Comment.Math.Univ.Carolin. 39,2 (1998)309–321

Multiplication of distributions

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Abstract. Multiplication by harmonic representations of distributions, introduced by Li Banghe, is an extension of a certain product by radial (rotationally symmetric) mollifiers and therefore a strict extension of the Kamiński and Colombeau product.

Keywords: distribution products, multiplication, harmonic representations, Kamiński product

Classification: 46F10

There exist several definitions of a multiplication of distributions. A popular approach to define a product for a pair of distributions is to approximate them by smooth functions, multiply these, and pass to a limit. Approximation is usually done by convolution with δ-sequences, i.e. sequences or nets of smooth functions, converging to the δ-distribution. The limit process should be related to the usual limit in distribution spaces, if the resulting object is required to be a distribution. Since most of these approaches extend the multiplication of continuous functions, regarded as distributions, these products cannot be defined for all pairs of distributions due to a well-known result of Schwartz [14]. The purpose of this paper is to give a reasonable enlargement of the definition area of some of these partial mappings, namely of products equivalent to the so-called Kamiński or Colombeau product.

In Section 1, multiplication by harmonic representations of distributions is defined, using only arguments of standard analysis. This localizable multiplication, introduced by Li Banghe in [8], [9] with the aid of methods of non-standard analysis, is in the one-dimensional case equivalent to the Tillmann product (see [16]) which is defined by analytic representations, but does not have a localization property in the multi-dimensional case.

Section 2 contains a proof, that multiplication by harmonic representations is an extension of a certain product defined only with radial δ-sequences and, therefore, is a strict extension of the Kamiński product by a given distinguishing example. This is an extended positive answer to a problem posed by Oberguggenberger in [13].

Throughout this paper, Ω denotes a non-empty, open subset of $\mathbb{R}^n$; $n \in \mathbb{N} \setminus \{0\}$ means the dimension. Elements of $\mathbb{R}^{n+1}$ are denoted by (x, y) for $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$, a lower index always refers to the last coordinate, for example $\mathbb{R}^{n+1}_{>0} = \{(x, y) \mid x \in \mathbb{R}^n, y > 0\}$. $\mathcal{E}(\Omega)$ (resp. $\mathcal{D}(\Omega)$) is the space of C^∞-functions in Ω

(with compact support) with its usual topology, where the letter Ω is sometimes omitted in case of $\Omega = \mathbb{R}^n$. The dual space $\mathcal{D}'(\Omega)$, equipped with the strong topology, denotes the space of distributions; $\mathcal{E}'(\Omega)$ is identified with the space of distributions with compact support. The mapping $\mathrm{Dis}\colon L^1_{\mathrm{loc}} \longrightarrow \mathcal{D}'$ denotes the usual embedding of locally integrable functions into the distribution space.

1. Multiplication by harmonic representations

1.1 Definition. A *harmonic representation of* $S \in \mathcal{D}'(\mathbb{R}^n)$ *in* Ω is a harmonic function $\sigma\colon \mathbb{R}^{n+1}_{>0} \longrightarrow \mathbb{C}$ which satisfies

$$\lim_{y\to 0_+} \int_\Omega \sigma(x,y)\varphi(x)\,dx = S(\bar{\varphi}) \tag{1.1}$$

for all $\varphi \in \mathcal{D}(\Omega)$, where $\bar{\varphi}$ denotes the trivial extension of φ to $\mathbb{R}^n$.

Since $\mathcal{D}'(\Omega)$ is a (semi-)Montel space, it is equivalent for the existence of the limes (1.1) to require $\lim_{y\to 0_+} \mathrm{Dis}\,\sigma(\cdot,y) = S_{|\Omega}$ in $\mathcal{D}'(\Omega)$. Hence harmonic representations of distributions (in $\mathbb{R}^n$) are solutions of the Dirichlet problem for the upper half-space $\mathbb{R}^{n+1}_{>0}$ with distributional boundary conditions. As in the classical case, the *Poisson kernel* $P(x,y) := p_y(x)$ with

$$p_y(x) := \frac{2}{\omega_{n+1}} \frac{y}{(y^2 + \|x\|^2)^{(n+1)/2}} \qquad ((x,y) \in \mathbb{R}^{n+1}_{\neq 0}) \tag{1.2}$$

(ω_{n+1} denotes the volume of the surface of the unit sphere in $\mathbb{R}^{n+1}$) plays an important rôle. P is harmonic with $\int p_y = y^{-n} \int p_1(\cdot/y) = 1$ for all $y \neq 0$. One proves by induction over $\varkappa \in \mathbb{N}^n$

$$\partial^\varkappa p_y(x) = y^{-n} \sum_{\substack{\frac{|\varkappa|}{2} \le i \le |\varkappa| \\ i\in\mathbb{N}}} \frac{1}{\left(\frac{\|x\|^2}{y^2} + 1\right)^{\frac{n+1}{2}+i}} \cdot \frac{a_i(x)}{y^{2i}}$$

with some homogeneous polynomials a_i of degree $2i - |\varkappa|$. Hence $\partial^\varkappa p_y \in C^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ for all $y \neq 0$ and $\varkappa \in \mathbb{N}^n$. Since $|a_i(x)/y^{2i}| \le C_\varkappa y^{-|\varkappa|}(1 + \|x\|^2/y^2)^{i-|\varkappa|/2}$, it follows that

$$|\partial_x^\varkappa (y^{-(n+1)} p_1(x/y))| \le C_\varkappa (\|x\|^2 + y^2)^{-(n+1+|\varkappa|)/2} \tag{1.3}$$

is bounded away from zero, independently of $y > 0$.

