Around splitting and reaping

Jörg Brendle

Abstract. We prove several results on some cardinal invariants of the continuum which are closely related to either the splitting number $\mathfrak{s}$ or its dual, the reaping number $\mathfrak{r}$.

Keywords: cardinal invariants of the continuum, splitting number, open splitting number, reaping number, σ-reaping number, Cichoń's diagram, Hechler forcing, finite support iteration

Classification: 03E05, 03E35

Introduction

We investigate, and give (partial) answers to, several questions related to splitting and reaping. Our work is motivated by recent work of Kamburelis and Węglorz [KW].

As usual $[S]^\omega$ denotes the countable subsets of an infinite set S. Given $A, X \in [\omega]^\omega$, we say X *splits* A if both $X \cap A$ and $A \setminus X$ are infinite. A family $\mathcal{F} \subseteq [\omega]^\omega$ such that every member of $[\omega]^\omega$ is split by an element of $\mathcal{F}$ is called a *splitting family.* The *splitting number* $\mathfrak{s}$ is the size of the smallest splitting family. Now let $\mathcal{B}_0$ be the standard base of the Cantor space 2^ω — that is, $\mathcal{B}_0$ consists of all clopen sets of the form $[\sigma] := \{f \in 2^\omega;\ \sigma \subseteq f\}$ where $\sigma \in 2^{<\omega}$ is a finite sequence of 0's and 1's. Given a sequence $\langle B_n;\ n \in \omega\rangle$ of pairwise disjoint members of $\mathcal{B}_0$, we say $X \subset 2^\omega$ *splits* $\langle B_n;\ n \in \omega\rangle$ if both $\{n;\ B_n \subset X\}$ and $\{n;\ B_n \cap X = \emptyset\}$ are infinite. A family $\mathcal{F} \subseteq P(2^\omega)$ is an *open splitting family* if each such $\langle B_n;\ n \in \omega\rangle$ is split by an element of $\mathcal{F}$ — and the *open splitting number* $\mathfrak{s}(\mathcal{B}_0)$ is the size of the least open splitting family. Note that we can assume all members of an open splitting family are themselves open, for going over to the interior of a subset of 2^ω does not change the phenomenon of open splitting. It is easy to see that $\mathfrak{s}(\mathcal{B}_0) \geq \mathfrak{s}$, and Kamburelis and Węglorz [KW, Proposition 3.6] characterized $\mathfrak{s}(\mathcal{B}_0)$ as the maximum of $\mathfrak{s}$ and another cardinal, the *separating number* $\mathfrak{sep}$, which we shall define below in § 1. We prove in Theorem 1.1 that $\mathfrak{sep}$ (and thus $\mathfrak{s}(\mathcal{B}_0)$) is at least the size of the smallest non-meager set. As a consequence, $\mathfrak{s}(\mathcal{B}_0)$ and $\mathfrak{sep}$ are equal (Corollary 1.2); this answers a question implicit in the work of Kamburelis and Węglorz [KW, p. 273].

Another consequence of Theorem 1.1 are new lower bounds for the *off-branch number* $\mathfrak{o}$, the minimum number of sets needed to blow up an almost disjoint family consisting of branches of a tree to a mad family. For example, one gets $\mathfrak{o} \geq \mathfrak{s}$ (Corollary 1.4). This complements results of Leathrum [Le].

In Section 2 of the present work, we show that the lower and upper bounds obtained for $\mathfrak{s}(\mathcal{B}_0)$ by Kamburelis, Węglorz and in our Theorem 1.1 are best possible when one compares it to cardinal invariants in Cichoń's diagram — i.e., to cardinals related to measure and category, see [BJ, Chapter 2]. This is done by using several well-known independence results and by proving a new one which shows the consistency of $\mathfrak{s}(\mathcal{B}_0) > \mathsf{cof}(\mathcal{M})$ in Theorem 2.3.

Here, given an ideal $\mathcal{I}$, $\mathsf{cof}(\mathcal{I})$, the *cofinality of* $\mathcal{I}$, is the size of the smallest $\mathcal{F} \subseteq \mathcal{I}$ such that every member of $\mathcal{I}$ is contained in a member of $\mathcal{F}$. We also let $\mathsf{non}(\mathcal{I})$, the *uniformity of* $\mathcal{I}$, denote the size of the least subset of $\bigcup \mathcal{I}$ not in $\mathcal{I}$; and $\mathsf{cov}(\mathcal{I})$, the *covering number of* $\mathcal{I}$, stands for the cardinality of the smallest $\mathcal{F} \subseteq \mathcal{I}$ with $\bigcup \mathcal{F} = \bigcup \mathcal{I}$. Finally, $\mathcal{M}$ is the meager ideal and $\mathcal{N}$ is the null ideal.

A family $\mathcal{F} \subseteq [\omega]^\omega$ is called a *reaping family* iff no $X \in [\omega]^\omega$ splits all members of $\mathcal{F}$ iff for all $X \in [\omega]^\omega$ there is $A \in \mathcal{F}$ with either $A \subseteq^* X$ or $A \cap X$ being finite. Here, we write $A \subseteq^* X$ (and say *A is almost contained in X*) iff $A \setminus X$ is finite. The *reaping number* (or *refinement number*) $\mathfrak{r}$ is the size of the least reaping family. $\mathcal{F} \subseteq [\omega]^\omega$ is said to be *σ-reaping* iff for no countable $\mathcal{X} \subseteq [\omega]^\omega$, every $A \in \mathcal{F}$ is split by some $X \in \mathcal{X}$ iff for any $\{X_n;\ n \in \omega\} \subseteq [\omega]^\omega$ there is $A \in \mathcal{F}$ such that for all n, either $A \subseteq^* X_n$ or $A \subseteq^* \omega \setminus X_n$. The *$\sigma$-reaping number* $\mathfrak{r}_\sigma$ is the cardinality of the smallest σ-reaping family. Clearly $\mathfrak{r} \leq \mathfrak{r}_\sigma$. The following, however, is unknown.

