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ωH-sets and cardinal invariants

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Abstract. A subset A of a Hausdorff space X is called an ωH-set in X if for every open family $\mathcal{U}$ in X such that $A \subset \bigcup \mathcal{U}$ there exists a countable subfamily $\mathcal{V}$ of $\mathcal{U}$ such that $A \subset \bigcup\{\overline{V} : V \in \mathcal{V}\}$. In this paper we introduce a new cardinal function $t_{s\theta}$ and show that $|A| \leq 2^{t_{s\theta}(X)\psi_c(X)}$ for every ωH-set A of a Hausdorff space X.

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All spaces considered in this paper are Hausdorff. We refer the reader to [8], [10] and [11] for notations and details on cardinal functions. Our approach to elementary submodels follows that of [15] (see also [5], [6], [9]). As usual $\psi(X)$ and $\chi(X)$ denote the pseudocharacter and the character of the space X. A Urysohn space is a space in which distinct points have disjoint closed neighbourhoods. Moreover, for any set S, we denote by $\mathcal{P}_m(S)$ the collection of all subsets of S whose cardinality is not greater than m.

A subset A of a space X is called an H-set (ωH-set) in X if for every open family $\mathcal{U}$ in X such that $A \subset \bigcup \mathcal{U}$ there exists a finite (countable) family $\mathcal{V} \subset \mathcal{U}$ satisfying $A \subset \bigcup \overline{\mathcal{V}}$. A space X is said to be quasi Lindelöf if for every open cover $\mathcal{U}$ of X there is a countable subfamily $\mathcal{V}$ of $\mathcal{U}$ satisfying $X = \bigcup\{\overline{U} : U \in \mathcal{V}\}$. It is clear that every H-set in a space X and every quasi Lindelöf space X is an ωH-set in X.

Let $x \in X$, a closed pseudobase for x in X is a family $\mathcal{V}$ of open neighbourhoods of x in X such that $\bigcap\{\overline{V} : V \in \mathcal{V}\} = \{x\}$. The closed pseudocharacter of X, denoted by $\psi_c(X)$, is the smallest infinite cardinal κ such that every point has a closed pseudobase of cardinality not greater than κ.

The θ-closure of a subset A of a space X, denoted by $cl_\theta(A)$, is the set of all points $x \in X$ such that $\overline{U} \cap A \neq \emptyset$ for every open neighbourhood U of x ([13]).

For any space X, $t_{s\theta}(X)$ will denote the smallest infinite cardinal κ such that for every $C \subset X$ and any $x \in cl_\theta(C)$ there is $S \subset C$ with $|S| \leq \kappa$, $x \in cl_\theta(S)$ and $|cl_\theta(S)| \leq 2^\kappa$ (see [12] for related concepts).

Remark 1. If X is a Urysohn space, then $t_{s\theta}(X)\psi_c(X) \leq \chi(X)$. Set $\chi(X) = \kappa$ and for every $x \in X$ let $\mathcal{B}(x)$ be a base for X at the point x such that $|\mathcal{B}(x)| \leq \kappa$. Now let $C \subset X$ and $p \in cl_\theta(C)$, for every $B \in \mathcal{B}(p)$ take a point $x_B \in \overline{B} \cap C$ and set $S = \{x_B : B \in \mathcal{B}(p)\}$. Clearly $|S| \leq \kappa$ and $p \in cl_\theta(S)$. Now let us show that $|cl_\theta(S)| \leq 2^\kappa$. Since X is a Urysohn space, it follows that $\bigcap\{cl_\theta(\overline{B} \cap S) :$

$B \in \mathcal{B}(x)\} = \{x\}$ for every $x \in cl_\theta(S)$. Set $\mathcal{G}(x) = \{\overline{B} \cap S : B \in \mathcal{B}(x)\}$ and let $\psi : cl_\theta(S) \to \mathcal{P}_\kappa(\mathcal{P}(S))$ be the map defined by $\psi(x) = \mathcal{G}(x)$. Since ψ is injective, it follows that $|cl_\theta(S)| \le |\mathcal{P}_\kappa(\mathcal{P}(S))| \le 2^\kappa$. Therefore $t_{s\theta}(X) \le \kappa$ and $t_{s\theta}(X)\psi_c(X) \le \chi(X)$.

Theorem 2. *If A is an ωH-set of the space X, then $|A| \le 2^{t_{s\theta}(X)\psi_c(X)}$.*

Proof: Let $t_{s\theta}(X)\psi_c(X) = \kappa$, let τ be the topology on X, and for every $x \in X$ let $\mathcal{B}(x)$ be a closed pseudobase for x in X such that $|\mathcal{B}(x)| \le \kappa$.

Let $f : X \to \mathcal{P}(\tau)$ be the map defined by $f(x) = \mathcal{B}(x)$ for every $x \in X$. Now let $T = 2^\kappa \cup \{A, X, \tau, 2^\kappa, f\}$ and take an elementary submodel $\mathcal{M}$ such that $T \subset \mathcal{M}$, $|\mathcal{M}| = 2^\kappa$, and which reflects enough formulas so that the following conditions are satisfied:

(i) $C \in \mathcal{M}$ for every $C \in \mathcal{P}_\kappa(\mathcal{M})$;
(ii) $\mathcal{B}(x) \in \mathcal{M}$ for every $x \in X \cap \mathcal{M}$;
(iii) if $C \subset X$ and $C \in \mathcal{M}$, then $\overline{C}, cl_\theta(C) \in \mathcal{M}$;
(iv) if $\mathcal{A} \in \mathcal{M}$, then $\bigcup \mathcal{A} \in \mathcal{M}$;
(v) if C, D are subsets of X such that $C \cap \mathcal{M} \subset D$ and $C, D \in \mathcal{M}$, then $C \subset D$;
(vi) if $E \in \mathcal{M}$ and $|E| \le 2^\kappa$, then $E \subset \mathcal{M}$.

