An elementary proof of a theorem on sublattices of finite codimension

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Abstract. This paper presents an elementary proof and a generalization of a theorem due to Abramovich and Lipecki, concerning the nonexistence of closed linear sublattices of finite codimension in nonatomic locally solid linear lattices with the Lebesgue property.

Keywords: linear lattice, Lebesgue property, lattice homomorphism
Classification: 46A40, 47B65

In 1990 Y.A. Abramovich and Z. Lipecki proved, by means of Boolean algebra techniques and Liapunov's convexity theorem, the following result ([1, Theorem 2]; cf. [3, Example 27.8]):

Let $X = (X, \tau)$ be a Hausdorff locally solid linear lattice such that:

(i) *X is nonatomic and Dedekind complete,*
(ii) *X has the Lebesgue property.*

Then X contains no proper closed sublattices of finite codimension.

(By a sublattice of a linear lattice we always mean a *linear* sublattice. X has the Lebesgue property (or, τ is a Lebesgue topology on X) provided that for every MS-sequence (x_α) in X with $x_\alpha \downarrow 0$ we have $x_\alpha \to 0(\tau)$. For other undefined notions and basic results concerning linear lattices (= Riesz spaces) in this paper we refer the reader to the monographs [2], [3]).

Here we give an elementary and short proof of a more general result, namely, we show that the two assumptions in the above theorem, i.e. that τ is Hausdorff and X is Dedekind complete are superfluous. It should be noted that every infinite dimensional linear lattice possesses sublattices of arbitrary finite codimension ([1, Theorem 3]).

Theorem. *Let X be a nonatomic linear lattice, and let Y be a sublattice of X with $\dim X/Y < \infty$. Then Y is order dense in X.*
If, additionally, τ is a Lebesgue topology on X, then Y is τ-dense in X.
In particular, the topological dual X' of any nonatomic locally solid linear lattice (X, τ) with the Lebesgue property is nonatomic (equivalently, X has no nontrivial continuous Riesz homomorphisms $X \to \mathbf{R}$).

Order denseness of Y in X is understood in the sense of ([2, Definition 1.9]), i.e. that $Y_+ \setminus \{0\}$ is cofinal in $X_+ \setminus \{0\}$.

Proof: Let Q denote the quotient map $X \to X/Y$. Since X is nonatomic, every principal ideal $A_e = \{x \in X : |x| \leq \lambda e$ for some $\lambda \geq 0\}$, $e \in X^+$, is of infinite dimension. If Y were not order dense in X, then $A_e \cap Y = \{0\}$ for some $e \in X^+$, and hence Q restricted to A_e would be a linear isomorphism; thus $\dim A_e \leq \dim Q(X) < \infty$, a contradiction. This proves the first part of the theorem, and since for Lebesgue topologies order denseness implies topological denseness, the second part also follows; the particular case is implied by ([2, Theorem 3.13]; [3, Theorem 18.3 (iii)]). □

Examples. 1. Let K be a topological Hausdorff space. The lattice $C(K)$ is nonatomic whenever K has no isolated points, thus every such lattice has the property described in the first part of the Theorem.

2. Let S_p denote the nonatomic sublattice, consisting of all step functions, of the (nonatomic) lattice $L_p = L_p(0,1)$, $0 < p < \infty$. It is easily seen that S_p endowed with the p-norm topology has the Lebesgue property, and hence S_p possesses the property described in the second part of the Theorem without being even σ-Dedekind complete (compare with (i) above).

3. This example seems to be known; we include it for completeness of the paper. Let $1 < p < \infty$. Since every continuous linear functional f on the lattice L_p is order continuous ([2, Theorems 9.1, 22.1 and 22.4]), any family of seminorms (q_f) of the form $q_f(x) = |f|(|x|)$, $x \in L_p$, determines a Lebesgue topology on L_p ([2, p. 40]). This topology is Hausdorff whenever the family (q_f) is total.

References

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(*Received* July 1, 1997, *revised* November 4, 1997)