Question (Vojtáš [Vo], see also [Va]). *Is $\mathfrak{r} < \mathfrak{r}_\sigma$ consistent?*

A related open problem is

Question (Miller [Mi 1]). *Is $cf(\mathfrak{r}) = \omega$ consistent?*

Note that $\mathfrak{r}_\sigma$ must have uncountable cofinality. $\mathfrak{r}$ and $\mathfrak{s}$ are dual in a natural way. There is a version of $\mathfrak{s}$, the *$\aleph_0$-splitting number* $\aleph_0 - \mathfrak{s}$ (the size of the smallest $\mathcal{F} \subseteq [\omega]^\omega$ such that for every countable $\mathcal{X} \subseteq [\omega]^\omega$, all members of $\mathcal{X}$ are split by a single member of $\mathcal{F}$), which has a definition similar to $\mathfrak{r}_\sigma$ even though they are strictly speaking not dual. Kamburelis and Węglorz [KW, Section 2] got some partial results on the question whether $\mathfrak{s} < \aleph_0 - \mathfrak{s}$ is consistent. We show how these results can be "dualized" to yield a partial answer to Vojtáš' question above. In particular we prove that if $\mathfrak{r} < \mathfrak{r}_\sigma$, then $\mathsf{non}(\mathcal{M})$ must be large while $\mathfrak{d}$ must be small (Corollaries 3.4 and 3.7).

Here, given $f, g \in \omega^\omega$ we write $f \leq^* g$ (and say *g eventually dominates f*) iff $f(n) \leq g(n)$ for all but finitely many n. The *dominating number* $\mathfrak{d}$ is the size of the least family $\mathcal{F} \subseteq \omega^\omega$ such that each $g \in \omega^\omega$ is eventually dominated by a member of $\mathcal{F}$. The dual *unbounding number* $\mathfrak{b}$ is the size of the least $\mathcal{F} \subseteq \omega^\omega$ such that no single $g \in \omega^\omega$ eventually dominates all members of $\mathcal{F}$.

Our notation is standard. Basic references for cardinal invariants are [vD], [Va] and [BJ].

Acknowledgments. I am grateful to Menachem Kojman for pointing out Shelah's result used in Theorem 3.6. I also thank Claude Laflamme for explaining why the

consistency of $\mathfrak{r} < \mathfrak{r}_\sigma$ cannot be proved by a countable support iteration (see end of § 3).

1. Open splitting versus separating

The phenomenon of open splitting defined in the Introduction turns out to be closely related to the one of *separating*, due to Kamburelis and Węglorz [KW, p. 271], which we shall explain shortly. The related cardinal invariant will figure prominently in the next section (on consistency results) as well.

Given a real $x \in 2^\omega$ and $n \in \omega$, let $r(x, n)$ denote the sequence of length $n+1$ which agrees with x in the first n places, but differs in the last, i.e. $r(x, n) \restriction n = x \restriction n$ and $r(x, n)(n) = 1 - x(n)$. We say that an open set $G \subseteq 2^\omega$ *separates* a pair (x, A) where $x \in 2^\omega$ and $A \in [\omega]^\omega$ iff $x \notin G$ but $[r(x, n)] \subseteq G$ for infinitely many $n \in A$. A family $\mathcal{G}$ of open subsets of 2^ω is a *separating family* iff each (x, A) is separated by a member of $\mathcal{G}$. We let

$$\mathfrak{sep} := \min\{|\mathcal{G}|;\ \mathcal{G} \text{ is a separating family}\},$$

the *separating number*. We show

1.1 Theorem. $\mathsf{non}(\mathcal{M}) \leq \mathfrak{sep}$.

Proof: Let $\mathcal{G}$ be a family of open sets of 2^ω of size less than $\mathsf{non}(\mathcal{M})$. For $\sigma \in 2^{<\omega}$ and $k > |\sigma|$ let $\tau_{\sigma,k} = \tau$ be such that $|\tau| = k$, $\sigma \subseteq \tau$ and $\tau(i) = 0$ for all $i \geq |\sigma|$. For $G \in \mathcal{G}$, we define a function $f_G : 2^{<\omega} \to \omega$ by

$$f_G(\sigma) := \begin{cases} \min\{k > |\sigma|;\ [\tau_{\sigma,k}] \subseteq G\} & \text{if such a } k \text{ exists} \\ |\sigma| + 1 & \text{otherwise.} \end{cases}$$

Next use Bartoszyński's classical characterization of the cardinal $\mathsf{non}(\mathcal{M})$ (see [Ba], [BJ, Lemma 2.4.8]) to find a function $g : 2^{<\omega} \to \omega$ with $g(\sigma) \neq f_G(\sigma)$ for all $G \in \mathcal{G}$ and almost all σ. Notice that we can assume without loss of generality that $g(\sigma) > |\sigma|$ for all σ (in fact, since all the f_G have this property, we can simply restrict ourselves to the space of such functions and apply Bartoszyński's result there). Now define recursively a sequence $\langle \sigma_n \in 2^{<\omega};\ n \in \omega \rangle$ with $\sigma_n \subset \sigma_{n+1}$ as follows:

$$\sigma_0 = \langle\rangle$$
$$\sigma_{n+1}(i) = \begin{cases} 0 & \text{if } |\sigma_n| \leq i < |\sigma_{n+1}| - 1 \\ 1 & \text{if } i = |\sigma_{n+1}| - 1 \end{cases}$$

where we put $|\sigma_{n+1}| = g(\sigma_n)$. Then $x := \bigcup_{n \in \omega} \sigma_n$ defines a real number. Put $A = \{i;\ x(i) = 1\}$. We claim that no $G \in \mathcal{G}$ separates (x, A). The proof of this claim will conclude our argument.

