Stepanoff's theorem in separable Banach spaces

Donatella Bongiorno[1]

Abstract. Stepanoff's theorem is extended to infinitely dimensional separable Banach spaces.

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1. Introduction

The classical Rademacher theorem [8], and [4, p. 216] has been extended to infinitely dimensional spaces by N. Aronszajn [1], J.P.R. Christensen [2], P. Mankiewicz [5], and R.R. Phelps [6]. These authors introduced different notions of exceptionality of a subset of a separable Banach space with respect to which Gateaux differentiability of Lipschitz mappings with values in spaces with Radon-Nikodym property (abbreviated as *RNP*) holds almost everywhere.

In $\mathbb{R}^n$, another extension of Rademacher theorem is the theorem of Stepanoff [8], [9], and [4, p. 218]:

Stepanoff's theorem. *If a map $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz at every point of a set $G \subset \mathbb{R}^n$, then it is differentiable a.e. in G.*

Our aim is to extend this theorem to separable Banach spaces. We will use Aronszajn's notion of exceptional sets, since it is the strongest one among those mentioned above.

We first establish, for any Banach space Y having *RNP*, Stepanoff's theorem for Y-valued functions of a real variable (Proposition 1).

In our main result (Theorem 1) we prove Stepanoff's theorem in the setting where X is a separable Banach space, Y is a Banach space having *RNP* and $f\colon X \to Y$ is Lipschitz at every point of a set $G \subset X$.

The main difference between the proof of the classical Stepanoff's theorem and our proof is that the Aronszajn's theorem on differentiability of Lipschitz functions applied to the distance function is used instead of the density theorem. The same application of Aronszajn's theorem is used also to overcome the difficulties related to the fact that the exceptional sets have to be Borel. Indeed, in our setting the

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set of points at which directional derivative exists may be non-measurable. For example, let S be a non-Borel subset of $\mathbb{R}$. Then the function

$$f(x,y) = \begin{cases} 0, & \text{if } x \in \mathbb{R} \text{ and } y \in S, \\ 0, & \text{if } x \leq 0 \text{ and } y \notin S, \\ x, & \text{if } x > 0 \text{ and } y \notin S \end{cases}$$

is Lipschitz at every point of the set $G = \{(x,y) \in \mathbb{R}^2 : x = 0\}$ but $\{(x,y) \in G : f'_x$ exists$\} = \{(x,y) \in \mathbb{R}^2 : x = 0, y \in S\}$ is not Borel.

2. Preliminaries

The sets of all natural, rational and real numbers are denoted by $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$, respectively. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space and $\mathcal{L}^n$ denotes the n-dimensional Lebesgue measure on $\mathbb{R}^n$. In all the paper, X and Y are real Banach spaces. Y^* denotes the topological dual of Y, i.e. the space of all bounded linear functionals on Y. By $\langle u, v^* \rangle$ we denote the value of the functional $v^* \in Y^*$ at the point $u \in Y$.

A set $V^* \subset Y^*$ is said to be *total* with respect to a set $V \subset Y$ whenever the condition $\langle u, \omega^* \rangle = 0$, for all $\omega^* \in V^*$ and $u \in V$, implies $u = 0$. For each non-zero $u \in X$ we define $\mathcal{A}(u)$ to be the family of all Borel sets $E \subset X$ such that

$$\mathcal{L}^1\{\lambda \in \mathbb{R} : x + \lambda u \in E\} = 0, \ \forall\, x \in X.$$

Moreover, if $\{u_n\} \subset X$ is a sequence of non-zero elements, we define

$$\mathcal{A}\{u_n\} = \{E \subset X : E = \cup E_n, \quad \text{with} \quad E_n \in \mathcal{A}(u_n)\}.$$

The Aronszajn exceptional class $\mathcal{A}$ is defined as the intersection of the families $\mathcal{A}\{u_n\}$, over all possible sequences $\{u_n\}$ of non-zero elements of X whose span is dense in X (following Aronszajn, we will call these sequences complete). Given a function $f\colon X \to Y$ and a set $E \subset X$, we denote by $f|_E$ the restriction of f to E. Given $u \in X$, the directional derivative of f at a point x is defined by

$$f'_u(x) = \lim_{t \to 0} \frac{f(x+tu) - f(x)}{t}.$$

For real valued functions we also define

$$\overline{f'_u}(x) = \limsup_{t \to 0} \frac{f(x+tu) - f(x)}{t}.$$

Definition 1. *A function $f\colon X \to Y$ is said to be Lipschitz at the point $x \in X$ if there exist two positive constants C and δ such that*

$$\|f(x+h) - f(x)\| \leq C\,\|h\|,$$

for all $h \in X$ with $\|h\| < \delta$.

Lemma 1. *Given $f\colon X \to Y$ and $L, \delta > 0$, let G be the set of all points $x \in X$ such that*

$$\|f(x+h) - f(x)\| \le L\,\|h\| \quad \textit{whenever} \quad \|h\| < \delta.$$

Then G is a closed set.

