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Some types of implicative ideals

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Abstract. This paper studies basic properties for five special types of implicative ideals (modular, pentagonal, even, rectangular and medial). The results are used to prove characterizations of modularity and distributivity.

Keywords: implicative ideals, modular ideals, pentagonal ideals, even ideals, rectangular ideals, medial ideals, modularity, distributivity

Classification: 06B10, 06C99, 06D99

1. Introduction

In this paper, we study lattice ideals that are implicative analogs of semiprime ideals. In particular, we focus upon a complete description of all possible inclusion relations between the corresponding classes. We also exhibit examples showing that the considered concepts define different classes. Since semiprime ideals occur as a natural tool for a description of Boolean algebras, it is not surprising that also these five classes of ideals play an interesting role in the theory of Boolean algebras (see [3] and the techniques of [5]). As a by-product we obtain new characterizations of modularity and distributivity.

First we note that the Rav's definition of a semiprime ideal ([4]) can be given in a slightly modified form: An ideal I of a lattice L is *semiprime* if

$$\forall\, a,b,c,d \in L \ \ (a \wedge b) \vee (a \wedge c) \in I \ \ \Rightarrow \ a \wedge (b \vee c) \in I. \tag{1.1}$$

The class of all semiprime ideals will be denoted by **Sp**.

The new approach to the semiprimeness suggests a definition of a more general class of ideals in the following way: An ideal I of L is called *implicative*, if there exist two lattice polynomials $p(x_1, x_2, \cdots, x_n)$ and $q(x_1, x_2, \cdots, x_n)$ such that

$$\forall\, a_1, a_2, \cdots, a_n \in L \ \ p(a_1, a_2, \cdots, a_n) \in I \ \ \Rightarrow \ q(a_1, a_2, \cdots, a_n) \in I.$$

As usual, for any a, b, c of a lattice L, the *upper median* $\overline{\mathrm{med}}(a, b, c)$ is the element $(a \vee b) \wedge (a \vee c) \wedge (b \vee c)$; the *lower median* is defined dually by $\underline{\mathrm{med}}(a, b, c) :=$ $= (a \wedge b) \vee (a \wedge c) \vee (b \wedge c)$.

2. Five classes of implicative ideals

To introduce the first class, we will need the following lemma.

Lemma 2.1. *The following are equivalent for an ideal I of a lattice L.*

$$(2.1)\qquad \forall\, a,b,c \in L\ \ [a \vee (b \wedge c) \in I \ \ \&\ \ a \leq c] \ \Rightarrow\ (a \vee b) \wedge c \in I;$$

$$(2.1')\ \forall\, a,b,c \in L\ \ (a \wedge c) \vee [b \wedge (a \vee c)] \in I \ \ \Rightarrow\ [(a \wedge c) \vee b] \wedge (a \vee c) \in I;$$

$$(2.1'')\qquad \forall\, a,b,c \in L\ \ (a \wedge b) \vee (a \wedge c) \in I \ \ \Rightarrow\ a \wedge [b \vee (a \wedge c)] \in I.$$

Proof: Immediate. □

An ideal I of L is said to be *modular* if it satisfies one of the conditions (2.1)–(2.1″). The class of modular ideals will be denoted by **M**.

An ideal I of L is called *pentagonal*, if

$$(2.2)\qquad \forall\, a,b,c \in L\ \ a \vee (b \wedge c) \in I \ \ \Rightarrow\ (a \vee b) \wedge (a \vee c) \in I.$$

The class of all such ideals will be denoted by **Pe**.

An ideal I of L is said to be *even*, if

$$(2.3)\qquad \forall\, a,b,c \in L\ \ [a \wedge (b \vee c)] \vee [b \wedge (a \vee c)] \in I \ \ \Rightarrow\ \overline{\mathrm{med}}(a,b,c) \in I.$$

We will use the letter **E** to denote the class of even ideals.

Remark 2.2. The ideal $(s]$ of the lattice L_9 pictured in Figure 1 (see [6, p. 192]) is modular. It is not even: If $a = r$, $b = u$ and $c = t$, then $[a \wedge (b \vee c)] \vee [b \wedge (a \vee c)] = 0$ and $\overline{\mathrm{med}}(a,b,c) = t$. Note that the same argument applies to the ideal $(0]$ of L_9. The ideal $(s]$ is not pentagonal, since $s \vee (u \wedge w) = s$ and $(s \vee u) \wedge (s \vee w) = x$.

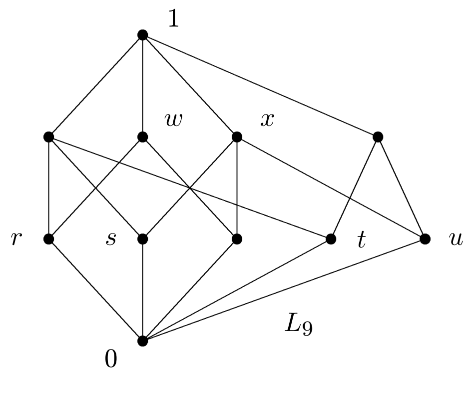

Figure 1

Remark 2.3. As noted above, the ideal $(0]$ of the lattice L_9 is not even. It can be verified that it is pentagonal. Hence the class **Pe** is not a subclass of the class **E**.

