelian Polish groups is also valid, and the proof is the same.

Theorem 1.1. *Let X be a separable Banach space, $A \subset X$ Borel measurable. The set A is* not *Haar null if and only if there exist $\delta > 0$ and $r > 0$ so that for every probability measure μ with* spt $\mu \subset B(0,r)$ *there exists $x \in X$ so that $\mu(A+x) \ge \delta$.*

Proof: Let A be a Borel subset of X, and suppose that for each $n \in \mathbb{N}$ there exists a probability measure μ_n with spt $\mu_n \subset B(0,\frac{1}{n})$ so that $\mu_n(A+x) < \frac{1}{n}$ for all $x \in X$. We will show that A is Haar null.

Observe that the sequence (μ_n) converges weakly to $\boldsymbol{\delta}_o$. By induction choose a subsequence ν_n of μ_n satisfying $d(\alpha, \alpha * \nu_n) < 1/2^n$ for each α which is a convolution of some ν_i's with different indices $i = 1, \dots, n-1$. Since $(\mathcal{M}, d)$ is a complete metric space, we can define

$$\nu = \lim_{k\to\infty} \nu_1 * \nu_2 * \dots * \nu_k \quad \text{and} \quad \beta_n = \lim_{k\to\infty} \nu_{n+1} * \nu_{n+2} * \dots * \nu_{n+k},$$

for $n \in \mathbb{N}$. Put also $\alpha_n = \nu_1 * \dots * \nu_{n-1}$, and $\gamma_n = \alpha_n * \beta_n$, for $n = 2, 3, \dots$. Then for any $z \in X$ and $n \ge 2$

$$\begin{aligned}\nu(A+z) &= (\alpha_n * \nu_n * \beta_n)(A+z) = (\nu_n * \gamma_n)(A+z)\\ &= \int \nu_n(A+z-x)\,d\gamma_n(x) \le \sup_{x\in X} \nu_n(A+z-x) \le 1/n.\end{aligned}$$

Therefore, $\nu(A+z)=0$ for each $z \in X$, and A is Haar null.

The other implication follows from the inner regularity of each $\mu \in \mathcal{M}$. There exists a compact set $K \subset X$ so that $\mu(K)>0$. By covering of K by finitely many open sets of diameter less than r, we can find also some $C \subset K$ with $\operatorname{diam} C<r$ for which $\mu(C)>0$. If we define $\mu'(F)=\mu(F \cap C)/\mu(C)$ for any Borel subset F of X, then there exists $x \in X$ so that $\mu'(A+x) \geq \delta$. This means that $\mu(A+x)>0$, and the set A is not Haar null. □

Easy examples on the real line show that in general one cannot achieve $\delta=1$ in the previous theorem. However, if we restrict ourselves to weakly compact sets, we can achieve δ arbitrarily close to 1 by shrinking r to zero. To prove this, let us state an auxiliary lemma first. If F is a finite set, we denote by $|F|$ the cardinality of F.

Lemma 1.2. *Let X be a Banach space, $A \subset X$ weakly compact, $F \subset X$ finite, and $n \in \mathbb{N}$ so that $|F \setminus (A+x)| \geq n$ for each $x \in X$. Then*

$$\alpha_F=\inf_{x \in X} \max\{\operatorname{dist}(E, A+x): \ E \subset F \setminus (A+x) \ \text{ and } \ |E|=n\}>0.$$

Proof: Suppose $\alpha_F=0$. Since F is finite, there exist $E \subset F$ with $|E|=n$, $z \in E$ and $x_n \in X$ so that

$$\{z\} \cup (F \setminus E) \subset (A+x_n)+B(0, \frac{1}{n}).$$

We claim that there exists $x \in X$ so that $\{z\} \cup (F \setminus E) \subset A+x$; this is a contradiction, since $F \setminus (A+x) \subset E \setminus \{z\}$ implies $|F \setminus (A+x)| \leq n-1$.