To see this is true, fix $G \in \mathcal{G}$. We know that $f_G(\sigma_n) \neq g(\sigma_n)$ for almost all n. Fix such an n and let $i := |\sigma_{n+1}| - 1 = g(\sigma_n) - 1$. Notice that all i's from A are of this form, so they are the only ones we have to deal with. Two cases may hold:

Case 1. $f_G(\sigma_n) > g(\sigma_n) = i + 1$. Then $r(x,i) = \tau_{\sigma_n,i+1}$ and $[r(x,i)] \not\subseteq G$ by definition of f_G.

Case 2. $f_G(\sigma_n) < g(\sigma_n) = i+1$. Then $\tau_{\sigma_n,f_G(\sigma_n)} \subseteq \sigma_{n+1}$. Since $[\tau_{\sigma_n,f_G(\sigma_n)}] \subseteq G$ by definition of f_G, we conclude $x \in G$.

If the second case holds at least once, then G does not separate (x, A) — and if the first case holds almost always, then G does not separate (x, A) either. Hence we are done. □

We immediately infer

1.2 Corollary. *$\mathfrak{sep} \geq \mathfrak{s}$; in particular, one has $\mathfrak{sep} = \mathfrak{s}(\mathcal{B}_0)$ as well as $\mathfrak{s}(\mathcal{B}_0) \geq \mathsf{non}(\mathcal{M})$.*

PROOF: It is well-known (and easy to see) that $\mathsf{non}(\mathcal{M}) \geq \mathfrak{s}$. The second part follows now from the characterization of $\mathfrak{s}(\mathcal{B}_0)$ as $\max\{\mathfrak{s}, \mathfrak{sep}\}$ due to Kamburelis and Węglorz which we mentioned in the Introduction. □

We now proceed to compare $\mathfrak{s}(\mathcal{B}_0)$ to other cardinal invariants of the continuum. Since the open splitting number equals the separating number by the Corollary, we may as well deal with $\mathfrak{sep}$ which seems to be combinatorially simpler. The two lower bounds for $\mathfrak{sep}$ which are known are $\mathsf{non}(\mathcal{M})$ (see above) and $\mathsf{cov}(\mathcal{M})$ [KW, Proposition 3.7] — other lower bounds for $\mathfrak{sep}$ which have been given previously (like $\mathsf{cov}(\mathcal{N})$) are subsumed by our Theorem 1.1; the only known upper bound is $\mathsf{cof}(\mathcal{N})$ [KW, Proposition 3.9]. Using the same argument, this upper bound can be improved to the modified version of localization $\mathsf{cov}(\mathcal{J}_\ell)$ discussed in [BS, Theorem 3.5(b)].

An upper bound of a different flavour can be got as follows. The branches in $\omega^{<\omega}$ form an almost disjoint family $\mathcal{A}$. The *off-branch number* $\mathfrak{o}$, introduced by Leathrum [Le] and further studied in [Br], is the size of the smallest almost disjoint family $\mathcal{B}$ of subsets of $\omega^{<\omega}$ needed to extend $\mathcal{A}$ to a mad (maximal almost disjoint) family. Families which are almost disjoint and each member of which meets each branch only finitely often, like $\mathcal{B}$, are called *off-branch families.* It is known that $\mathfrak{a} \leq \mathfrak{o}$ [Le, Theorem 4.1] where $\mathfrak{a}$ is the (standard) *almost-disjointness number.* The following is easy to see.

1.3 Proposition. $\mathfrak{sep} \leq \mathfrak{o}$.

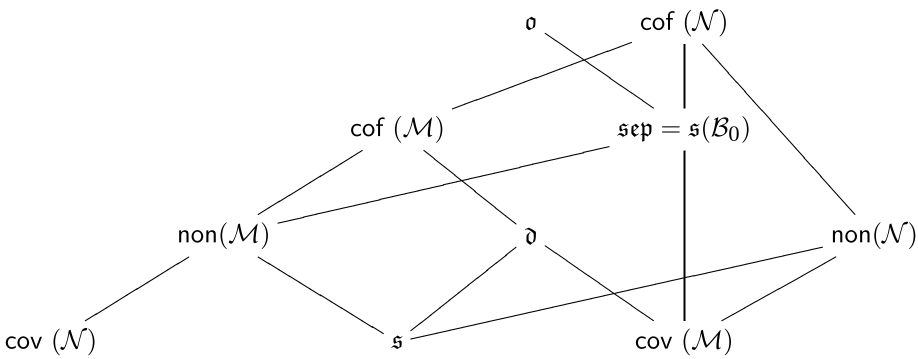
PROOF: Let us work with $2^{<\omega}$ instead of $\omega^{<\omega}$ (this does not affect $\mathfrak{o}$, see [Le, Lemma 3.1]). Given $A \subseteq 2^{<\omega}$, define open sets $G_{A,n} = \bigcup_{s\in A_n}[s]$ where A_n is A with the first n elements removed. We claim that if $\mathcal{A}$ is a maximal off-branch family, then $\{G_{A,n};\ A \in \mathcal{A} \text{ and } n \in \omega\}$ is a separating family.