Proof: Let $\|x_k - x\| \to 0$, with $x_k \in G$. Given $h \in X$ with $\|h\| < \delta$ and given $0 < \varepsilon < \delta - \|h\|$ we can find x_k such that $\|x - x_k\| < \varepsilon$. Then

$$\|x + h - x_k\| \le \|x - x_k\| + \|h\| < \varepsilon + \|h\| < \delta.$$

Thus

$$\begin{aligned}\|f(x+h) - f(x)\| &\le \|f(x+h) - f(x_k)\| + \|f(x_k) - f(x)\| \\ &\le L(\|x + h - x_k\| + \|x - x_k\|) \\ &< L(\|h\| + 2\|x - x_k\|) < L\|h\| + 2L\varepsilon.\end{aligned}$$

By the arbitrariness of ε it follows $\|f(x+h) - f(x)\| \le L\|h\|$ for each $\|h\| < \delta$. Hence $x \in G$. □

Definition 2. *A function $f\colon X \to Y$ is said to be Gateaux differentiable at a point $x \in X$ if $f'_u(x)$ exists for each $u \in X$ and if the map $u \mapsto f'_u(x)$ is linear and bounded from X to Y. This map is called the Gateaux differential of f at the point x.*

Definition 3 (See [3], p. 217)**.** *It is said that Y has the Radon-Nikodym property if every function $f\colon [0,1] \longrightarrow Y$ of bounded variation is differentiable a.e. in $[0,1]$.*

Proposition 1. *Let Y have the RNP and let $f\colon \mathbb{R} \to Y$. Denote by G the set of all points $x \in \mathbb{R}$ at which f is Lipschitz. Then G is an F_σ-set and f is differentiable a.e. in G.*

Proof: For each natural n, let G_n denote the set of all $x \in G \cap [-n, n]$ such that

$$\|f(x+h) - f(x)\| \le n\,\|h\| \quad \text{whenever} \quad \|h\| < \frac{1}{n}.$$

Clearly $G = \cup G_n$. Moreover, G_n is a closed set by Lemma 1.

Let f_n be the extension of $f\,|_{G_n}$ to $[-n, n]$ such that f_n is linear on each contiguous interval of G_n. It is easy to prove that f_n is a Lipschitz function on $[-n, n]$. Thus, since Y has the *RNP*, there exists $\Gamma_n \subset [-n, n]$ such that f_n is differentiable on Γ_n and $\mathcal{L}^1([-n, n] \setminus \Gamma_n) = 0$. Let $\tilde{G}_n$ be the set of all points $x \in G_n$ at which the distance function $\operatorname{dist}(x, G_n)$ is differentiable. Since $\operatorname{dist}(x, G_n)$ is Lipschitz, then $\mathcal{L}^1(G_n \setminus \tilde{G}_n) = 0$. Hence $\mathcal{L}^1(G_n \setminus (\Gamma_n \cap \tilde{G}_n)) = 0$.

Define $N = \bigcup_n (G_n \setminus (\Gamma_n \cap \tilde{G}_n))$ and let $x \in G \setminus N$. Then there exists n such that $x \in \Gamma_n \cap \tilde{G}_n$. We will prove that f is differentiable at the point x.

Let $0 < \varepsilon < 2n$. By the differentiability of f_n and $\operatorname{dist}(x, G_n)$ at the point x, there exists $\delta_\varepsilon \in (0, n^{-1})$ such that

$$\left\| \frac{f_n(x+t) - f_n(x)}{t} - f_n'(x) \right\| < \frac{\varepsilon}{2} \tag{1}$$

and

$$\operatorname{dist}(x+t, G_n) < \frac{\varepsilon}{2(\|f_n'(x)\| + n)} |t|$$

for each $0 < |t| < \delta_\varepsilon$.

Then, for any fixed $0 < |t| < \delta_\varepsilon$, we can find $y \in G_n$ such that

$$|x + t - y| < \frac{\varepsilon}{2(\|f_n'(x)\| + n)} |t|. \tag{2}$$

Now $f_n = f$ on G_n. Therefore, by (1) and (2) we have

$$\begin{aligned} &\|f(x+t) - f(x) - f_n'(x)\, t\| \\ &\qquad \le \|f(y) - f(x) - f_n'(x)\,(y - x)\| + \|f(y) - f(x+t)\| \\ &\qquad\quad + \|f_n'(x)\|\, |x + t - y| \\ &\qquad \le \frac{\varepsilon}{2}|y - x| + n|x + t - y| + \|f_n'(x)\|\, |x + t - y| \\ &\qquad < \frac{\varepsilon}{2}|t| + \frac{\varepsilon}{2}|t| = \varepsilon|t|. \end{aligned}$$

Since this holds for any $0 < |t| < \delta_\varepsilon$, we conclude that f is differentiable at the point x. □

3. Lemmas

In this section we assume that X is a separable Banach space, $G \subset X$ is a closed set, Y is a Banach space satisfying the *RNP* and $f\colon X \to Y$ is a function such that there exist $L, \delta > 0$ with

$$\|f(y) - f(x)\| \le L\, \|y - x\| \tag{3}$$

whenever $x \in X$, $y \in G$ and $\|y - x\| < \delta$. We also assume that D is a Borel subset of G such that the distance function $\operatorname{dist}(x, G)$ is Gateaux differentiable at each point of D.