To prove the claim, choose for each $y \in \{z\} \cup (F \setminus E)$ a point $a_n^y \in A$ so that $\|y-(a_n^y+x_n)\|<1/n$. Since A is weakly compact and F finite, we can suppose that for each $y \in \{z\} \cup (F \setminus E)$, the sequence (a_n^y) converges weakly to some $a^y \in A$. Consequently, the sequence (x_n) converges weakly to $y-a^y$ which is necessarily the same point for all $y \in \{z\} \cup (F \setminus E)$; denote it by x. Clearly, $y \in A+x$ whenever $y \in \{z\} \cup (F \setminus E)$. □

Recall that if X is a separable metric space and D is a dense subset of X then the set of probability measures whose supports are finite subsets of D is dense in the set $\mathcal{M}$ of all probability measures on X (c.f. [P, p. 44]). In our setting, X will be a separable Banach space, and since it contains no isolated points, the set of all measures of the form $\frac{1}{|F|}\sum_{x \in F} \boldsymbol{\delta}_x$, where $F \subset D$ is finite, is dense in $\mathcal{M}$. This enables to test Haar "nullness" of weakly compact sets only by this special class of measures (see (ii) and (iii) of the next theorem). Let us remark, that in each separable Banach space X there is a closed and bounded Haar null set satisfying (iii) of the next theorem. To see this, take a countable, dense subset D of the unit ball of X; let F_n, $n \in \mathbb{N}$ be all finite subsets of D. Take a countable, bounded set $C=(x_n)$ such that the distance of any two different points in it is at least 3. Put $A = \bigcup (F_n + x_n)$. Then A clearly satisfies (iii), but A is also Haar null since it is countable. The positive cone of ℓ_2 provides an example of a weakly closed Haar null set satisfying (iii), (see e.g. [MS]). In connection with Theorem 3 of [MS] it would be interesting to know, whether in the case of A convex one can achieve $\delta = 1$ in the next theorem. Notice also, that according to the remark at the end of this paper, if (i) of the next theorem is valid then the Banach space X is necessarily reflexive.

Theorem 1.3. *Let X be a separable Banach space, D be a dense subset of X, $A \subset X$ be weakly compact. Then the following are equivalent:*

(i) *the set A is* not *Haar null;*
(ii) *there exist $0 < \delta < 1$ and $0 < r$ such that for each finite $F \subset D \cap B(0, r)$ there exists $x \in X$ so that $|F \cap (A + x)| \geq \delta |F|$;*
(iii) *for each $0 < \delta < 1$ there exists $0 < r$ such that for each finite $F \subset D \cap B(0, r)$ there exists $x \in X$ so that $|F \cap (A + x)| \geq \delta |F|$;*
(iv) *for each $0 < \delta < 1$ there exists $0 < r$ such that for each probability measure μ on X with* spt *$\mu \subset B(0, r)$ there exists $x \in X$ so that $\mu(A + x) \geq \delta$.*

Proof: The implications (iv)$\Rightarrow$(iii)$\Rightarrow$(ii) are obvious. The implications (i)$\Rightarrow$(ii) and (iv)$\Rightarrow$(i) are contained in Theorem 1.1. To prove the rest, let us show first, that if $0 < \delta < 1$ and $0 < r$ are such that for each finite $F \subset D \cap B(0, r)$ there exists $x \in X$ so that $|F \cap (A + x)| \geq \delta |F|$, then for each probability measure μ on X with spt $\mu \subset B(0, r)$ there exists $x \in X$ so that $\mu(A + x) \geq \delta$. This, of course, covers the implication (iii)$\Rightarrow$(iv). Let $\mu \in \mathcal{M}$ with spt $\mu \subset B(0, r)$ be given. Choose finite sets $F_n \subset B(0, r)$ so that the sequence of measures $\mu_n = \frac{1}{|F_n|} \sum_{x \in F_n} \boldsymbol{\delta}_x$ converges in the weak topology to μ. Since the set $\{\mu_n\}_{n \in \mathbb{N}}$ is relatively compact, there exists $K \subset X$ compact so that $\mu_n(K) > 1 - \delta$ for each $n \in \mathbb{N}$ (c.f. [P, p. 47]). Choose $x_n \in X$ so that $\mu_n(A + x_n) > \delta$. Hence, $\mu((A + x_n) \cap K) > 0$ and $(A + x_n) \cap K \neq \emptyset$. Choose $k_n \in K$ and $a_n \in A$ so that $x_n = k_n - a_n$. Since K is compact, and A is weakly compact we can suppose that k_n converge to some $k \in K$, and a_n converge weakly to some $a \in A$. The sequence (x_n) converges weakly to $x = k - a$; we will show that $\mu(A + x) \geq \delta$.