To see this, take a pair (x, B) with $x \in 2^\omega$ and $B \subseteq \omega$. By maximality of $\mathcal{A}$, there must be $A \in \mathcal{A}$ such that $r(x,n) \in A$ for infinitely many $n \in B$. Since A is *off-branch*, it can contain only finitely many initial segments of x. Hence there is m such that $x \notin G_{A,m}$ as well as $[r(x,n)] \subseteq G_{A,m}$ for infinitely many $n \in B$, as required. □

1.4 Corollary. *$\mathfrak{o} \geq \mathsf{non}(\mathcal{M})$, and hence $\mathfrak{o} \geq \mathfrak{s}$.* □

The inequality $\mathfrak{o} \geq \mathfrak{s}$ answers a question implicitly asked in [Le, Section 8]. Note that Proposition 1.3 and Corollary 1.4 improve the lower bounds given for $\mathfrak{o}$ in [Le].

The known *ZFC*-results about the cardinals discussed here can be subsumed in the following diagram where cardinals increase as one moves upwards along the lines (see above or the standard references [vD], [Va] and [BJ] for the arguments).



Let us note that the cardinal $\mathsf{cov}(\mathcal{J})$ discussed in [BS, 3.5] sits in a similar place as $\mathfrak{sep}$ in the diagram. We therefore ask

1.5 *Question. What is the relationship between $\mathfrak{sep}$ and $\mathsf{cov}(\mathcal{J})$? Can one prove $\mathsf{cov}(\mathcal{J}) \geq \mathfrak{sep}$ in ZFC?*

2. Some consistency results concerning the separating number

By results of Kamburelis and Węglorz and of the preceding section, $\mathfrak{sep}$ is comparable to most of the cardinals in Cichoń's diagram — the only ones which are not covered by these results being $\mathfrak{d}$, $\mathsf{cof}(\mathcal{M})$ and $\mathsf{non}(\mathcal{N})$. We proceed to show that any of those may be both larger and smaller than $\mathfrak{sep}$.

Let us deal first with $\mathsf{non}(\mathcal{N})$: the consistency of $\mathfrak{sep} > \mathsf{non}(\mathcal{N})$ follows from the well-known consistency of $\mathsf{non}(\mathcal{M}) > \mathsf{non}(\mathcal{N})$ [BJ] and Theorem 1.1 while the consistency of $\mathfrak{sep} < \mathsf{non}(\mathcal{N})$ follows from the one of $\mathsf{cov}(\mathcal{J}_\ell) < \mathsf{non}(\mathcal{N})$ (cf. [BS]) and the remark in Section 1 saying that $\mathfrak{sep} \leq \mathsf{cov}(\mathcal{J}_\ell)$ — alternatively, using a standard argument, one can show that $\mathfrak{sep} = \omega_1$ in Miller's infinitely often equal reals model [Mi] which generically blows up $\mathsf{non}(\mathcal{N})$.

Since $\mathfrak{d} \leq \mathsf{cof}(\mathcal{M})$ (see [BJ, Theorem 2.2.11]), it suffices to show the consistency of $\mathfrak{sep} < \mathfrak{d}$ as well as the one of $\mathfrak{sep} > \mathsf{cof}(\mathcal{M})$. The former follows from the consistency of $\mathfrak{o} < \mathfrak{d}$ [Br, Section 1], and Proposition 1.3. For the latter we shall use a modified version $\mathbb{D}$ of *Hechler forcing.* The reason for using the modification is that it makes rank arguments much simpler (see [Br 1] for similar forcing notions). Apart from that it has the same effect as Hechler forcing on cardinal invariants of the continuum.

Conditions in $\mathbb{D}$ are pairs (s, ϕ) where $s \in \omega^{<\omega}$ is strictly increasing and $\phi : \omega^{<\omega} \to \omega$ is such that $\phi(s) > s(|s|-1)$. We put $(s, \phi) \leq (t, \psi)$ iff $s \supseteq t$, $\phi \geq \psi$ everywhere and $s(i) \geq \psi(s \restriction i)$ for all $|t| \leq i < |s|$. To show the required consistency, we shall use an ω_1-iteration of $\mathbb{D}$ with finite supports over a model of $MA + \mathfrak{c} = \kappa$ where $\kappa \geq \omega_2$ is an arbitrary regular cardinal. It is well-known that the extension satisfies $\mathsf{cof}(\mathcal{M}) = \omega_1$ [BJ, 7.6.10]. So it suffices to show it also satisfies $\mathfrak{c} = \mathfrak{sep} = \kappa$. The crucial point is:

2.1 Main Lemma. *Let $\dot{G}$ be a $\mathbb{D}$-name for an open set. Then we can find countably many open sets $\{G_i;\ i \in \omega\}$ such that whenever no G_i separates (x, A), then*

$$\Vdash_{\mathbb{D}} \text{“}\dot{G} \textit{ does not separate } (x, A)\text{”}.$$

Proof: Fix $\tau \in 2^{<\omega}$. For $s \in \omega^{<\omega}$ strictly increasing, we define the rank $rk(s, \tau)$ by induction on the ordinals.

$\alpha = 0$. We say $rk(s, \tau) = 0$ iff $(s, \psi) \Vdash$ “$[\tau] \subseteq \dot{G}$” for some ψ.