The following proposition ensures the existence of harmonic representations of a large class of distributions, in particular of those with compact support, hence for each distribution a 'local harmonic representation'.

1.2 Proposition. *Let $S \in \mathcal{D}'(\mathbb{R}^n)$ with $S = \sum_{|\varkappa|\le k} \partial^\varkappa \operatorname{Dis} f_\varkappa$, $f_\varkappa \in L^1(\mathbb{R}^n)$. Then*

$$\sigma(x,y) := (S * p_y)(x) = \sum_{|\varkappa|\le k} (f_\varkappa * (\partial^\varkappa p_y))(x)$$

for $(x,y) \in \mathbb{R}^{n+1}_{\neq 0}$ is a harmonic representation of S with $\sigma(x,y) = -\sigma(x,-y)$. σ can be continued as a harmonic function to $(\mathbb{R}^n \setminus \operatorname{supp} S) \times \{0\}$ satisfying $\lim_{y\to 0_+} \sigma(x,y) = 0$ for $x \notin \operatorname{supp} S$.

If $S = \operatorname{Dis} f$, $f \in L^1$, this is immediate by using the reflexion principle of harmonic functions. From

$$\int \varphi(x)\partial_x^\varkappa \sigma(x,y)\,dx = (-1)^{|\varkappa|} \int \sigma(x,y)\partial^\varkappa \varphi(x)\,dx$$
$$\xrightarrow{y\to 0_+} (-1)^{|\varkappa|} S(\partial^\varkappa\varphi) = \partial^\varkappa S(\varphi).$$

for $\varphi \in \mathcal{D}$ one concludes, that a derivated harmonic representation of a certain distribution is again a harmonic representation of the derivated distribution.

In general, the convolution $S * p_y$ is not defined. One may use a partition of unity and the real analyticity of harmonic functions for a constructive proof (Liu Shangping [11]), or a general existence result for certain hypoelliptic operators (Langenbruch [7]) to obtain the following theorem.

1.3 Theorem. *For every $S \in \mathcal{D}'(\mathbb{R}^n)$ there exists a harmonic representation σ, harmonic in $\mathbb{R}^{n+1} \setminus (\operatorname{supp} S \times \{0\})$, and satisfying $\sigma(x,y) = -\sigma(x,-y)$ for $(x,y) \in \mathbb{R}^{n+1}_{\neq 0}$.*

If σ_1 and σ_2 are harmonic representations of a distribution $S \in \mathcal{D}'(\mathbb{R}^n)$, then $\sigma_1 - \sigma_2$ is a harmonic representation of the zero distribution, and even an entire harmonic function:

1.4 Theorem. *For each harmonic representation σ of the zero distribution in Ω there exists a harmonic continuation $\bar\sigma$ of σ onto $\bar\Omega := (\Omega \times \{0\}) \cup \mathbb{R}^{n+1}_{\neq 0}$ satisfying $\bar\sigma(x,y) = -\bar\sigma(x,-y)$ for $(x,y) \in \Omega \times \mathbb{R}$.*

1.5 Corollary (Li Banghe, Li Yaqing [10]). *Two harmonic representations of a distribution differ only by a function σ, which is harmonic in $\mathbb{R}^{n+1}$ and satisfies $\sigma(x,y) = -\sigma(x,-y)$ and in particular $\sigma(x,0) = 0$.* □

Proof of Theorem 1.4: Let $\Omega_0 := \Omega \times (-\infty,\infty)$. The linear functional

$$S(\varphi) := \int_0^\infty \int_\Omega \sigma(x,y)\varphi(x,y)\,dx\,dy - \int_{-\infty}^0 \int_\Omega \sigma(x,-y)\varphi(x,y)\,dx\,dy$$

is well-defined for $\varphi \in \mathcal{D}(\Omega_0)$: The inner integrals are continuous functions with limit 0 for $y \to 0$ (by assumption), because $\{\varphi(\cdot,y) \mid y \in \mathbb{R}\}$ is bounded in $\mathcal{D}(\Omega)$.

A similar argument establishes the continuity of S on $\mathcal{D}(\Omega_0)$, since all elements of a converging sequence have their support in one compact subset of Ω_0. From Green's formula follows for all $\varepsilon > 0$

$$\int_\varepsilon^\infty \int_\Omega \sigma(x,y)\Delta_{x,y}\varphi(x,y)\,dx\,dy = \int_\Omega \varphi(x,\varepsilon)\sigma_y(x,\varepsilon)\,dx - \int \sigma(x,\varepsilon)\varphi_y(x,\varepsilon)\,dx.$$

As above, the second integral converges to 0 for $\varepsilon \to 0_+$. One obtains

$$S(\Delta\varphi) = \lim_{\varepsilon\to 0_+} \int_\Omega \sigma_y(x,\varepsilon)\big(\varphi(x,\varepsilon) - \varphi(x,-\varepsilon)\big)\,dx = 0$$

by Lebesgue's Theorem, since the essential integration domain is bounded. Hence $\Delta S = 0$ in $\mathcal{D}'(\Omega_0)$.