Lemma 2. *For each $u \in X$, the set*

$$\Delta = \{x \in G \cap D : f_u'(x) \ \ \textit{does not exist}\}$$

belongs to $\mathcal{A}(u)$.

Proof: By Proposition 1, applied to the restriction of f to the line $x + \mathbb{R}u$, we have $\mathcal{L}^1(\{\lambda \in \mathbb{R} : x + \lambda u \in \Delta\}) = 0$ for each $x \in X$. So we need only to show that Δ is a Borel set.

For each $n, m \in \mathbb{N}$, let F_{nm} be the set of all points $x \in G$ such that

$$\left\| \frac{f(x+tu)-f(x)}{t} - \frac{f(x+su)-f(x)}{s} \right\| \le \frac{1}{n} \tag{4}$$

for all $0 < |t| < 1/m$ and for all $0 < |s| < 1/m$. Moreover, let E_{nmk} be the set of all points $x \in G$ such that

$$\left\| \frac{f(y)-f(x)}{t} - \frac{f(z)-f(x)}{s} \right\| \le \frac{1}{n}$$

for all $0 < |t| < 1/m$, for all $0 < |s| < 1/m$ and for all $y, z \in G$ with $\|y-(x+tu)\| < |t|/k$, $\|z-(x+su)\| < |s|/k$.

Step 1. E_{nmk} is a closed set for each $n, m, k \in \mathbb{N}$.

Let $\{x_\nu\} \subset E_{nmk}$ and let $x \in X$ with $\|x_\nu - x\| \to 0$ when $\nu \to \infty$. Since G is a closed set, we have $x \in G$. Moreover, by (3), $f|_G$ is continuous, thus $f(x_\nu) \to f(x)$ when $\nu \to \infty$.
Fix $0 < |t| < 1/m$, $0 < |s| < 1/m$ and fix $y, z \in G$ with $\|y-(x+tu)\| < |t|/k$, $\|z-(x+su)\| < |s|/k$. Since $\lim_\nu \|x_\nu - x\| = 0$ there exists $\overline{\nu} \in \mathbb{N}$ such that $\|y-(x_\nu+tu)\| < |t|/k$ and $\|z-(x_\nu+su)\| < |s|/k$ for all $\nu > \overline{\nu}$.
Thus

$$\left\| \frac{f(y)-f(x)}{t} - \frac{f(z)-f(x)}{s} \right\| = \lim_\nu \left\| \frac{f(y)-f(x_\nu)}{t} - \frac{f(z)-f(x_\nu)}{s} \right\| \le \frac{1}{n};$$

hence $x \in E_{nmk}$.

Step 2. $(G \cap D) \setminus \Delta = \bigcap_n \bigcup_m (F_{nm} \cap D)$.

Let $x \in (G \cap D) \setminus \Delta$. Since $f'_u(x)$ exists, given $\varepsilon = 1/n$ there is $\eta_\varepsilon > 0$ such that

$$\left\| \frac{f(x+tu)-f(x)}{t} - \frac{f(x+su)-f(x)}{s} \right\| \le \varepsilon \tag{5}$$

for all $0 < |t| < \eta_\varepsilon$ and $0 < |s| < \eta_\varepsilon$. Let $1/m < \eta_\varepsilon$. Then $x \in F_{nm} \cap D$; hence $x \in \bigcap_n \bigcup_m (F_{nm} \cap D)$.
Let $x \in \bigcap_n \bigcup_m (F_{nm} \cap D)$. For each $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $\varepsilon > 1/n$. Now, since $x \in \bigcup_m F_{nm}$, there exists $m \in \mathbb{N}$ such that (4) and thus (5) hold for each $0 < |t| < 1/m$ and $0 < |s| < 1/m$. Therefore $f'_u(x)$ exists.

Step 3. $(G \cap D) \setminus \Delta = \bigcap_n \bigcup_m \bigcup_k (E_{nmk} \cap D)$.