$\alpha > 0$. We say $rk(s, \tau) \leq \alpha$ iff there are infinitely many j such that $rk(s\hat{\ }j, \tau) < \alpha$.

For $s \in \omega^{<\omega}$, define $G_s = \bigcup\{[\tau];\ rk(s, \tau) < \infty\}$ and also $H_{s,i} = \bigcup\{[\tau];\ rk(s\hat{\ }j, \tau) < \infty$ for some $j \geq i\}$, for $i \in \omega$. We claim the collection $\mathcal{G} = \{G_s, H_{s,i};\ s \in \omega^{<\omega}, i \in \omega\}$ is as required. To see this take (x, A) such that no $G \in \mathcal{G}$ separates it. We have to show that

$$\Vdash_{\mathbb{D}} \text{“}\dot{G} \text{ does not separate } (x, A)\text{”}.$$

Take $(s, \phi) \in \mathbb{D}$. Without loss of generality assume $(s, \phi) \Vdash x \notin \dot{G}$. Note that this means $x \notin G_s$. Hence there are only finitely many $n \in A$ with $[r(x, n)] \subseteq G_s$. Let n_0 be their maximum $+1$. We shall construct $\psi \geq \phi$ such that

$$(s, \psi) \Vdash \text{“}[r(x, n)] \not\subseteq \dot{G} \text{ for all } n \geq n_0 \text{ with } n \in A\text{”}.$$

Clearly this is sufficient.

The construction of ψ proceeds by recursion. We start by defining $\psi(s)$. We know that $x \notin H_{s,\phi(s)}$ — otherwise we could find a condition stronger than (s, ϕ) which forces $x \in \dot{G}$, a contradiction. Hence there are only finitely many $n \in A$, $n \geq n_0$, with $[r(x, n)] \subseteq H_{s,\phi(s)}$. Now note that, since $[r(x, n)] \not\subseteq G_s$ for any $n \geq n_0$ with $n \in A$, for each such n there can be only finitely many i with $[r(x, n)] \subseteq H_{s,i}$. Thus we can find $\psi(s) \geq \phi(s)$ such that $[r(x, n)] \not\subseteq H_{s,\psi(s)}$ for any $n \in A$, $n \geq n_0$. This means that $[r(x, n)] \not\subseteq G_{s\hat{\ }j}$ for any $n \in A$, $n \geq n_0$ and $j \geq \psi(s)$. Therefore we can proceed with the recursive construction in exactly the same fashion.

Now, (s,ψ) forces the required statement because for any $t \supseteq s$ with $t(i) \geq \psi(t{\restriction}i)$ for $|s| \leq i < |t|$, we will have $rk(t, r(x,n)) = \infty$ for any $n \in A$, $n \geq n_0$ — i.e. no $(t,\chi) \leq (s,\psi)$ can force $[r(x,n)] \subseteq \dot{G}$. $\square$

Let us say a p.o. has property $(\star)$ iff it shares with $\mathbb{D}$ the property exhibited in 2.1.

2.2 Iteration Lemma. *Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha;\ \alpha < \delta\rangle$ be a finite support iteration of ccc p.o.'s. Assume that all $\mathbb{P}_\alpha$'s have property $(\star)$. Then also $\mathbb{P}_\delta$ has property $(\star)$.*

Proof: Let $\dot{G}$ be a $\mathbb{P}_\delta$-name for an open set. Without loss of generality $\delta = \omega$. Step into $V_n = V^{\mathbb{P}_n}$. Let $G_n = \bigcup\{[\tau];\ p \Vdash [\tau] \subseteq \dot{G}$ for some $p \in \mathbb{P}_\omega/\mathbb{P}_n\}$. Find, by assumption, sets $G_n^k \in V$ such that whenever no G_n^k, $k \in \omega$, separates (x, A), then

$$\Vdash_{\mathbb{P}_n} \text{“}\dot{G}_n \text{ does not separate } (x,A)\text{”}.$$

Take (x, A) such that no G_n^k, $k, n \in \omega$, separates it. We claim that

$$\Vdash_{\mathbb{P}_\omega} \text{“}\dot{G} \text{ does not separate } (x,A)\text{”}.$$

Let $p \in \mathbb{P}_\omega$. Without loss of generality assume that

$$p \Vdash_{\mathbb{P}_\omega} \text{“}x \notin \dot{G}\text{”}.$$

Find n such that $p \in \mathbb{P}_n$, and step into V_n (with $p \in G_n$, $\mathbb{P}_n$-generic over V). We know G_n does not separate (x, A). By assumption we must have $x \notin G_n$. Hence there are only finitely many $k \in A$ with $[r(x,k)] \subseteq G_n$. Thus we have that

$$\Vdash_{\mathbb{P}_\omega/\mathbb{P}_n} \text{“there are only finitely many } k \text{ with } [r(x,k)] \subseteq \dot{G}\text{”}$$

as required. $\square$

Putting everything together we now see

2.3 Theorem. *It is consistent to assume $\mathsf{cof}(\mathcal{M}) = \omega_1$ and $\mathfrak{sep} = \kappa$ where $\kappa \geq \omega_2$ is an arbitrary regular cardinal.*