Choose a countable set $\{x_n^*\}$ in the unit sphere of X^* which norms X. Let $\mathcal{T}$ be the weak topology on X induced by span $\{x_n^*\}$. The topological vector space $(X, \mathcal{T})$ is metrizable. The families of Borel sets on X defined by $\mathcal{T}$ and the norm topology coincide: each norm closed ball in X is $\mathcal{T}$-closed ($\bar{B}(0, 1) = \{x \in X : |\langle x_n^*, x \rangle| \leq 1\}$), and every norm open set is a union of countably many $\mathcal{T}$-closed balls. Therefore, each norm Borel probability measure on X can be considered also as a $\mathcal{T}$-Borel probability measure. Denote by τ and $\tau_{\mathcal{T}}$ the weak topologies on the set $\mathcal{M}$ of Borel probability measures on X derived from the norm topology, respectively the topology $\mathcal{T}$. The topology τ is stronger, therefore (μ_n) converges to μ also in $\tau_{\mathcal{T}}$. Since (x_n) converges weakly to x, it converges to x also in $\mathcal{T}$, and $(\boldsymbol{\delta}_{x_n})$ converges in $\tau_{\mathcal{T}}$ to $\boldsymbol{\delta}_x$. Define measures $\nu_n = \mu_n * \boldsymbol{\delta}_{x_n}$, and $\nu = \mu * \boldsymbol{\delta}_x$. The space $(\mathcal{M}, \tau_{\mathcal{T}})$ with the convolution is a topological semigroup (see [P, p. 57]), hence ν_n converge to ν in $\tau_{\mathcal{T}}$. Since A is weakly compact it is also $\mathcal{T}$-compact. Therefore, $\limsup \nu_n(A) \le \nu(A)$ (c.f. [P, p. 40]), and

$$\mu(A+x) = \nu(a) \ge \limsup \nu_n(A) = \limsup \mu_n(A+x_n) \ge \delta.$$

Finally, let us show that (ii)$\Rightarrow$(iii). Suppose for contradiction that there is some $0 < \delta' < 1$ so that for each $0 < r$ there is a finite set $F \subset B(0,r)$ so that $|F \cap (A+x)| < \delta'|F|$ for each $x \in X$. We will show that for each $\varepsilon > 0$ and each $r > 0$ there is a finite set $F \subset B(0,r)$ so that $|F \cap (A+x)| < \varepsilon |F|$ for each $x \in X$. That will contradict (ii), since (ii), as shown above, implies that for each probability measure μ on X with spt $\mu \subset B(0,r)$ there exists $x \in X$ so that $\mu(A+x) \ge \delta$. The existence of such a set F follows by induction (find $k \in \mathbb{N}$ so large that $\delta^{2^k} < \varepsilon$) from the following claim:

Suppose $0 < \delta < 1$ is such that for each $r > 0$ there exists a finite set $F \subset B(0,r)$ so that $|F \cap (A+x)| \le \delta |F|$, whenever $x \in X$. Then, for each $r > 0$, there is a finite set $F \subset B(0,r)$ so that $|F \cap (A+x)| \le \delta^2 |F|$, whenever $x \in X$.