Remark 2.4. The ideal $(e]$ of the lattice L_7 (see Figure 2) is even and it is not pentagonal: Here $a \vee (b \wedge c) = e$ and $(a \vee b) \wedge (a \vee c) = 1$. Therefore, the class **E** is not a subclass of **Pe**.

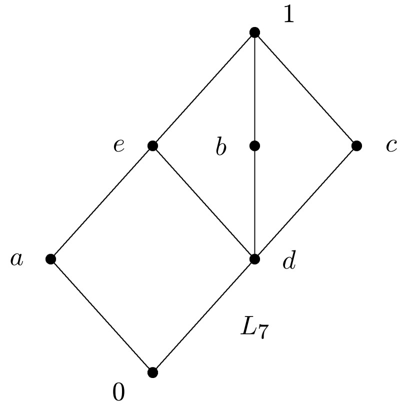

Figure 2

Theorem 2.5. *Let I be an ideal of a lattice L. Then*
(i) *if I is pentagonal, it is modular;*
(ii) *if I is even, it is modular.*

Proof: (i) Combine the definition of a pentagonal ideal with (2.1).

(ii) Suppose $(x\wedge y)\vee(x\wedge z) \in I$. Putting $a := x\wedge z$, $b := y$, $c := x$, we get $[a\wedge(b\vee \vee c)] \vee [b \wedge (a \vee c)] = (x \wedge y) \vee (x \wedge z) \in I$, and, by (2.3), $I \ni \overline{\text{med}}(a, b, c) = = [(x \wedge z) \vee y] \wedge x$. Thus I is modular by (2.1$''$). $\square$

Theorem 2.6. *Let L be a lattice. Then the following conditions are equivalent.*
(i) *The lattice L is modular.*
(ii) *Every ideal of L is modular.*
(iii) *Every ideal of L is even.*

Proof: (i) $\Rightarrow$ (iii): Let $s := [a \wedge (b \vee c)] \vee [b \wedge (a \vee c)]$ be an element of an ideal I. By modularity,

$$I \ni s = \{[b \wedge (a \vee c)] \vee a\} \wedge (b \vee c) = (a \vee b) \wedge (a \vee c) \wedge (b \vee c).$$

(iii) $\Rightarrow$ (ii): This follows from Theorem 2.5.

(ii) $\Rightarrow$ (i): Suppose $a \leq c$. Since $J := (a \vee (b \wedge c)]$ is modular for any $b \in L$, $(a\vee b)\wedge c \in J$ by (2.1). Therefore, $(a\vee b)\wedge c \leq a\vee(b\wedge c)$ and so $a\vee(b\wedge c) = (a\vee b)\wedge c$. $\square$

An ideal I of a lattice L is called *rectangular*, if

$$\forall a,b,c \in L \ \ (a \wedge c) \vee [b \wedge (a \vee c)] \in I \ \ \Rightarrow \overline{\mathrm{med}}(a,b,c) \in I. \tag{2.4}$$

The class of all rectangular ideals will be denoted by **Re**.

Lemma 2.7. *Any rectangular ideal of a lattice is modular.*

Proof: Suppose $a \leq c$ and let $a \vee (b \wedge c) \in I$. Note that $a \leq c$ implies $a \wedge (b \vee c) = a$. Consequently, $I \ni a \vee (b \wedge c) = (b \wedge c) \vee [a \wedge (b \vee c)]$. Since I is rectangular, $I \ni \overline{\mathrm{med}}(a,b,c)$. However, $\overline{\mathrm{med}}(a,b,c) \geq (a \vee b) \wedge c$. Thus $(a \vee b) \wedge c \in I$ and I is modular by (2.1). □

Theorem 2.8. *Let I be a rectangular ideal of a lattice L. Then I is pentagonal and even.*

Proof: 1. Let $i \in I$ and $b \wedge c \in I$. Then $(b \wedge c) \vee [i \wedge (b \vee c)] \in I$ and, by the definition of a rectangular ideal, we have $(b \vee c) \wedge (b \vee i) \wedge (i \vee c) \in I$. Put $A := i \vee (b \wedge c)$, $B := b \vee c$ and $C := (b \vee i) \wedge (i \vee c)$ so that $A \in I$ and $B \wedge C \in I$. Clearly, $A \leq C$ and $A \vee (B \wedge C) \in I$. By Lemma 2.7, (2.1) and $A \vee B \geq C$ we can see that $I \ni (A \vee B) \wedge C = (b \vee i) \wedge (i \vee c)$. It follows that I is pentagonal.