Proof: As mentioned before we use an ω_1-iteration of $\mathbb{D}$ with finite supports over a model of $MA + \mathfrak{c} = \kappa$, $\kappa \geq \omega_2$ regular. We still have to argue that $\mathfrak{sep} = \kappa$. $\mathfrak{sep} \leq \kappa$ is obvious because $\mathfrak{c} = \kappa$. To see $\mathfrak{sep} \geq \kappa$, let $\mathcal{G}$ be a family of less than κ many open sets. By the Main Lemma 2.1 and the Iteration Lemma 2.2 we can find, in the ground model, a family $\mathcal{H}$ of less than κ many open sets such that whenever no $H \in \mathcal{H}$ separates (x, A), then also no $G \in \mathcal{G}$ separates (x, A). Since MA holds in the ground model, we easily find (x, A) such that no $H \in \mathcal{H}$ separates it, and we are done. $\square$

In fact, if we replace the ω_1-iteration of $\mathbb{D}$ by a λ-iteration where $\lambda < \kappa$ is an arbitrary uncountable regular cardinal, we get the consistency of $\mathsf{cof}(\mathcal{M}) = \lambda < \kappa = \mathfrak{sep}$.

3. Reaping versus σ-reaping

Let us quickly review the results of Kamburelis and Węglorz on splitting and $\aleph_0$-splitting to motivate how they can be dualized to get analogous results on reaping and on Vojtáš' notion of σ-reaping. Let $\bar{X} = \langle X_n;\ n \in \omega\rangle$ be a partition of ω into finite sets. Say that $A \in [\omega]^\omega$ *splits* $\bar{X}$ iff both $\{n;\ X_n \subseteq A\}$ and $\{n;\ X_n \cap A = \emptyset\}$ are infinite. Put

$$\mathfrak{fs} := \min\{|\mathcal{F}|;\ \mathcal{F} \subseteq [\omega]^\omega \text{ and every partition is split by a member of } \mathcal{F}\},$$

the *finitely splitting number*, and

$$\begin{aligned}\mathfrak{fr} := \ & \min\{|\mathcal{F}|;\ \mathcal{F} \text{ consists of partitions} \\ & \quad \text{and no single } A \in [\omega]^\omega \text{ splits all members of } \mathcal{F}\},\end{aligned}$$

the *finitely reaping number*. Similarly we put

$$\begin{aligned}\aleph_0 - \mathfrak{fs} := \ & \min\{|\mathcal{F}|;\ \mathcal{F} \subseteq [\omega]^\omega \text{ and every countable set} \\ & \quad \text{of partitions is split by a member of } \mathcal{F}\},\\ \mathfrak{fr}_\sigma := \ & \min\{|\mathcal{F}|;\ \mathcal{F} \text{ consists of partitions and} \\ & \quad \text{no countable } \mathcal{A} \subseteq [\omega]^\omega \text{ splits all members of } \mathcal{F}\}.\end{aligned}$$

Now, Kamburelis and Węglorz showed that $\mathfrak{fs} = \max\{\mathfrak{b}, \mathfrak{s}\}$ [KW, Proposition 2.1]. Similarly, one shows that $\aleph_0 - \mathfrak{fs} = \max\{\mathfrak{b}, \aleph_0 - \mathfrak{s}\}$, but, in fact, one can easily argue that $\aleph_0 - \mathfrak{fs} = \mathfrak{fs}$. Dualizing this, we get

3.1 Proposition. $\mathfrak{fr} = \min\{\mathfrak{d}, \mathfrak{r}\}$.

Proof: $\mathfrak{r} \geq \mathfrak{fr}$ is obvious. To see $\mathfrak{d} \geq \mathfrak{fr}$, take $\mathcal{F} \subseteq \omega^\omega$ dominating. Given $f \in \mathcal{F}$, define a partition $\bar{X}^f = \langle X_n^f;\ n \in \omega\rangle$ with $X_n^f = [f^n(0), f^{n+1}(0))$ where $f^0(0) = 0$ and $f^{n+1}(0) = f(f^n(0))$. It remains to check that no $A \in [\omega]^\omega$ splits all $\bar{X}^f$: for such A, define $g_A \in \omega^\omega$ such that both A and its complement meet any of the intervals $[n, g_A(n))$; if $g_A \leq^* f$, then both A and its complement meet almost all of the X_n^f, and we are done.

We finally prove that $\mathfrak{fr} \geq \min\{\mathfrak{d}, \mathfrak{r}\}$. Take $\kappa < \min\{\mathfrak{d}, \mathfrak{r}\}$ and a family of partitions $\{\bar{X}^\alpha = \langle X_n^\alpha;\ n \in \omega\rangle;\ \alpha < \kappa\}$. Given $\alpha < \kappa$, define $g^\alpha \in \omega^\omega$ such that each interval $[k, g^\alpha(k))$ contains (at least) one X_n^α. Since $\kappa < \mathfrak{d}$ find $f \in \omega^\omega$ increasing such that for all α, we have $f(k) \geq g^\alpha(g^\alpha(k))$ for infinitely many k.

Now we check that for all α there are infinitely many n with $X_n^\alpha \subseteq [f^i(0), f^{i+1}(0))$ for some i: indeed, if k is such that $f(k) \geq g^\alpha(g^\alpha(k))$, then either $[k, g^\alpha(k)) \subseteq [f^i(0), f^{i+1}(0))$ for some i, or $f^i(0) \in (k, g^\alpha(k))$ for some i in which case $f^{i+1}(0) \geq f(k) \geq g^\alpha(g^\alpha(k))$ so that $[g^\alpha(k), g^\alpha(g^\alpha(k))) \subseteq [f^i(0), f^{i+1}(0))$. Since each of the intervals defined by g^α contains some X_n^α, we are done.