To see this, let $r > 0$ be given. Choose a finite set $F_1 \subset B(0, r/2)$, so that $|F_1 \cap (A+x)| \le \delta |F_1|$, whenever $x \in X$. Let

$$r_1 = \frac{1}{4} \min\{\alpha_{F_1},\ \min\{\|x-y\| :\ x, y \in F_1,\ x \ne y\}\},$$

where α_{F_1} is the positive number assigned by Lemma 1.2 to A, F_1 and $n = \lceil |F_1|(1-\delta) \rceil$. Choose a finite set $F_2 \subset B(0, r_1)$, so that $|F_2 \cap (A+x)| \le \delta |F_2|$, whenever $x \in X$. Put $F = F_1 + F_2$. Then, $F \subset B(0,r)$ and $|F| = |F_1| \cdot |F_2|$. Let $x \in X$ be given. By Lemma 1.2 there exists $E \subset F_1$ so that $|E| \ge n$ and $\operatorname{dist}(E, A+x) \ge \alpha_{F_1}$. Also,

$$(E + F_2) \cap (A+x) = \emptyset, \tag{1}$$

since $F_2 \subset B(0, \frac{1}{4}\alpha_{F_1})$. Let $z \in F_1 \setminus E$ be given. Then there exists $E_z \subset F_2$ so that

$$E_z \cap (A + x - z) = \emptyset, \tag{2}$$

and $|E_z| \ge (1-\delta)|F_2|$. By (1) and (2),

$$\left((E + F_2) \cup \bigcup_{z \in F_1 \setminus E} E_z + z \right) \cap (A+x) = \emptyset.$$

Since

$$\begin{aligned} |(E+F_2) \cup \bigcup_{z \in F_1 \setminus E} E_z + z| &= |F_2||E| + \sum_{z \in F_1 \setminus E} |E_z| \\ &\ge |F_2||E| + (|F_1| - |E|)(1-\delta)|F_2|, \end{aligned}$$

we have

$$\begin{aligned}|F\cap(A+x)| &\le |F_1||F_2| - |F_2||E| - (|F_1|-|E|)(1-\delta)|F_2|\\ &= |F_2|(|F_1|-|E|-(1-\delta)(|F_1|-|E|)) = |F_2|(|F_1|-|E|)\delta\\ &\le |F_2|(|F_1|-n)\delta \le |F_2|(|F_1|-|F_1|(1-\delta))\delta = \delta^2|F_1||F_2| = \delta^2|F|.\end{aligned}$$

□

2. Banach-Saks property of the dual implies smallness of convex sets

Recall (see [BL]) that a basic sequence (x_n) in a Banach space X is called almost unconditional if there is $c>0$ so that for each $k\in\mathbb{N}$ and $k\le n_1<n_2<\dots<n_k$, the projection P from $\overline{\operatorname{span}}\,(x_n)_{n\in\mathbb{N}}$ onto $\operatorname{span}\,(x_{n_i})_{i=1}^k$ defined by $P(\sum_{n=1}^\infty \alpha_n x_n) = (\sum_{i=1}^k \alpha_{n_i} x_{n_i})$ has norm at most c. If (x_n) is a weakly null basic sequence in a Banach space X, then there is a subsequence of (x_n) which is almost unconditional ([BL, p. 61]).

A sequence (y_n) in a Banach space X is said to converge to infinity uniformly with respect to subseries if

$$\lim_{k\to\infty}\inf_{n_1<\dots<n_k}\|\sum_{i=1}^k y_{n_i}\| = \infty.$$

Proposition 2.1. *Let X be a Banach space with an almost unconditional normalized Schauder basis (x_n). Suppose there is a subsequence (x_{n_j}) of (x_n) which converges to infinity uniformly w.r.t. subseries. Then the positive cone $Q=\{\sum_{n=1}^\infty \alpha_n x_n\in X:\ \alpha_n\ge 0\}$ of X is Haar null.*

Proof: Since Haar null sets are closed under isomorphisms and countable union, it is enough to show that $A=Q\cap B(0,1)$ is Haar null. For each $0<r<1$ and $0<\delta$ we will find a probability measure on X with $\operatorname{spt}\mu\subset \bar B(0,r)$ so that $\mu(A+x)<\delta$ for each $x\in X$; by Theorem 1.1, A is Haar null.