Let $x \in (G \cap D) \setminus \Delta$. By Step 2, for every n there exists $m \in \mathbb{N}$ such that $x \in F_{(n+1)m} \cap D$. Let $k \in \mathbb{N}$ be such that

$$\frac{1}{km} < \delta \quad \text{and} \quad \frac{1}{n+1} + \frac{2L}{k} < \frac{1}{n}. \tag{6}$$

Thus, by (3) and (6), for all $0 < |t| < 1/m$, $0 < |s| < 1/m$ and $y, z \in G$ with $\|y - (x + tu)\| < |t|/k$ and $\|z - (x + su)\| < |s|/k$, we have

$$\begin{aligned}\left\|\frac{f(y)-f(x)}{t} - \frac{f(z)-f(x)}{s}\right\| &\leq \left\|\frac{f(x+tu)-f(x)}{t} - \frac{f(x+su)-f(x)}{s}\right\| \\ &\quad + \left\|\frac{f(x+tu)-f(y)}{t}\right\| + \left\|\frac{f(x+su)-f(z)}{s}\right\| \\ &\leq \frac{1}{n+1} + \frac{L}{k} + \frac{L}{k} < \frac{1}{n}\,.\end{aligned}$$

Hence $x \in E_{nmk} \cap D$ and, consequently, $x \in \bigcap_n \bigcup_m \bigcup_k (E_{nmk} \cap D)$.

Now let $x \in \bigcap_n \bigcup_m \bigcup_k (E_{nmk} \cap D)$. Then, for each n there exist $m, k \in \mathbb{N}$ such that $x \in E_{(n+1)mk} \cap D$. Then

$$\left\|\frac{f(y)-f(x)}{t} - \frac{f(z)-f(x)}{s}\right\| \leq \frac{1}{n+1}$$

for all $0 < |t| < 1/m$, $0 < |s| < 1/m$ and $y, z \in G$ with $\|y - (x + tu)\| < |t|/k$ and $\|z - (x + su)\| < |s|/k$. Let $\varepsilon > 0$ be such that

$$\frac{1}{n+1} + 2L\varepsilon < \frac{1}{n}\,.$$

Since $x \in D$, there exists $p > m$ such that

$$\operatorname{dist}(x + \tau u, G) < \varepsilon|\tau| \quad \text{for each} \quad 0 < |\tau| < \frac{1}{p}\,.$$

Given any $0 < |t|, |s| < \min(1/p, \delta/\varepsilon)$, we find $y \in G$ with $\|y - (x + tu)\| < \varepsilon|t|$ and $z \in G$ with $\|z - (x + su)\| < \varepsilon|s|$, and conclude that

$$\begin{aligned}&\left\|\frac{f(x+tu)-f(x)}{t} - \frac{f(x+su)-f(x)}{s}\right\| \\ &\qquad \leq \left\|\frac{f(y)-f(x)}{t} - \frac{f(z)-f(x)}{s}\right\| \\ &\qquad\quad + \left\|\frac{f(y)-f(x+tu)}{t}\right\| + \left\|\frac{f(z)-f(x+su)}{s}\right\| \\ &\qquad \leq \frac{1}{n+1} + \frac{L}{|t|}\|y - (x+tu)\| + \frac{L}{|s|}\|z - (x+su)\| \\ &\qquad < \frac{1}{n+1} + 2L\varepsilon < \frac{1}{n}\,.\end{aligned}$$

Then $x \in F_{np} \cap D$ and, consequently, $x \in \bigcap_n \bigcup_m (F_{nm} \cap D)$. □

Lemma 3. *If f is a real valued function and $u \in X$ then the mapping*

$$x \in G \cap D \to \overline{f'_u}(x)$$

is Borel.

PROOF: Let $c \in \mathbb{R}$. We have to prove that $\{x \in G \cap D : \overline{f'_u}(x) \leq c\}$ is a Borel set. For each $n, m, k \in \mathbb{N}$, let E_{nmk} be the set of all points $x \in G$ such that

$$\frac{f(y) - f(x)}{t} \leq c + \frac{1}{n}$$

for all $0 < |t| < 1/m$ and $y \in G$ with $\|y - (x + tu)\| < |t|/k$.

First of all we prove that E_{nmk} is a closed set for each $n, m, k \in \mathbb{N}$.
Indeed, let $\{x_\nu\} \subset E_{nmk}$ and let $x \in X$ with $\|x_\nu - x\| \to 0$ when $\nu \to \infty$. Since G is a closed set, we have $x \in G$. Moreover, by (3), $f|_G$ is continuous, thus $f(x_\nu) \to f(x)$ when $\nu \to \infty$.
Fix $0 < |t| < 1/m$ and fix $y \in G$ with $\|y-(x+tu)\| < |t|/k$. Since $\lim_\nu \|x_\nu - x\| = 0$ there exists $\overline{\nu} \in \mathbb{N}$ such that $\|y - (x_\nu + tu)\| < |t|/k$ for all $\nu > \overline{\nu}$.
Thus

$$\frac{f(y) - f(x)}{t} = \lim_\nu \frac{f(y) - f(x_\nu)}{t} \leq c + \frac{1}{n};$$

hence $x \in E_{nmk}$.