Let us define $A^\alpha = \{i;\ X_n^\alpha \subseteq [f^i(0), f^{i+1}(0)) \text{ for some } n\}$. By what we just proved, the A^α are all infinite. Since $\kappa < \mathfrak{r}$, we find $B \in [\omega]^\omega$ splitting all the A^α. Putting $C = \bigcup_{i \in B} [f^i(0), f^{i+1}(0))$ we easily see that C splits all $\bar{X}^\alpha$, so that the $\bar{X}^\alpha$ do not form a finitely reaping family. $\square$

Similarly, one has

3.2 Proposition. $\mathfrak{fr}_\sigma = \min\{\mathfrak{d}, \mathfrak{r}_\sigma\}$. $\square$

3.3 Proposition. $\mathfrak{fr} \leq \mathfrak{fr}_\sigma \leq \mathsf{cof}([\mathfrak{fr}]^\omega)$.

PROOF: The first inequality is obvious. To see the second, let $\{\bar{X}^\alpha;\ \alpha < \mathfrak{fr}\}$ be a finitely reaping family. With each countable subset A of $\mathfrak{fr}$ we associate a partition $\bar{X}^A$ such that for each $\alpha \in A$, almost all members of $\bar{X}^A$ contain some member of $\bar{X}^\alpha$. This is done easily. By construction, the $\bar{X}^A$ form a finitely σ-reaping family, and we are done. $\square$

3.4 Corollary. *If $\mathfrak{r}_\sigma \leq \mathfrak{d}$, then $\mathfrak{r}_\sigma \leq \mathsf{cof}([\mathfrak{r}]^\omega)$.* $\square$

3.5 *Questions.* (1) *Is $\mathfrak{fr} < \mathfrak{fr}_\sigma$ consistent?*
(2) *Is it consistent that $cf(\mathfrak{fr}) = \omega$?*

These two questions correspond (and are related) to Vojtáš' and Miller's questions on $\mathfrak{r}$ and $\mathfrak{r}_\sigma$, respectively. Let us notice that from large cardinals one can get the consistency of $\mathsf{cof}([\mathfrak{fr}]^\omega) > \mathfrak{fr}_\sigma$. On the other hand, if the covering lemma holds, one has $\mathsf{cof}([\mathfrak{fr}]^\omega) = \mathfrak{fr}$ and, in particular, $\mathfrak{fr} = \mathfrak{fr}_\sigma$ unless $cf(\mathfrak{fr}) = \omega$ in which case one would have $\mathsf{cof}([\mathfrak{fr}]^\omega) = \mathfrak{fr}_\sigma = \mathfrak{fr}^+$. Note that $cf(\mathfrak{fr}_\sigma)$ is necessarily uncountable.

Kamburelis and Węglorz also proved [KW, Proposition 2.3] that $\mathfrak{s} \geq \min\{\aleph_0\text{–}\mathfrak{s}, \mathsf{cov}(\mathcal{M})\}$. Dualizing this is more intricate.

3.6 Theorem. $\mathfrak{r}_\sigma \leq \max\{\mathsf{cof}([\mathfrak{r}]^\omega), \mathsf{non}(\mathcal{M})\}$.

PROOF: Let $\kappa = \max\{\mathsf{cof}([\mathfrak{r}]^\omega), \mathsf{non}(\mathcal{M})\}$. Let $\{B_\beta;\ \beta < \mathfrak{r}\}$ be a reaping family. Without loss of generality, we can assume that for each $\beta < \mathfrak{r}$, $\{B_\delta;\ B_\delta \subseteq B_\beta\}$ is reaping below B_β. Let $\{A_\alpha;\ \alpha < \kappa\}$ be stationary in $[\mathfrak{r}]^\omega$. We use here a deep result of Shelah [Sh, Theorem 2.6], saying that $\mathsf{cof}([\lambda]^\omega) = \min\{|X|;\ X \subseteq [\lambda]^\omega$ is stationary$\}$ (the inequality $\leq$ is straightforward, but $\geq$ is not and uses some pcf-theory). For $\alpha < \kappa$ fix a bijection $f_\alpha : A_\alpha \to \omega$. Finally let $\{g_\gamma;\ \gamma < \kappa\} \subseteq \omega^\omega$ be non-meager. Given α and γ construct $C_{\alpha,\gamma}$, an infinite subset of ω, recursively as follows:

$$C^0_{\alpha,\gamma} = \omega$$

$$C^{n+1}_{\alpha,\gamma} = \begin{cases} B_{f_\alpha^{-1}(g_\gamma(n))} & \text{if this set is almost contained in } C^n_{\alpha,\gamma} \\ C^n_{\alpha,\gamma} & \text{otherwise.} \end{cases}$$

In the end let $C_{\alpha,\gamma}$ be an infinite pseudointersection of the $C^n_{\alpha,\gamma}$. We claim that the $C_{\alpha,\gamma}$ form a σ-reaping family.

To see this, fix $\{D_n;\ n \in \omega\} \subseteq [\omega]^\omega$. We have to find $\alpha, \gamma < \kappa$ such that for all n we have either $C_{\alpha,\gamma} \subseteq^* D_n$ or $C_{\alpha,\gamma} \cap D_n$ is finite. Let us form the set $E = \{F \subseteq \mathfrak{r};\ F$ is countable and for all $n \in \omega$ and $\beta \in F$ there is $\delta \in F$ such that either $B_\delta \subseteq^* B_\beta \cap D_n$ or $B_\delta \subseteq^* B_\beta \setminus D_n\}$. Note that E is club in $[\mathfrak{r}]^\omega$ by choice of the B_β. Hence we find $\alpha < \kappa$ such that $A_\alpha \in E$. Let M be a countable model such that $\{B_\beta;\ \beta < \mathfrak{r}\}, f_\alpha \in M$ and $\{D_n;\ n \in \omega\}, A_\alpha \subseteq M$. There is $\gamma < \kappa$ such that g_γ is Cohen over M. We check the pair α, γ works.