Let $c>0$ be the almost unconditional basis constant of (x_n). Let $0<r<1$ and $0<\delta<1$ be given. Put $y_j=x_{n_j}$ and choose $k\in\mathbb{N}$ so that

$$\|\sum_{i=1}^{k'} y_{j_i}\| > \frac{4c^2}{r}\quad\text{whenever}\quad k\le k',\quad\text{and}\quad j_1<\dots<j_{k'}.$$

Choose $K\in\mathbb{N}$ so that $k<\delta K$. Observe that if $k<j_1<\dots<j_k$ then the natural projection on $\operatorname{span}\,(y_{j_i})_{i=1}^k$ has norm at most c, and

$$\|\sum_{i=1}^k \alpha_i y_{j_i}\| \le 2c\|\sum_{i=1}^k \beta_i y_{j_i}\|$$

whenever $|\alpha_i| < |\beta_i|$ (this follows from c-unconditionality of $(y_{j_i})_{i=1}^k$; for exact computations see e.g. [GD, p. 22]). Put $F = \{-ry_{k+1}, -ry_{k+2}, \dots, -ry_{k+K}\} \subset \bar{B}(0,r)$, and $\mu = \frac{1}{|F|}\sum_{x\in F}\boldsymbol{\delta}_x$. Suppose there exists $x = \sum \alpha_n x_n \in X$ so that $\mu(A - x) \geq \delta$; equivalently, $|A \cap (x + F)| \geq \delta|F|$. Then there exists $E' \subset \{k+1, k+2, \dots, k+K\}$ so that $|E'| \geq \delta|F| = \delta K > k$, and $\alpha_{n_j} \geq r$ for each $j \in E'$. Choose some $E \subset E'$ so that $|E| = k$. Then

$$\|x\| \geq \frac{1}{c}\|\sum_{j\in E}\alpha_{n_j}y_j\| \geq \frac{1}{2c^2}\|\sum_{j\in E} ry_j\| \geq \frac{1}{2c^2}r\frac{4c^2}{r} = 2.$$

Since $F \subset \bar{B}(0,r)$, $A \subset B(0,1)$, and $r < 1$, the above inequality implies that $A \cap (F + x) = \emptyset$, which contradicts the fact that $|A \cap (F + x)| \geq \delta|F| > 0$. □

To prove Proposition 2.3 which strengthens slightly Lemma 1 of [M] we will need the following elementary lemma; we include a proof since we could not find a reference for it.

Lemma 2.2. *Let X be a Banach space, (x_n) an almost unconditional basis of X. Then the biorthogonal sequence (x_n^*) of (x_n) is also almost unconditional.*

Proof: Let $c > 0$ be such that (x_n) is c-almost unconditional. Let $k \in \mathbb{N}$ and $k \leq n_1 < \dots < n_k$ be given. Let $P : X \to \operatorname{span}(x_{n_i})_{i=1}^k$ be the projection defined by $P(\sum \alpha_n x_n) = \sum_{i=1}^k \alpha_{n_i}x_{n_i}$, and $\tilde{P} : \overline{\operatorname{span}}(x_n^*)_{n\in\mathbb{N}} \to (x_{n_i}^*)_{i=1}^k$ be the projection defined by $\tilde{P}(\sum \beta_n x_n^*) = \sum_{i=1}^k \beta_{n_i}x_{n_i}^*$. Then $\|P\| \leq c$. Let $x^* = \sum_{\mathbb{N}} \beta_n x_n^* \in \overline{\operatorname{span}}(x_n^*)_{n\in\mathbb{N}}$ be given. Then