By Weyl's Lemma for distributions (see Schwartz [15, p. 216]), there is a harmonic function σ_0 on Ω_0 with $S = \operatorname{Dis}\sigma_0$. It follows $\sigma_0 = \sigma$ in $\Omega \times (0,\infty)$, resp. $\sigma_0(x,y) = -\sigma(x,-y)$ for $(x,y) \in \Omega \times (-\infty, 0)$ by continuity. σ_0 can be continued by $\sigma(x,y)$ on $\mathbb{R}^{n+1}_{>0}$ and by $-\sigma(x,-y)$ on $\mathbb{R}^{n+1}_{<0}$ to a function $\bar\sigma$, harmonic on $\bar\Omega$, which completes the proof. □

Denoting the vector space of harmonic representations in $\mathcal{D}'(\mathbb{R}^n)$ by $\mathcal{H}_n$ and the subspace of harmonic representations of the zero distribution by $\mathcal{H}_0$, there is an algebraic isomorphism

$$\mathcal{D}'(\mathbb{R}^n) \cong \mathcal{H}_n/\mathcal{H}_0$$

which may by extended topologically (see Langenbruch [7]). This motivates

1.6 Definition. Let $S, T \in \mathcal{D}'(\mathbb{R}^n)$ with harmonic representations σ and τ. Their *product by harmonic representations* is defined by

$$S \cdot T := \lim_{y\to 0_+} \operatorname{Dis}\big(\sigma(\cdot,y)\tau(\cdot,y)\big), \tag{1.4}$$

if this limit exists in $\mathcal{D}'(\mathbb{R}^n)$.

Again, due to the Montel property, it is equivalent to require the existence of the weak limit in (1.4), that is

$$S \cdot T(\varphi) := \lim_{y\to 0_+} \int \sigma(x,y)\tau(x,y)\varphi(x)\,dx \tag{1.5}$$

for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$.

This product has been introduced by Li Banghe in [8] for $n = 1$ and for higher dimensions in [9] applying methods of non-standard analysis. The resulting product was defined for all pairs of distributions, but the range was extended to functions, mapping from $\mathcal{D}'(\mathbb{R}^n)$ into a non-standard model of the complex numbers.

The product (1.4) is independent from the chosen harmonic representation:

1.7 Theorem. *Let σ_1, σ_2 and τ_1, τ_2 be harmonic representations in Ω of distributions $S, T \in \mathcal{D}'(\mathbb{R}^n)$ respectively. Then*

$$\int_\Omega \sigma_1(x,y)\tau_1(x,y)\varphi(x)\,dx - \int_\Omega \sigma_2(x,y)\tau_2(x,y)\varphi(x)\,dx \xrightarrow{y\to 0_+} 0$$

holds for every $\varphi \in \mathcal{D}(\mathbb{R}^n)$ with $\operatorname{supp}\varphi \subseteq \Omega$.

Proof: Let $(y_i)_i \in {\mathbb{R}_{>0}}^{\mathbb{N}}$ with $(y_i)_i \to 0$. From Theorem 1.4 one infers the existence of harmonic functions σ, τ on $\bar{\Omega} = (\Omega \times \{0\}) \cup \mathbb{R}^{n+1}_{\neq 0}$ with $\sigma(x,0) = 0 = \tau(x,0)$ for all $x \in \Omega$ as well as $\sigma = \sigma_1 - \sigma_2$ and $\tau = \tau_1 - \tau_2$. For every $x \in \Omega$ and $i \in \mathbb{N}$ holds

$$\begin{aligned}\int_\Omega \sigma_1(x,y_i)\tau_1(x,y_i)\varphi(x)\,dx - \int_\Omega \sigma_2(x,y_i)\tau_2(x,y_i)\varphi(x)\,dx = \\ \int_\Omega \sigma(x,y_i)\tau(x,y_i)\varphi(x)\,dx + \int_\Omega \sigma(x,y_i)\tau_2(x,y_i)\varphi(x)\,dx + \\ \int_\Omega \sigma_2(x,y_i)\tau(x,y_i)\varphi(x)\,dx.\end{aligned}$$

The first summand converges to 0 for $i \to \infty$ due to the continuity of σ and τ. Furthermore, the sequence $(\sigma(\cdot, y_i)\varphi_{\,|\,\Omega})_i$ converges to 0 in $\mathcal{D}(\Omega)$. Since $\operatorname{Dis}(\tau_2(\cdot, y_i))$ converges to $T_{|\Omega}$ in $\mathcal{D}'(\Omega)$, the second summand (as well as the third one) converges for $i \to \infty$ to $T(0) = 0$.