For this, by a straightforward genericity argument as well as by the definition of $C_{\alpha,\gamma}$ and the $C^n_{\alpha,\gamma}$, it suffices to show that given $n \in \omega$, $s \in \omega^{<\omega}$ and $k < |s|$ with $C^{|s|}_{\alpha,s} = B_{f_\alpha^{-1}(s(k))} =: B$ (which lies in M), there is (in M) $t \supset s$ with $|t| = |s| + 1$ such that $C^{|t|}_{\alpha,t} = B_{f_\alpha^{-1}(t(|s|))}$ is either almost contained in $B \cap D_n$ or almost contained in $B \setminus D_n$. This, however, is easy: since $A_\alpha \in E$, there is $\delta \in A_\alpha$ such that $B_\delta \subseteq^* B \cap D_n$ or $B_\delta \subseteq^* B \setminus D_n$. Hence, we can put $t(|s|) = f_\alpha(\delta)$, and we are done. $\square$

We immediately infer

3.7 Corollary. *If* $\mathsf{non}(\mathcal{M}) < \mathfrak{r}_\sigma$, *then* $\mathfrak{r}_\sigma \leq \mathsf{cof}([\mathfrak{r}]^\omega)$. $\square$

As a consequence of their results, Kamburelis and Węglorz got that if $\mathfrak{s} < \aleph_0 - \mathfrak{s}$, then $\mathsf{cov}(\mathcal{M}) \leq \mathfrak{s} < \aleph_0 - \mathfrak{s} \leq \mathfrak{b}$; a fortiori, the consistency of $\mathfrak{s} < \aleph_0 - \mathfrak{s}$ cannot be got with a finite support iteration because such an iteration forces $\mathsf{cov}(\mathcal{M}) \geq \mathsf{non}(\mathcal{M})$ and one has $\mathfrak{b} \leq \mathsf{non}(\mathcal{M})$ and $\mathfrak{d} \geq \mathsf{cov}(\mathcal{M})$ in *ZFC*. Our results about $\mathfrak{r}$ and $\mathfrak{r}_\sigma$ are somewhat weaker, but we still get, e.g., that if $\mathfrak{r}_\sigma = \omega_2 > \omega_1 = \mathfrak{r}$, then $\mathfrak{d} = \omega_1$ and $\mathsf{non}(\mathcal{M}) = \omega_2$ so that this consistency cannot be got with a finite support iteration either. On the other hand, Laflamme (unpublished) has shown that the latter consistency cannot be got by a countable support iteration of proper forcing over a model for *CH*. So, if $\mathfrak{r} = \omega_1 < \omega_2 = \mathfrak{r}_\sigma$ is consistent at all, a completely new forcing technique would be needed for the proof, and there may well be a *ZFC*-result lurking behind.

References

[Ba] Bartoszyński T., *Combinatorial aspects of measure and category*, Fundamenta Mathematicae **127** (1987), 225–239.

[BJ] Bartoszyński T., Judah H., *Set Theory. On the Structure of the Real Line*, A.K. Peters, Wellesley, Massachusetts, 1995.

[Br] Brendle J., *Mob families and mad families*, Archive for Mathematical Logic, to appear.

[Br 1] Brendle J., *Dow's principle and Q-sets*, Canadian Math. Bull., to appear.

[BS] Brendle J., Shelah S., *Evasion and prediction II*, Journal of the London Mathematical Society **53** (1996), 19–27.

[KW] Kamburelis A., Węglorz B., *Splittings*, Archive for Mathematical Logic **35** (1996), 263–277.

[Le] Leathrum T., *A special class of almost disjoint families*, Journal of Symbolic Logic **60** (1995), 879–891.

[Mi] Miller A., *Some properties of measure and category*, Transactions of the American Mathematical Society **266** (1981), 93–114 and **271** (1982), 347–348.

[Mi 1] Miller A., *A characterization of the least cardinal for which the Baire category theorem fails*, Proceedings of the American Mathematical Society **86** (1982), 498–502.

[Sh] Shelah S., *Advances in cardinal arithmetic*, Finite and infinite combinatorics in sets and logic (N.W. Sauer, R.E. Woodrow and B. Sands, eds.), Kluwer, Dordrecht, 1993, pp. 355–383.

[vD] van Douwen E.K., *The integers and topology*, Handbook of Set-theoretic Topology (K. Kunen and J.E. Vaughan, eds.), North-Holland, Amsterdam, 1984, pp. 111-167.

[Va] Vaughan J., *Small uncountable cardinals and topology*, Open problems in topology (J. van Mill and G. Reed, eds.), North-Holland, Amsterdam, 1990, pp. 195-218.

[Vo] Vojtáš P., *Cardinalities of noncentered systems of subsets of* ω, Discrete Mathematics **108** (1992), 125–129.

Department of Mathematics, Bradley Hall, Dartmouth College, Hanover, NH 03755, USA
and
Graduate School of Science and Technology, Kobe University, Rokko-dai 1-1, Nada, Kobe 657, Japan

E-mail: brendle@pascal.seg.kobe-u.ac.jp

(*Received* June 23, 1997)