2. Now suppose that $a \wedge (b \vee c)$ and $b \wedge (a \vee c)$ belong to I. A fortiori, $a \wedge c \in I$ and $b \wedge (a \vee c) \in I$. By the definition of a rectangular ideal we therefore have $\overline{\mathrm{med}}(a,b,c) \in I$ and we conclude that I is even. □

Remark 2.9. The ideal $(0]$ of the lattice L_6 shown in Figure 3 is even and pentagonal. It is not rectangular, since $(a \wedge c) \vee [b \wedge (a \vee c)] = 0$ and $\overline{\mathrm{med}}(a,b,c) = d$. It follows that the class **Re** is a proper subclass of the class **Pe** $\cap$ **E**.

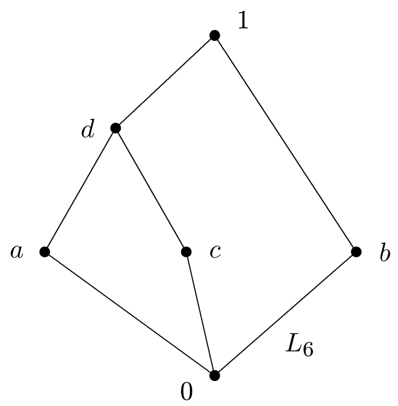

Figure 3

An ideal I of L is said to be *medial*, if the implication

$$\underline{\mathrm{med}}(a,b,c) \in I \quad \Rightarrow \quad \overline{\mathrm{med}}(a,b,c) \in I$$

is true for any $a, b, c \in L$. The class of medial ideals will be denoted by **Me**.

Theorem 2.10. *In any lattice,*

(i) *every medial ideal is rectangular;*
(ii) *every semiprime ideal is medial.*

PROOF: (i) Note that $\underline{\mathrm{med}}(a,b,c) \leq (a \wedge c) \vee [b \wedge (a \vee c)]$ holds for any $a, b, c \in L$.

(ii) Let $I \in \mathbf{Sp}$ and suppose that $\underline{\mathrm{med}}(a,b,c) \in I$. Now, $\underline{\mathrm{med}}(a,b,c) = \overline{\mathrm{med}}(a,b,c)$ in any distributive lattice. By [1, Lemma 2.1], we can see that $(\underline{\mathrm{med}}(a,b,c), \overline{\mathrm{med}}(a,b,c)) \in \hat{C}(L)$ where $\hat{C}(L)$ denotes the smallest congruence of L such that $L/\hat{C}(L)$ is distributive (see [2]).

We claim that $\overline{\mathrm{med}}(a,b,c) \in I$. Were this false, we would have an allele p/i such that $i \in I$ and $p \notin I$. Consequently, by [1, Main Theorem], the ideal I is not semiprime, and a contradiction ensues. □

Remark 2.11. The ideal $(0]$ of the lattice M_5 given in Figure 4 is rectangular. It is not medial, since $\underline{\mathrm{med}}(a,b,c) = 0$ and $\overline{\mathrm{med}}(a,b,c) = 1$.

Note that the ideal $(0]$ in the lattice L_7 represented in Figure 2 is medial. However, it is not semiprime, since $(a \wedge b) \vee (a \wedge c) = 0$ and $a \wedge (b \vee c) = a$.

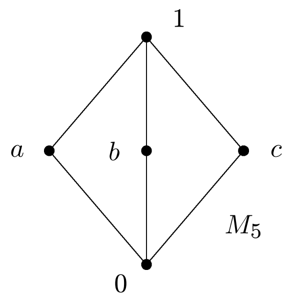

Figure 4

The theorems and the remarks mentioned above lead to a complete description of all inclusion relations between the studied classes of ideals, as indicated in Figure 5.

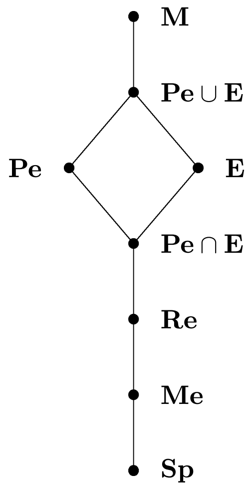

Figure 5

The following result can be viewed as alternative characterizations of distributivity.

Theorem 2.12. *For any lattice L, the following are equivalent:*

(i) *the lattice L is distributive;*
(ii) *any ideal of L is medial;*
(iii) *any ideal of L is rectangular;*
(iv) *any ideal of L is pentagonal.*

PROOF: Using the preceding theorems, one can easily see that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ $\Rightarrow$ (iv).

Assume that L satisfies (iv). First note that there is no sublattice of L isomorphic to the lattice N_5 of Figure 6. Indeed, were this false, let $I = (a]$. Clearly $a \vee (b \wedge c) \in I$ and, by assumption, $I \ni (a \vee b) \wedge (a \vee c) = c$, a contradiction.

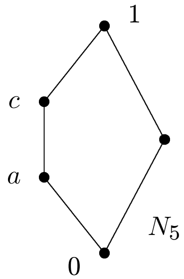

Figure 6

We are now reduced to proving that there is no sublattice M_5 (see Figure 4) in L. If this were false, then let $I = (a]$ and, similarly as above, we would have a contradiction. Thus L is distributive. $\square$

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