□

Obviously, multiplication by harmonic representations is commutative and bilinear. In particular,

$$\lim_{y\to 0_+} \int_\Omega \sigma(x,y)\tau(x,y)\varphi(x) = 0$$

holds for each harmonic representation σ of the zero distribution in Ω and any harmonic representation of a distribution T in Ω, since this is true for the special representation $\sigma \equiv 0$. This implies a localization property: If distributions S_1, S_2 coincide in Ω, the product $S_1 \cdot T$ exists locally (i.e. the limit in (1.5) exists for all $\varphi \in \mathcal{D}(\Omega)$), if and only if this is true for $S_2 \cdot T$, in which case both products coincide. This property implies the validity of the 'support formula' $\operatorname{supp}(S \cdot T) \subseteq \operatorname{supp} S \cap \operatorname{supp} T$ (see Oberguggenberger [13, p. 38]). Presuming the existence of harmonic representations of distributions, an equivalent definition for the product (1.4) can be given:

1.8 Corollary. *$S \cdot T$ exists if and only if for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$ there is a function $\chi \in \mathcal{D}(\mathbb{R}^n)$, $\chi \equiv \mathbf{1}$ on a neighbourhood of* $\operatorname{supp}\varphi$, *such that*

$$\int \big((\chi S) * p_y\big)\big((\chi T) * p_y\big)\varphi \tag{1.6}$$

converges for $y \to 0_+$. In this case, the limit (1.6) *is independent of χ and equals $S \cdot T(\varphi)$.* $\square$

Historically, multiplication by harmonic representations arose from the inadequacy of multiplication by analytic representations in the multidimensional case. An analytic representation of a distribution $S \in \mathcal{D}'(\mathbb{R})$ is a function σ, analytic in $\mathbb{C} \setminus \mathbb{R}$, satisfying

$$\lim_{y \to 0_+} \operatorname{Dis}\bigl(\sigma(x+iy) - \sigma(x-iy)\bigr) = S$$

in $\mathcal{D}'(\mathbb{R})$. If σ is an analytic representation of a distribution S, the function $\bar{\sigma}(x,y) := \sigma(x+iy) - \sigma(x-iy)$ for $(x,y) \in \mathbb{R} \times \mathbb{R}_{\neq 0}$ is a harmonic representation of S. Conversely, for a harmonic representation σ of S with harmonic conjugate τ defined by

$$\tau(x,y) := \int_{y_0}^{y} \frac{\partial \sigma}{\partial x}(x,t)\,dt - \int_{x_0}^{x} \frac{\partial \sigma}{\partial y}(t,y_0)\,dt, \qquad (x,y) \in \mathbb{R} \times \mathbb{R}_{\neq 0},$$

(where (x_0,y_0) is a fixed element of $\mathbb{R} \times \mathbb{R}_{\neq 0}$), the function $\bar{\sigma}$

$$\bar{\sigma}(x+iy) := \begin{cases} \frac{1}{2}(\sigma(x,y) + i\tau(x,y)) & \text{for } y > 0 \\ \frac{1}{2}(-\sigma(x,-y) + i\tau(x,-y)) & \text{for } y < 0, \end{cases}$$

is analytic in $\mathbb{C} \setminus \mathbb{R}$ and $\bar{\sigma}(x+iy) - \bar{\sigma}(x-iy) = \sigma(x,y)$ holds for $y > 0$. Hence $\bar{\sigma}$ is an analytic representation of S. Therefore, the corresponding product by analytic representations

$$S \cdot T := \lim_{y \to 0_+} \operatorname{Dis}\bigl((\sigma(x+iy) - \sigma(x-iy))(\tau(x+iy) - \tau(x-iy))\bigr),$$

of distributions from $\mathcal{D}'(\mathbb{R})$ is equivalent to (1.4). Multiplication defined by analytic representations was introduced by Tillmann [16], and can be generalized to higher dimensions, where analytic representations are analytic functions in $(\mathbb{C} \setminus \mathbb{R})^n$, but then with the disadvantage of not having the localization property: Itano has shown in [2], that for $n = 2$ the product of two Dirac measures of different points is not the zero distribution, hence the support formula as a consequence of the localization property is not valid.

2. Some extension results

2.1 Definition. A sequence $(\sigma_i)_i \in \mathcal{D}(\mathbb{R}^n)^{\mathbb{N}}$ with $\sigma_i \geq 0$ and $\int \sigma_i = 1$ for all $i \in \mathbb{N}$ satisfying

(2.1) $\qquad \exists\, \sigma \in \mathcal{D}(\mathbb{R}^n)\, \exists\, (s_i)_i \in (0,\infty)^{\mathbb{N}} : (s_i)_i \to 0_+ \ : \sigma_i = s_i^{-n} \sigma(\cdot/s_i)$

is called a *model sequence*. A model sequence is called *radial*, if the defining function σ in (2.1) is a radial function.

Herein, a function f is radial (or rotationally symmetric), if it is invariant under all orthogonal transformations of $\mathbb{R}^n$, or equivalently, if there is a function g satisfying $f(x) = g(\|x\|)$ for all $x \in \mathbb{R}^n$.

Model sequences are special types of mollifiers, namely δ-sequences, as they converge as distributions to the δ-distribution. They were introduced by Kamiński in [6] in context with the following definition:

2.2 Definition. The *Kamiński product* $S \odot T$ of $S, T \in \mathcal{D}'$ is defined by

$$\lim_{i\to\infty} \operatorname{Dis}\big((S * \sigma_i)(T * \sigma_i)\big), \tag{2.2}$$

if this limes exists in $\mathcal{D}'$ for all model sequences $(\sigma_i)_i$ and does not depend on the chosen sequence.