$$\begin{aligned}
\|\tilde{P}(x^*)\| &= \sup\nolimits_{0\neq x\in X} \frac{1}{\|x\|}\langle \tilde{P}(\sum \beta_n x_n^*), x\rangle \\
&= \sup\nolimits_{0\neq \sum \alpha_n x_n = x\in X} \frac{1}{\|x\|}\langle \sum\nolimits_{i=1}^k \beta_{n_i}x_{n_i}^*, \sum \alpha_n x_n\rangle \\
&= \sup\nolimits_{0\neq \sum \alpha_n x_n = x\in X} \frac{1}{\|x\|}\langle \sum \beta_n x_n^*, \sum\nolimits_{i=1}^k \alpha_{n_i}x_{n_i}\rangle \\
&= \sup\nolimits_{0\neq \sum \alpha_n x_n = x\in X} \frac{1}{\|x\|}\langle \sum \beta_n x_n^*, P(\sum \alpha_n x_n)\rangle \\
&\leq \sup\nolimits_{0\neq x\in X} \frac{1}{\|x\|}\|x^*\|\cdot\|P\|\cdot\|x\| \leq c\|x^*\|.
\end{aligned}$$

Therefore, $\|\tilde{P}\| \leq c$. □

We say that a convex subset K of a Banach space X is spanning if it contains a line segment in every direction, that is $\bigcup_{t>0} t(K - K) = X$.

Proposition 2.3. *Let X be a reflexive Banach space, and $K \subset X$ closed, convex, bounded and spanning set with empty interior. Then there exist $v \in X$ and an* *infinite dimensional quotient space Y of X with an almost unconditional Schauder basis (y_n) so that the image of $K+v$ under the quotient mapping is contained in the positive cone $Q=\{\sum_{n=1}^{\infty}\alpha_n y_n\in Y:\ \alpha_n\ge 0\}$ of Y.*

Proof: Since X is reflexive, it is a consequence of the theorem of Amir and Lindenstrauss (see Lemma 3 of [M]) that there exists a projection P of X onto a separable subspace Z of X so that the interior of $\overline{P(K)}$ in Z is empty. Clearly, $\overline{P(K)}$ is spanning in Z. Therefore we can suppose that X is separable. By the proof of Lemma 1 of [M] there exists $v\in X$ and a weakly null sequence (x_n^*) in the unit sphere of X^* so that $\langle x_n^*, x+v\rangle\ge 0$ for each $x\in K$.

By [LT, p. 11] (proof of the theorem that every separable Banach space has a quotient with a Schauder basis), if (x_n^*) is a normalized w^*-null sequence in the dual of a separable Banach space X, then there is a subsequence (u_n^*) of (x_n^*) so that

- (i) each subsequence (y_n^*) of (u_n^*) is a basic sequence;
- (ii) if we denote by $(y_n)\subset(\overline{\operatorname{span}}(y_n^*))^*$ the functionals biorthogonal to (y_n^*), then the operator $T: X\to(\overline{\operatorname{span}}(y_n^*))^*$ defined by $\langle T(x), x^*\rangle=\langle x^*, x\rangle$, $x^*\in\overline{\operatorname{span}}(y_n^*)$ maps X onto $\overline{\operatorname{span}}(y_n)$.

Let (u_n^*) be such a subsequence of our weakly null sequence (x_n^*). As mentioned above, there is a subsequence (y_n^*) of (u_n^*) which is almost unconditional. By Lemma 2.2 the functionals $(y_n)\subset(\overline{\operatorname{span}}(y_n^*))^*$ biorthogonal to (y_n^*) form an almost unconditional basic sequence. Put $Y=\overline{\operatorname{span}}(y_n)$, and let (f_n) be the coefficient (biorthogonal) functionals of (y_n) in Y^*. Then $T^*(f_n)=y_n^*$, and since

$$\langle f_n, T(x+v)\rangle=\langle T^*(f_n), x+v\rangle=\langle y_n^*, x+v\rangle\ge 0$$

for each $x\in K$, Y is the desired quotient space. □

The following lemma is contained in the proof of Theorem 4 of [MS]. For an easy reference we include it with a proof.