To finish the proof it is enough to show that $\{x \in G \cap D : \overline{f'_u}(x) \leq c\} = \bigcap_n \bigcup_m \bigcup_k (E_{nmk} \cap D)$.
Let $\overline{f'_u}(x) \leq c$. Then, for each n there exist $m \in \mathbb{N}$ such that

$$\frac{f(x + tu) - f(x)}{t} \leq c + \frac{1}{n+1}$$

for all $0 < |t| < 1/m$. Let $k \in \mathbb{N}$ be such that

$$\frac{1}{km} < \delta \quad \text{and} \quad c + \frac{1}{n+1} + \frac{L}{k} < c + \frac{1}{n}. \tag{7}$$

Thus, by (3) and (7), for all $0 < |t| < 1/m$ and for all $y \in G$ with $\|y - (x + tu)\| < |t|/k$, we have

$$\frac{f(y) - f(x)}{t} = \frac{f(x + tu) - f(x)}{t} + \frac{f(y) - f(x + tu)}{t} \leq c + \frac{1}{n+1} + \frac{L}{k} < c + \frac{1}{n}.$$

Hence $x \in E_{nmk}$ and, consequently, $x \in \bigcap_n \bigcup_m \bigcup_k E_{nmk}$.
Now let $x \in \bigcap_n \bigcup_m \bigcup_k (E_{nmk} \cap D)$. For each n there exist $m, k \in \mathbb{N}$ such that $x \in E_{(n+1)mk} \cap D$. Then

$$\frac{f(y) - f(x)}{t} \leq c + \frac{1}{n+1}$$

for all $0 < |t| < 1/m$ and $y \in G$ with $\|y - (x + tu)\| < |t|/k$. Let $\varepsilon > 0$ be such that

$$\frac{1}{n+1} + L\varepsilon < \frac{1}{n}\,.$$

Since $x \in D$, there exists $p > m$ such that

$$\operatorname{dist}(x + \tau u, G) < \varepsilon|\tau| \quad \text{for each} \quad 0 < |\tau| < \frac{1}{p}\,.$$

Thus, for each $0 < |t| < \min(1/p, \varepsilon/\delta)$, we find $y \in G$ with $\|y - (x+tu)\| < \varepsilon|t|$ and conclude that

$$\begin{aligned}\frac{f(x+tu) - f(x)}{t} &= \frac{f(y) - f(x)}{t} + \frac{f(x+tu) - f(y)}{t}\\ &\le c + \frac{1}{n+1} + \frac{L}{|t|}\|y - (x+tu)\|\\ &< c + \frac{1}{n+1} + L\varepsilon < c + \frac{1}{n}\,.\end{aligned}$$

Consequently, $\overline{f'_u}(x) \le c$. □

Lemma 4. *Let $u, v \in X$ and let $\omega^* \in Y^*$ with $\|\omega^*\| \le 1$. Denote $g = \langle f, \omega^* \rangle$. Then the set*

$$\Gamma = \{x \in G \cap D :\ g'_u(x), g'_v(x), g'_{u+v}(x) \ \text{ exist and } \ g'_u(x) + g'_v(x) \ne g'_{u+v}(x)\}$$

belongs to $\mathcal{A}\{u, v\}$.

Proof: By application of Lemma 3 to g and $-g$ we get easily that g'_u, g'_v and g'_{u+v} are Borel functions (on its sets of existence), hence Γ is a Borel set. By Stepanoff's theorem, applied to the restriction of g to the plane $x + \mathbb{R}u + \mathbb{R}v$, we have $\mathcal{L}^2\{(t,s) \in \mathbb{R}^2 :\ x + tu + sv \in \Gamma\} = 0$ for each $x \in X$. Then by [1, Chapter 1, Section 1, Proposition 1] we get $\Gamma \in \mathcal{A}\{u, v\}$. □

Lemma 5. *Let $\{u_n\}$ be a complete sequence in X, let $\{v_n\}$ be the sequence of all finite linear combinations with rational coefficients of $\{u_n\}$ and let $\{\omega^*_n\} \subset Y^*$ have $\|\omega^*_n\| \le 1$. Denote $g_k = \langle f, \omega^*_k \rangle$. Then the set*

$$\begin{aligned}\Omega = \{x \in G \cap D : f'_{v_n}(x) \ &\text{does not exist for some } n \ \text{ or}\\ &(g_k)'_{v_n}(x) + (g_k)'_{v_m}(x) \ne (g_k)'_{v_n+v_m}(x) \ \text{ for some } \ k, n, m\}\end{aligned}$$

belongs to $\mathcal{A}\{u_n\}$. Moreover, in the case in which the sequence $\{\omega^*_n\}$ is total with respect to the linear span of all derivatives $\{f'_{v_n}(x) : x \in (G \cap D) \setminus \Omega\}$, the mapping $T_x : v \to f'_v(x)$ from $\{v_n\}$ into Y is additive for each $x \in (G \cap D) \setminus \Omega$, and satisfies the following conditions:

(i) $f'_{\lambda v_n}(x) = \lambda f'_{v_n}(x)\ \forall\, \lambda \in \mathbb{Q},\ \forall\, n \in \mathbb{N},\ \forall\, x \in (G \cap D) \setminus \Omega$,
(ii) $\|f'_{v_n}(x)\| \le L\|v_n\|\ \forall\, n \in \mathbb{N},\ \forall\, x \in (G \cap D) \setminus \Omega$.