The Kamiński product is sometimes also called Colombeau product due to a result of Jelínek [5], since it exists for $S, T \in \mathcal{D}'$ if and only if the product of S and T in certain Colombeau algebras admits a so-called associated distribution, namely $S \odot T$. It is well-known, that this partial multiplication is a generalization of the multiplication of continuous functions, regarded as distributions, of the multiplications $\mathcal{E} \cdot \mathcal{D}'$ and $\mathcal{O}_M \cdot \mathcal{S}'$, and furthermore $L^p \cdot L^q$, $1/p + 1/q = 1$, or even more general, of the Sobolev spaces $W^{k,p} \cdot W^{-k,q}$ which are based on duality.

The Kamiński product for $S, T \in \mathcal{D}'$ exists if and only if

$$\lim_{i\to\infty} \frac{1}{2} \operatorname{Dis}\big((S * \sigma_i)(T * \tau_i) + (S * \tau_i)(T * \sigma_i)\big) \tag{2.3}$$

exists in $\mathcal{D}'$ for all model sequences $(\sigma_i)_i, (\tau_i)_i$ and does not depend on the chosen sequences, in which case it equals $S \odot T$. This can be obtained by using arguments of Wawak [17] and Jelínek [4]; for a complete proof see [1]. Furthermore, one may enlarge the class of mollifiers in (2.2) or (2.3) by requiring only

$$\operatorname{supp} \sigma_i \to \{0\} \text{ and } \forall\, \varkappa \in \mathbb{N}^n\, \exists\, C_\varkappa > 0\, \forall\, i \in \mathbb{N} : \int \|x\|^{|\varkappa|} |\partial^\varkappa \sigma_i(x)|\, dx \le C_\varkappa \tag{2.4}$$

instead of (2.1).

2.3 Definition. The *radial product* $S \overset{\text{rad}}{\odot} T$ of $S, T \in \mathcal{D}'$ is defined by

$$\lim_{i\to\infty} \frac{1}{2} \operatorname{Dis}\big((S * \sigma_i)(T * \tau_i) + (S * \tau_i)(T * \sigma_i)\big) \tag{2.5}$$

if this limit exists in $\mathcal{D}'$ for all radial model sequences $(\sigma_i)_i, (\tau_i)_i$ and does not depend on the chosen sequences.

Equivalently, one may require the existence of the weak limit in (2.5), which in view of the support properties of the delta sequences implies the localization property noted in Section 1. The same applies obviously for the Kamiński product.

From the preceding discussion, it follows that the radial product is an extension of the Kamiński product. At least in the multidimensional case, the contrary is not true:

2.4 Example. For $S := \operatorname{Dis} H \otimes \underbrace{\delta \otimes \cdots \otimes \delta}_{(n-1)\text{-times}}$ and $T := \delta \otimes \underbrace{\operatorname{Dis} H \otimes \cdots \otimes \operatorname{Dis} H}_{(n-1)\text{-times}}$, where $n \geq 2$ and H denotes the Heaviside function on $\mathbb{R}$,

$$\lim_{i\to\infty} \operatorname{Dis}\big((S * \sigma_i)(T * \tau_i)\big) = 2^{-n} \underbrace{\delta \otimes \cdots \otimes \delta}_{n\text{-times}}$$

holds for all radial model sequences $(\sigma_i)_i, (\tau_i)_i$, but the Kamiński product $S \odot T$ is not defined.

Proof: Let $\sigma_i := s_i^{-n}\sigma(\cdot/s_i)$ and $\tau_i := t_i^{-n}\tau(\cdot/t_i)$ for radial functions $\sigma, \tau \in \mathcal{D}$ with $\int \sigma = 1 = \int \tau$. Suppose $(s_i/t_i)_i \xrightarrow{i\to\infty} r \in \mathbb{R}$. For $\varphi \in \mathcal{D}$ one has

$$\begin{aligned}
&\operatorname{Dis}\big((S * \sigma_i) \cdot (T * \tau_i)\big)(\varphi) = \\
&= \int_{\mathbb{R}^n} \int_0^\infty \cdots \int_0^\infty \sigma_i(x_1, \ldots, x_n) \cdot \tau_i(x_1 + y_1, x_2 - y_2, \ldots, x_n - y_n) \cdot \\
&\qquad\qquad \cdot \varphi(x_1 + y_1, x_2, \ldots, x_n)\, dy_1 \ldots dy_n\, d(x_1, \ldots, x_n).
\end{aligned}$$

By substitution $\left(\frac{x_1}{s_i}, \ldots, \frac{x_n}{s_i}, \frac{y_1}{t_i}, \ldots, \frac{y_n}{t_i}\right) \mapsto (x_1, \ldots, x_n, y_1, \ldots, y_n)$ this equals

$$\begin{aligned}
&\int_{\mathbb{R}^n} \int_0^\infty \cdots \int_0^\infty \sigma(x_1, \ldots, x_n) \cdot \tau(x_1 \frac{s_i}{t_i} + y_1, x_2 \frac{s_i}{t_i} - y_2, \ldots, x_n \frac{s_i}{t_i} - y_n) \cdot \\
&\qquad\qquad \cdot \varphi(s_i x_1 + t_i y_1, s_i x_2, \ldots, s_i x_n)\, dy_1 \ldots dy_n\, d(x_1, \ldots, x_n),
\end{aligned}$$

an integral, which converges to $\varphi(0, \ldots, 0) \cdot I$ for $i \to \infty$, where

$$I := \int_{\mathbb{R}^n} \underbrace{\int_0^\infty \cdots \int_0^\infty}_{n\text{-times}} \psi(x_1, \ldots, x_n, y_1, \ldots, y_n)\, dy_1 \ldots dy_n\, dx_1 \ldots dx_n$$

and

$$\psi(x_1, \ldots, x_n, y_1, \ldots, y_n) := \sigma(x_1, \ldots, x_n)\tau(rx_1 + y_1, rx_2 - y_2, \ldots, rx_n - y_n).$$