Lemma 2.4. *Let X be a Banach space, Y be a quotient of X (T be the quotient mapping), and A be a Borel subset of Y. Let ν be a Radon probability measure on Y so that $\nu(A+y)=0$ for each $y\in Y$. Then there is a Radon probability measure μ on X so that $\mu(T^{-1}(A)+x)=0$ for each $x\in X$.*

Proof: By the theorem of Bartle and Graves (see e.g. [BP]), the inverse of T admits a continuous selection f. Consequently, the formula $\mu(B)=\nu(f^{-1}(B))$ for Borel subsets B of X defines a Radon probability measure on X. Let $x\in X$ be given. Then $f^{-1}(T^{-1}(A)+x)\subset A+T(x)$, hence $\mu(T^{-1}(A)+x)\le\nu(A+T(x))$ $=0$. □

A Banach space X is said to have the Banach-Saks property if every bounded sequence (x_n) in X contains a subsequence (x'_n) such that the sequence of Cesaro means $(n^{-1}\sum_1^n x'_k)_{n\in\mathbb{N}}$ is norm-convergent. Every superreflexive Banach space has the Banach-Saks property. A Banach space with the Banach-Saks property is necessarily reflexive, but there exists a reflexive Banach space without the Banach-Saks property ([Ba]).

In the proof of the next theorem we will need the following facts.

A Banach space X has the Banach-Saks property if and only if X is reflexive and no quotient of X^* has a spreading model isomorphic to c_o ([BL, p. 82]).

A Banach space X does not have a spreading model isomorphic to c_o if and only if every bounded non convergent sequence in X contains a subsequence which converges to infinity uniformly with respect to subseries ([BL, p. 77]).

Theorem 2.5. *Let X be a Banach space such that X^* has the Banach-Saks property, and $K \subset X$ be a closed and convex set with empty interior. Then there exists a Radon probability measure μ on X so that $\mu(K + x) = 0$ for all $x \in X$.*

Proof: Since a countable union of Haar null sets is Haar null, we can suppose that K is bounded. If K is not spanning, there exists $x \in X$ so that the intersection of any translate of the line segment $[0, x]$ and K contains at most one point. Consequently, any translate of K is a null set for the Lebesgue measure sitting on $[0, x]$. Suppose that K is spanning. Since X^* has the Banach-Saks property the Banach space X is reflexive. By Proposition 2.3 there is a quotient space Y of X with an almost unconditional basis (y_n) and $v \in X$ so that the image of $K + v$ under the quotient mapping is contained in the positive cone of Y. By considering $(y_n/\|y_n\|)$ instead of (y_n), we can suppose that the basis of Y is normalized. Since X^* has the Banach-Saks property, Y does not have a spreading model isomorphic to c_o. Hence there exists a subsequence of (y_n) which converges to infinity uniformly w.r.t. subseries, and by Proposition 2.1 the positive cone of Y is Haar null. Lemma 2.4 finishes the proof. □

Notice that each closed, convex, *symmetric* and nowhere dense subset C of a Banach space X is Haar null. This is an easy consequence of the Baire category theorem: there exist $x \in X \setminus \bigcup_{\mathbb{N}} nC$. Since C is convex and symmetric each line parallel to x intersects C in at most one point, and the Lebesgue measure on $[0, x]$ shows that C is Haar null. As a corollary, if X is a nonreflexive Banach space and A is a weakly compact subset of X then A is Haar null. Indeed, the set $C = \overline{\mathrm{conv}}(A \cup -A)$ is convex, symmetric, and weakly compact by the theorem of Krein. Since X is not reflexive, C has empty interior, hence C, and also $A \subset C$ are Haar null.

If instead of Haar null sets we consider Aronszajn null sets (for the definition see [A]), closed, convex, and nowhere dense sets are in general not "small" any more.