Proof: For each $n \in \mathbb{N}$, set

$$\Delta_n = \{x \in G \cap D : f'_{v_n}(x) \ \text{ does not exist}\}$$

and for each $n, m, k \in \mathbb{N}$, set

$$\begin{aligned}\Gamma_{n,m,k} = \{x \in G \cap D : \ & (g_k)'_{v_n}(x), (g_k)'_{v_m}(x), (g_k)'_{v_n+v_m}(x) \ \text{ exist and} \\ & (g_k)'_{v_n}(x) + (g_k)'_{v_m}(x) \neq (g_k)'_{v_n+v_m}(x)\}.\end{aligned}$$

We have by Lemma 2 $\Delta_n \in \mathcal{A}(v_n)$ and $\Gamma_{n,m,k} \in \mathcal{A}\{v_n, v_m\}$ by Lemma 4. Thus, since

$$\Omega = \left(\bigcup_n \Delta_n\right) \cup \left(\bigcup_{n,m,k} \Gamma_{n,m,k}\right),$$

we get $\Omega \in \mathcal{A}\{v_n\}$. Hence, by [1, Chapter 1, Section 1, Remark 1] we have $\Omega \in \mathcal{A}\{u_n\}$.

Now let $x \in (G \cap D) \setminus \Omega$. Then

$$\begin{aligned}\langle f'_{v_n+v_m}(x),\, \omega_k^*\rangle &= \left\langle \lim_{\lambda\to 0} \lambda^{-1}(f(x+\lambda(v_n+v_m)) - f(x)),\, \omega_k^* \right\rangle \\ &= \lim_{\lambda\to 0} \lambda^{-1}(\langle f(x+\lambda(v_n+v_m)),\, \omega_k^*\rangle - \langle f(x),\, \omega_k^*\rangle) \\ &= (g_k)'_{v_n+v_m}(x) = (g_k)'_{v_n}(x) + (g_k)'_{v_m}(x) \\ &= \lim_{t\to 0} t^{-1}(\langle f(x+t\,v_n),\, \omega_k^*\rangle - \langle f(x),\, \omega_k^*\rangle) \\ &\quad + \lim_{t\to 0} t^{-1}(\langle f(x+t\,v_m),\, \omega_k^*\rangle - \langle f(x),\, \omega_k^*\rangle) \\ &= \langle f'_{v_n}(x),\, \omega_k^*\rangle + \langle f'_{v_m}(x),\, \omega_k^*\rangle = \langle f'_{v_n}(x) + f'_{v_m}(x), \omega_k^*\rangle.\end{aligned}$$

Thus, if $\{\omega_k^*\}$ is total with respect to the linear span of all derivatives $\{f'_{v_n}(x) : x \in (G \cap D) \setminus \Omega\}$, we have:

$$f'_{v_n+v_m}(x) = f'_{v_n}(x) + f'_{v_m}(x).$$

Condition (i) can be proved in a similar way. Condition (ii) follows directly by (3):

$$\|f'_{v_n}(x)\| = \lim_{t\to 0} \left\| \frac{f(x+t\,v_n) - f(x)}{t} \right\| \leq L\|v_n\|.$$

$\square$

Lemma 6. *Let $\{v_n\}$ be a sequence in X and let Γ be a subset of $G \cap D$ such that $f'_{v_n}(x)$ exists for each $n \in \mathbb{N}$ and for each $x \in \Gamma$. Denote by V the linear span of all derivatives $\{f'_{v_n}(x) : x \in \Gamma\}$. Then there exists a sequence $\{\omega_n^*\} \subset Y^*$ which is total with respect to the linear space V.*

Proof: Let $\{x_n\}$ be dense in Γ and let $\{w_n\}$ be the sequence of all finite linear combinations with rational coefficients of the set

$$\left\{k\left(f\left(x_m+\frac{v_h}{k}\right)-f(x_m)\right):\ k,m,h\in\mathbb{N}\right\}.$$

The sequence $\{w_n\}$ is dense in V. In fact, it is enough to prove that for any $x \in \Gamma$, $n \in \mathbb{N}$ and $\varepsilon > 0$ there exist $k, m \in \mathbb{N}$ such that

$$\left\|f'_{v_n}(x)-k\left(f\left(x_m+\frac{v_n}{k}\right)-f(x_m)\right)\right\|<\varepsilon.$$