Since σ and τ are radial,

$$I + I = \int_{\mathbb{R}^{n+1}} \underbrace{\int_0^\infty \cdots \int_0^\infty}_{(n-1)\text{-times}} \psi(x_1, \ldots, x_n, y_1, \ldots, y_n)\, dy_1 \ldots dy_n\, dx_1 \ldots dx_n$$

results from the substitution $(x_1, y_1) \mapsto -(x_1, y_1)$. Therefore,

$$2^n I = \int_{\mathbb{R}^{2n}} \psi = \Big(\int \sigma\Big) \cdot \Big(\int \tau\Big) = 1$$

is straightforward. If $(t_i/s_i)_i$ converges in $\mathbb{R}$, a similar argumentation applies. Now, the assumption that

$$\lim_{i\to\infty} \operatorname{Dis}\big((S * \sigma_i)(T * \tau_i)\big)(\varphi) \neq 2^{-n}\varphi(0, \ldots, 0)$$

for at least one function $\varphi \in \mathcal{D}(\mathbb{R}^n)$ yields the existence of a strictly increasing sequence $(i_j)_j \in \mathbb{N}^{\mathbb{N}}$ such that no subsequence of $\big(\operatorname{Dis}\big((S * \sigma_{i_j})(T * \tau_{i_j})\big)(\varphi)\big)_j$ converges to $2^{-n}\delta(\varphi)$. But at least one of the non-negative sequences $(s_{i_j}/t_{i_j})_j$ and $(t_{i_j}/s_{i_j})_j$ has a converging subsequence, which generates a contradiction.

Finally, the Kamiński product $S \odot T$ is not defined, because of the fact that the integral I depends on the choice of the non-radial functions $\sigma = \tau$ in (2.3) taking $(s_i/t_i)_i \to 0$. □

Referring to the multiplication by harmonic representations, the Poisson kernel p_y as a locally regularizing element does not have compact support. Therefore, from the existence of the Kamiński product one cannot infer directly the existence of the product by harmonic representations. However, Oberguggenberger [12] has shown that in the one-dimensional case the equivalent Tillmann product by analytic representations extends the Kamiński product. In the part of his proof concerning the local existence of this product, it suffices to take distributions having compact support, in particular finite order, and to construct a δ-net $(\sigma_\varepsilon)_\varepsilon$ satisfying (2.4) for $\varepsilon > 0$, which approximates the Poisson kernel p_ε uniformly on $\mathbb{R}$ up to a certain derivation order.

In [13, § 27] Oberguggenberger conjectured, that in any dimension $n \in \mathbb{N}_{\geq 1}$ the multiplication by harmonic representations extends the Kamiński product. In the following it is proved that multiplication by harmonic representations even extends the radial product.

In the sequel, for $S, T \in \mathcal{E}'(\mathbb{R}^n)$, $\varphi \in \mathcal{D}$, $\varepsilon, \delta > 0$ and $\sigma, \tau \in C^\infty$ let $F_{(\varepsilon,\delta)}(\sigma, \tau)$ denote

$$\text{(2.6)} \quad \begin{aligned} &\frac{1}{2}\int \Big((S * \sigma_\varepsilon)\cdot(T * \tau_\delta) + (S * \tau_\delta)\cdot(T * \sigma_\varepsilon)\Big)(x)\varphi(x)\,dx \\ &\quad = \frac{1}{2}\big(S_y \otimes T_z\big)\Big(\int \big(\sigma_\varepsilon(x-y)\tau_\delta(x-z) + \sigma_\varepsilon(x-z)\tau_\delta(x-y)\big)\varphi(x)\,dx\Big) \end{aligned}$$

(equality follows from Fubini's theorem), where $\psi_\alpha := \alpha^{-n}\psi(\cdot/\alpha)$ for $\psi \in C^\infty$ and $\alpha > 0$.

2.5 *Remark.* The bilinear form $F_{(\varepsilon,\delta)}$ is separately continuous on $\mathcal{E}\times\mathcal{E}$. For $\lambda > 0$, $\tau \in C^\infty \cap L^1$ and a bounded subset B of $\mathcal{E}$,

$$\{F_{(\varepsilon,\delta)}(\sigma, \tau) \mid \varepsilon > \lambda, \delta > 0, \sigma \in B\}$$

is bounded in $\mathbb{C}$. The latter follows from $\int \tau_\delta = \int \tau_1$ and the second equation in (2.6) with integration by parts.

2.6 *Remark.* Let E be one of the spaces $\mathcal{D}$, $\mathcal{E}$ or $\mathcal{D}_K$, which is the space of C^∞-functions, having their support in the compact subset K of $\mathbb{R}^n$. Denote by E^{rad} the subspace of radial functions in E. This is a closed subspace, since a converging net in E converges pointwise.