Lemma 2.6. *Let X be a separable Banach space, and $K \subset X$ closed, convex set such that $\overline{\mathrm{span}}\, K = X$. Then K is not Aronszajn null.*

Proof: We can suppose that $0 \in K$. Observe that there exists $Q \subset X$ closed, convex, symmetric set such that $\overline{\mathrm{span}}\, Q = X$, and $v \in X$ so that $Q + v \subset K$. To see this, choose a countable set $D = (x_n)$ so that D is dense in K. Choose $\alpha_n > 0$ with $\sum_1^\infty \alpha_n = 1$. Then

$$v = \sum \alpha_i \frac{x_i}{2} \in K,$$

and, also, $v \pm \alpha_n \frac{x_n}{2} \in K$ for each $n \in N$. Put $Q = \overline{\text{conv}}(\pm \frac{\alpha_n}{2} x_n)$. Then Q is the required set.

Since Aronszajn null sets are translationally invariant, to finish the proof it is enough to show that each closed, convex, *symmetric* set $K \in X$ such that $\overline{\text{span}}\, K = X$ is not Aronszajn null. Again, choose $(y_n) \subset K$ dense in K. By the proof of Proposition 1.f.3 of [LT, p. 43], there is a Markushevich basis (x_n) of X, so that $(x_n) \subset \text{span}\,(y_n)$. Since K is convex and symmetric, for each $n \in N$ there exists $0 < \beta_n$ so that $\beta_n x_n \in K$. Choose $\alpha_n > 0$ with $\sum_1^\infty \alpha_n = 1$. Put $C = \{\sum_1^\infty \gamma_n \beta_n x_n : 0 \leq \gamma_n \leq \alpha_n\}$. Then $C \subset K$ and by the proof of Theorem 1 of [A, p. 154], the set C is not Aronszajn null. Consequently K is not Aronszajn null. □

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References

[A] Aronszajn N., *Differentiability of Lipschitzian mappings between Banach spaces*, Studia Math. **LVII** (1976), 147–190.

[Ba] Baernstein A., *On reflexivity and summability*, Studia Math. **42** (1972), 91–94.

[BL] Beauzamy B., Lapreste J.T., *Modèles étalés des espaces de Banach*, Travaux en cours, Hermann, Paris, 1984.

[BP] Bessaga C., Pelczynski A., *Selected Topics in Infinite-Dimensional Topology*, PWN, Warszawa, 1975.

[C] Christensen J.P.R., *Topology and Borel Structure*, North-Holland, Amsterdam, 1974.

[GD] Guerre-Delabrière S., *Classical Sequences in Banach Spaces*, Marcel Dekker, New York, 1992.

[H] Hunt B.R., *The prevalence of continuous nowhere differentiable functions*, Proc. Amer. Math. Soc. **122** (1994), 711–717.

[HSY] Hunt B.R., Sauer T., Yorke J.A., *Prevalence: A translation-invariant "almost every" on infinite-dimensional spaces*, Bull. Amer. Math. Soc. (N.S.) **27** (1992), 217–238.

[LT] Lindenstrauss J., Tzafriri L., *Clasical Banach Spaces I.*, Springer-Verlag, Berlin, 1977.

[M] Matoušková E., *Convexity and Haar null sets*, Proc. Amer. Math. Soc. **125** (1997), no. 6, 1793–1799.

[MS] Matoušková E., Stegall C., *A characterization of reflexive Banach spaces*, Proc. Amer. Math. Soc **124** (1996), no. 4, 1083–1090.

[My] Mycielski J., *Some unsolved problems on the prevalence of ergodicity, instability, and algebraic independence*, Ulam Quarterly, vol 1, no. 3 (1992), 30–37.

[P] Parthasarathy K.R., *Probability Measures on Metric Spaces*, Academic Press, New York, 1967.

[S] Solecki S., *On Haar null sets*, Fundam. Math. **149** (1996), 205–210.

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