By the existence of the directional derivative $f'_{v_n}(x)$ and by the Gateaux differentiability of the distance function $\operatorname{dist}(x, G)$ at the point x, there exists $k > \varepsilon/4L\delta$ such that

$$\left\|f'_{v_n}(x)-k\left(f\left(x+\frac{v_n}{k}\right)-f(x)\right)\right\|<\frac{\varepsilon}{2} \tag{8}$$

and

$$\operatorname{dist}\left(x+\frac{v_n}{k},G\right)<\frac{\varepsilon}{8kL}\,.$$

Then there exists $y \in G$ such that

$$\left\|x+\frac{v_n}{k}-y\right\|<\frac{\varepsilon}{8kL}<\frac{\delta}{2}\,.$$

Moreover, since $\{x_n\}$ is dense in Γ, there exists $m \in \mathbb{N}$ such that

$$\|x-x_m\|<\frac{\varepsilon}{8kL}<\frac{\delta}{2}\,.$$

Hence

$$\left\|x_m+\frac{v_n}{k}-y\right\|\le\|x-x_m\|+\left\|x+\frac{v_n}{k}-y\right\|<\frac{\delta}{2}+\frac{\delta}{2}=\delta$$

and

$$\begin{aligned}
&\|(f(x+\frac{v_n}{k})-f(x))-(f(x_m+\frac{v_n}{k})-f(x_m))\|\\
&\qquad\le\|f(x)-f(x_m)\|+\|f(x+\frac{v_n}{k})-f(y)\|+\|f(x_m+\frac{v_n}{k})-f(y)\|\\
&\qquad\le L(\|x-x_m\|+\|x+\frac{v_n}{k}-y\|+\|x_m+\frac{v_n}{k}-y\|)\\
&\qquad\le 2L(\|x-x_m\|+\|x+\frac{v_n}{k}-y\|)\\
&\qquad\le\frac{\varepsilon}{2k}\,.
\end{aligned}\tag{9}$$

Thus, by (8) and (9), we get

$$\begin{aligned}
&\|f'_{v_n}(x_m) - k(f(x_m + \frac{v_n}{k}) - f(x_m))\| \\
&\qquad \le \|f'_{v_n}(x) - k(f(x + \frac{v_n}{k}) - f(x))\| \\
&\qquad\quad + k\|(f(x + \frac{v_n}{k}) - f(x)) - (f(x_m + \frac{v_n}{k}) - f(x_m))\| \\
&\qquad < \frac{\varepsilon}{2} + k\,\frac{\varepsilon}{2k} = \varepsilon.
\end{aligned}$$

Therefore $\{w_n\}$ is dense in V. Now, by Hahn-Banach theorem, there exists $\{\omega_n^*\} \subset Y^*$ with $\|\omega_n^*\| = 1$ and $\langle w_n, \omega_n^*\rangle = \|w_n\|$. The sequence $\{\omega_n^*\}$ is the required sequence. In fact, let $w \in V$ with $\langle w, \omega_n^*\rangle = 0$, for each n. Since $\{w_n\}$ is dense in V, there exists $\{w_{n_k}\} \subset \{w_n\}$ such that $\|w_{n_k} - w\| \to 0$. Then

$$\begin{aligned}
\|w\| &= \lim_{k\to\infty} \|w_{n_k}\| = \lim_{k\to\infty} \langle w_{n_k}, \omega_{n_k}^*\rangle \\
&= \lim_{k\to\infty} \langle w_{n_k} - w, \omega_{n_k}^*\rangle \\
&\le \lim_{k\to\infty} \|w_{n_k} - w\| = 0.
\end{aligned}$$

Thus $w = 0$. □

Lemma 7. *The set of all points $x \in G \cap D$ at which f is not Gateaux differentiable belongs to $\mathcal{A}$.*

Proof: Take $\{u_n\}$ and $\{v_n\}$ as in Lemma 5. Denote by Γ the set of all points $x \in G \cap D$ such that $f'_{v_n}(x)$ exists for each $n \in \mathbb{N}$. By Lemma 6 there exists a sequence $\{\omega_n^*\} \subset Y^*$ which is total with respect to the linear span of all derivatives $\{f'_{v_n}(x) : x \in \Gamma\}$. Define Ω as in Lemma 5. Then $\Omega \in \mathcal{A}\{u_n\}$ and the mapping $T_x\colon v \mapsto f'_v(x)$ from $\{v_n\}$ into Y is additive and satisfies conditions (i) and (ii) for each $x \in (G \cap D) \setminus \Omega$.

We will prove that f is Gateaux differentiable on $(G \cap D) \setminus \Omega$. In fact, let $x \in (G \cap D) \setminus \Omega$. By condition (ii) and by the density of $\{v_n\}$ in X, it follows that there exists a unique continuous mapping $\tilde{T}_x$ from X into Y such that $\tilde{T}_x(v_n) = T_x(v_n)$ for each $n \in \mathbb{N}$. Condition (i) and the additivity of T_x on $\{v_n\}$ imply that $\tilde{T}_x$ is linear, so that we have only to prove that

$$f'_u(x) = \tilde{T}_x(u) \tag{10}$$

for each $u \in X \setminus \{v_n\}$.