2.7 Theorem. *Let $\tau \in \mathcal{E}^{\mathrm{rad}} \cap L^1$ and $\chi \in \mathcal{D}^{\mathrm{rad}}$ with $\chi(x) = 1$ for $\|x\| \le 1$ and $\chi(x) = 0$ for $\|x\| \ge 2$. Suppose that for all nets $(t_\varepsilon)_\varepsilon \in (\mathbb{R}_{>0})^{(0,\infty)}$ with $(t_\varepsilon)_\varepsilon \to 0$ for $\varepsilon \to 0_+$ and all functions $\sigma \in \mathcal{D}^{\mathrm{rad}}$*

$$\lim_{\varepsilon\to 0_+} F_{(\varepsilon,t_\varepsilon)}(\sigma,\tau) = (S \overset{\mathrm{rad}}{\odot} T)(\varphi)\cdot \int \sigma \cdot \int \tau \tag{2.7}$$

holds. Then for all nets $(t_\varepsilon)_\varepsilon \in (\mathbb{R}_{>0})^{(0,\infty)}$ with $(t_\varepsilon)_\varepsilon \to 0$ for $\varepsilon \to 0_+$

$$\lim_{\varepsilon\to 0_+} F_{(\varepsilon,t_\varepsilon)}((\mathbf{1}-\chi)p,\tau) = (S \overset{\mathrm{rad}}{\odot} T)(\varphi)\cdot \Big(\int (\mathbf{1}-\chi)p\Big)\cdot \int \tau$$

holds, where $p := p_1 = P(\cdot,1)$ denotes the Poisson kernel (1.2).

PROOF: Let $(t_\varepsilon)_\varepsilon \in (\mathbb{R}_{>0})^{(0,\infty)}$ with $(t_\varepsilon)_\varepsilon \to 0$ for $\varepsilon \to 0_+$ and $p(k,\cdot)$ for $k \in \mathbb{N}$ the function $p(k,x) := p(x)\big(\chi(x/2^{k+1}) - \chi(x/2^k)\big)$. Thus,

$$(1-\chi)p = \sum_{k=0}^{\infty} p(k,\cdot), \tag{2.8}$$

where the right side means the limit in $\mathcal{E}$. Because of the separate continuity of $F_{(\varepsilon,t_\varepsilon)}$,

$$F_{(\varepsilon,t_\varepsilon)}((\mathbf{1}-\chi)p,\tau) = \sum_{k=0}^{\infty} F_{(\varepsilon,t_\varepsilon)}(p(k,\cdot),\tau). \tag{2.9}$$

Now, if summation and the limit for $\varepsilon \to 0$ are interchangeable, then the theorem is established by assumption (2.7), since equality (2.8) holds also in L^1.

For $K := \{x \in \mathbb{R}^n \mid 1 \le \|x\| \le 4\}$ the set of functions[1]

$$\{x \longmapsto 2^{k(n+1)} p(k, 2^k x) \mid k \in \mathbb{N}\} \tag{2.10}$$

is bounded (and contained) in $\mathcal{D}_K^{\mathrm{rad}}$, because, confirming (1.3), each derivative of the functions $\varepsilon^{-(n+1)} p(\cdot/\varepsilon)$, $\varepsilon > 0$ is bounded away from zero. The set

$$\{F_{(\varepsilon,t_{(\varepsilon/2^k)})}(2^{k(n+1)} p(k, 2^k\cdot),\tau) \mid k \in \mathbb{N}, \varepsilon > 0\} \tag{2.11}$$

[1] This splitting of the Poisson kernel has been adapted from Jelínek [3].

is bounded in $\mathbb{C}$: Otherwise, there are sequences $(\varepsilon_l)_l$, $(k_l)_l \in \mathbb{R}^{\mathbb{N}}$ satisfying

$$\left|F_{(\varepsilon_l, t_{(\varepsilon_l/2^{k_l})})}(2^{k_l(n+1)} p(k_l, 2^{k_l}\cdot), \tau)\right| \xrightarrow{l\to\infty} \infty. \tag{2.12}$$

Let $\bar{\varepsilon} := \liminf_{l\to\infty} \varepsilon_l$ in $[0,\infty]$ and without loss of generality $(\varepsilon_l)_l \to \bar{\varepsilon}$. In case of $\bar{\varepsilon} > 0$ a contradiction to (2.12) would follow according to Remark 2.5, because the set (2.10) is also bounded in $\mathcal{E}$. Since $(\varepsilon_l)_l \to 0$ and $(t_{(\varepsilon_l/2^{k_l})})_l \to 0$ for $l \to \infty$, by assumption the set of distributions F_l, $l \in \mathbb{N}$, defined by

$$F_l(\sigma) := F_{(\varepsilon_l, t_{(\varepsilon_l/2^{k_l})})}(\sigma, \tau) \qquad (\sigma \in \mathcal{D}_K^{\mathrm{rad}}),$$

is weakly bounded on $\mathcal{D}_K^{\mathrm{rad}}$, a Fréchet space confirming Remark 2.6. Applying the uniform boundedness principle, $\{F_l \mid l \in \mathbb{N}\}$ would be equicontinuous, hence bounded on bounded sets of $\mathcal{D}_K^{\mathrm{rad}}$ in contradiction to (2.12).