Given $\varepsilon > 0$, by the density of $\{v_n\}$ in X and by the continuity of $\tilde{T}_x$, there exists v_m such that

$$\|u - v_m\| < \frac{\varepsilon}{9L} \quad\text{and}\quad \|\tilde{T}_x(u - v_m)\| < \frac{\varepsilon}{3}\,. \tag{11}$$

Moreover, by the existence of $f'_{v_m}(x)$ and by the differentiability of the distance function $\operatorname{dist}(x, G)$ at the point x, there exists τ_ε such that

$$\left\| \frac{f(x+t\,v_m)-f(x)}{t} - f'_{v_m}(x) \right\| < \frac{\varepsilon}{3} \tag{12}$$

and

$$\operatorname{dist}(x+t\,u,\, G) < \frac{\varepsilon}{9L}\,|t|$$

for each $0 < |t| < \tau_\varepsilon$.

Let $0 < |t| < \min(\tau_\varepsilon, 9\delta L/2\varepsilon)$ and let $y \in G$ be such that

$$\|x+t\,u-y\| < \frac{\varepsilon}{9L}\,|t|.$$

Then,

$$\|x+tv_m-y\| \le \frac{2\,\varepsilon}{9L}\,|t|.$$

Thus we have

$$\begin{aligned}\left\|\frac{f(x+t\,u)-f(x+t\,v_m)}{t}\right\| &\le \left\|\frac{f(x+t\,u)-f(y)}{t}\right\| \\ &\quad + \left\|\frac{f(x+t\,v_m)-f(y)}{t}\right\| \\ &\le \frac{\varepsilon}{9} + \frac{2\varepsilon}{9} = \frac{\varepsilon}{3}\,.\end{aligned} \tag{13}$$

Now, since $f'_{v_m}(x) = \tilde{T}_x(v_m)$, by (11), (12) and (13) it follows that

$$\begin{aligned}\left\|\frac{f(x+t\,u)-f(x)}{t} - \tilde{T}_x(u)\right\| &\le \left\|\frac{f(x+t\,v_m)-f(x)}{t} - f'_{v_m}(x)\right\| \\ &\quad + \left\|\frac{f(x+t\,u)-f(x+t\,v_m)}{t}\right\| + \|\tilde{T}_x(u-v_m)\| \\ &\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,\end{aligned}$$

for each $0 < |t| < \tau_\varepsilon$. This proves that $f'_u(x)$ exists and (10) holds true. Thus f is Gateaux differentiable at x.

To end the proof, let Ψ be the set of all points $x \in G \cap D$ at which f is not Gateaux differentiable. We have just shown that

$$(G \cap D) \setminus \Omega \subset (G \cap D) \setminus \Psi.$$

Since the opposite inclusion is obvious, we infer that $\Psi = \Omega$, and, hence, $\Psi \in \mathcal{A}\{u_n\}$. As $\{u_n\}$ is an arbitrary complete sequence, we conclude that $\Psi \in \mathcal{A}$. $\square$

4. The main theorem

The following theorem is an extension of Stepanoff's theorem to separable Banach spaces.

Theorem 1. *Let X be a separable real Banach space and let Y be a real Banach space with the Radon-Nikodym property. Given $f: X \to Y$, let G be the set of all points $x \in X$ at which f is Lipschitz. Then there exists a set $E \in \mathcal{A}$ such that f is Gateaux differentiable at every point of $G \setminus E$.*

PROOF: For each natural n, let G_n be the set of all $x \in G_n$ such that

$$\|f(x+h) - f(x)\| \leq n\,\|h\| \quad \text{whenever} \quad \|h\| < \frac{1}{n}\,.$$

G_n is a closed set by Lemma 1 and $\cup G_n = G$. Since the distance function $\operatorname{dist}(x, G_n)$ is Lipschitz on X, by Aronszajn's theorem ([1, Theorem 1]; see also [6, Theorem 6]), there exists a Borel set D_n such that $X \setminus D_n \in \mathcal{A}$ and $\operatorname{dist}(x, G_n)$ is Gateaux differentiable on D_n. Then, in particular, $G_n \setminus D_n \in \mathcal{A}$. Denote by Ω_n the set of all points $x \in G_n \cap D_n$ at which f is not Gateaux differentiable. By Lemma 7 we have $\Omega_n \in \mathcal{A}$.

Define $E = (\bigcup_n \Omega_n) \cup (\bigcup_n (G_n \setminus D_n))$. Then $E \in \mathcal{A}$. Now let $x \in G \setminus E$. There exists $n \in \mathbb{N}$ such that $x \in G_n \setminus E$. The condition $x \notin E$ implies $x \notin G_n \setminus D_n$ (from which we get $x \in D_n$) and $x \notin \Omega_n$. Therefore $x \in (G_n \cap D_n) \setminus \Omega_n$, hence f is Gateaux differentiable at x. □

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PALERMO, VIA ARCHIRAFI 34, 90123 PALERMO, ITALY

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