A simple calculation for $\varepsilon > 0$ and $k \in \mathbb{N}$ gives

$$[(2^{k(n+1)} p(k, 2^k\cdot))_\varepsilon = 2^k (p(k,\cdot))_{\bar{\varepsilon}}$$

with $\bar{\varepsilon} := \varepsilon/2^k$. From $F_{(\varepsilon,\delta)}(\sigma,\tau) = F_{(1,1)}(\sigma_\varepsilon, \tau_\delta)$, the bilinearity of $F_{(1,1)}$, and the boundedness of the set (2.11), by replacing $\varepsilon/2^k$ by $\bar{\varepsilon}$ one obtains

$$\left|F_{(\bar{\varepsilon}, t_{\bar{\varepsilon}})}(p(k,\cdot), \tau)\right| \le C 2^{-k}$$

for some $C > 0$ which is independent of $\bar{\varepsilon} \in (0,\infty)$ and $k \in \mathbb{N}$. Hence the series (2.9) is majorized independently of ε, summation and passing to the limit can be interchanged, and the proof is complete. □

2.8 Corollary. *Let $S, T \in \mathcal{D}'(\mathbb{R}^n)$. If their radial product $S \overset{\mathrm{rad}}{\odot} T$ exists, then the product by harmonic representations $S \cdot T$ exists and equals $S \overset{\mathrm{rad}}{\odot} T$.*

Proof: Let $\varphi \in \mathcal{D}$ and $\chi \in \mathcal{D}^{\mathrm{rad}}$ with $\chi(x) = 1$ for $\|x\| \le 1$ and $\chi(x) = 0$ for $\|x\| \ge 2$. By the localization property of both products, S, T are assumed to have compact support. The bilinear form $F_{(\varepsilon,\delta)}$ for S, T, and φ is defined according to (2.6). Equality

$$\lim_{\varepsilon\to 0_+} F_{(\varepsilon, t_\varepsilon)}(\sigma,\tau) = (S \overset{\mathrm{rad}}{\odot} T)(\varphi) \cdot \Big(\int \sigma\Big) \cdot \Big(\int \tau\Big) \tag{2.13}$$

for all $\sigma, \tau \in \mathcal{D}^{\mathrm{rad}}$ is easily inferred from the existence of $(S \overset{\mathrm{rad}}{\odot} T)(\varphi)$ for all nets $(t_\varepsilon)_\varepsilon \in (\mathbb{R}_{>0})^{(0,\infty)}$ with $(t_\varepsilon)_\varepsilon \to 0$ for $\varepsilon \to 0_+$, because this is true for all subsequences $(\varepsilon_i, t_{\varepsilon_i})_i$. An application of Theorem 2.7 yields

$$\lim_{\varepsilon\to 0_+} F_{(\varepsilon, t_\varepsilon)}((1-\chi)p, \tau) = (S \overset{\mathrm{rad}}{\odot} T)(\varphi) \cdot \Big(\int (1-\chi)p\Big) \cdot \Big(\int \tau\Big) \tag{2.14}$$

and, by another subsequence argumentation for $F_{(\varepsilon,\delta)}(\sigma,\tau) = F_{(\delta,\varepsilon)}(\tau,\sigma)$,

$$\lim_{\varepsilon\to 0_+} F_{(\varepsilon,t_\varepsilon)}(\tau,(1-\chi)p) = (S \overset{\mathrm{rad}}{\odot} T)(\varphi)\cdot\Big(\int \tau\Big)\cdot\Big(\int (1-\chi)p\Big) \tag{2.15}$$

for all such nets $(t_\varepsilon)_\varepsilon$ and all $\tau \in \mathcal{D}^{\mathrm{rad}}$. Again, by Theorem 2.7

$$\lim_{\varepsilon\to 0_+} F_{(\varepsilon,t_\varepsilon)}((1-\chi)p,(1-\chi)p) = (S \overset{\mathrm{rad}}{\odot} T)(\varphi)\Big(\int (1-\chi)p\Big)\Big(\int (1-\chi)p\Big)$$

and now with $\sigma = \tau = \chi p$ in (2.13)–(2.15)

$$\begin{aligned} F_{(\varepsilon,t_\varepsilon)}(p,p) = {} & F_{(\varepsilon,t_\varepsilon)}(\chi p,\chi p) + F_{(\varepsilon,t_\varepsilon)}(\chi p,(1-\chi)p) + \\ & + F_{(\varepsilon,t_\varepsilon)}((1-\chi)p,\chi p) + F_{(\varepsilon,t_\varepsilon)}((1-\chi)p,(1-\chi)p) \\ & \xrightarrow{\varepsilon\to 0_+} (S \overset{\mathrm{rad}}{\odot} T)(\varphi)\cdot\int p\cdot\int p = (S \overset{\mathrm{rad}}{\odot} T)(\varphi) \end{aligned}$$

for all nets $(t_\varepsilon)_\varepsilon$ and in particular for $t_\varepsilon = \varepsilon$. In view of Corollary 1.8, this completes the proof. □

A verification of an example of Jelínek [3] shows, that in the one-dimensional case the product by harmonic representations of certain distributions exists, whereas their radial product is not defined. Therefore, in any dimension the multiplication by harmonic representations is a strict extension of the Kamiński product.

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(*Received* December 13, 1996, *revised* October 2, 1997)