Observe also that by (ii) and (vi) $\mathcal{B}(x) \subset \mathcal{M}$ for every $x \in X \cap \mathcal{M}$.

Now let us show that $cl_\theta(X \cap \mathcal{M}) = X \cap \mathcal{M}$. Let $x \in cl_\theta(X \cap \mathcal{M})$, since $t_{s\theta}(X) \le \kappa$, there is a subset S of $X \cap \mathcal{M}$ such that $|S| \le \kappa$, $x \in cl_\theta(S)$ and $|cl_\theta(S)| \le 2^\kappa$. Since $S \in \mathcal{M}$ (by (i)), it follows by (iii) that $cl_\theta(S) \in \mathcal{M}$. So by (vi) $cl_\theta(S) \subset \mathcal{M}$ and $x \in X \cap \mathcal{M}$.

The last step is to show that $A \subset X \cap \mathcal{M}$ (and hence $|A| \le 2^{t_{s\theta}(X)\psi_c(X)}$).

Suppose there is a point $p \in A \setminus \mathcal{M}$. For every $x \in A \cap \mathcal{M}$ let $U_x \in \mathcal{B}_x$ (observe that $U_x \in \mathcal{M}$) such that $p \notin \overline{U}_x$, and for every $x \in A \setminus \mathcal{M}$ let U_x be an open neighbourhood of x such that $\overline{U}_x \cap \mathcal{M} = \emptyset$. Since A is an ωH-set in X, it follows that there is a $C \in \mathcal{P}_\omega(A)$ such that $A \subset \bigcup\{\overline{U}_x : x \in C\}$. Now let $G = \bigcup\{\overline{U}_x : x \in C \cap \mathcal{M}\}$. If $y \in A \cap \mathcal{M}$, then there is some $x \in C$ such that $y \in \overline{U}_x$; hence $\overline{U}_x \cap \mathcal{M} \ne \emptyset$ and $x \in C \cap \mathcal{M}$. Therefore $y \in \overline{U}_x \subset G$ and $A \cap \mathcal{M} \subset G$.

Now $\overline{U}_x \in \mathcal{M}$ for every $x \in C \cap \mathcal{M}$ (by (iii)), therefore $\{\overline{U}_x : x \in C \cap \mathcal{M}\} \subset \mathcal{M}$. By (i) it follows that $\{\overline{U}_x : x \in C \cap \mathcal{M}\} \in \mathcal{M}$. Hence $G \in \mathcal{M}$ (by (iv)).

Therefore, by (v), it follows that $A \subset G$. Since $p \notin G$, we have a contradiction. □

Corollary 3 (see [1]). *If A is an H-set in the Urysohn space X, then $|A| \le 2^{\chi(X)}$.*

Corollary 4 (see [2]). *If X is a quasi Lindelöf Urysohn space, then $|X| \le 2^{\chi(X)}$.*

Example 5. Let τ be the euclidean topology on R and let X be R with the topology $\sigma = \{V \setminus C : V \in \tau, C \in \mathcal{P}_\omega(R)\}$. X is a Urysohn hereditarily Lindelöf space (so, a fortiori, every subset of X is an ωH-set in X). Observe that $\psi_c(X) = \omega$ and $\chi(X) = c$. Now let us show that $t_{s\theta}(X) = \omega$. First note that $cl_\sigma(V \setminus C) =$

$cl_\sigma(V)$ for every $V \setminus C \in \sigma$. In fact, let $x \in cl_\sigma(V)$ and take $W \setminus K \in \sigma$ such that $x \in W \setminus K$. Since $(W \setminus K) \cap V \neq \emptyset$, it follows that $|(W \cap V) \setminus K| = c$ (observe that $W \cap V$ is a non-empty open set of the euclidean line). Therefore $\emptyset \neq (W \cap V) \setminus (K \cup C) = (W \setminus K) \cap (V \setminus C)$, and $x \in cl_\sigma(V \setminus C)$. Now let $B \subset R$ and $x \in cl_\theta(B)$. Set $V_n = (x - \frac{1}{n}, x + \frac{1}{n})$ for every $n \in N$ and take a point $x_n \in cl_\sigma(V_n) \cap B$. The set $S = \{x_n : n \in N\}$ is a countable subset of B such that $|cl_\theta(S)| \leq c$. It remains to show that $x \in cl_\theta(S)$. Let $G = V \setminus C \in \sigma$ such that $x \in G$ and let n be such that $V_n \subset V$. Then $x_n \in cl_\sigma(V_n) \subset cl_\sigma(V) = cl_\sigma(V \setminus C) = cl_\sigma(G)$ and $cl_\sigma(G) \cap S \neq \emptyset$. So $x \in cl_\theta(S)$. Therefore $|X| = 2^{t_{s\theta}(X)\psi_c(X)} < 2^{\chi(X)}$.

Recently A. Bella and I.V. Yaschenko have shown that "Urysohn" cannot be omitted in the above corollaries ([3]).

Remark 6. A space X is H-closed if every open cover of X has a finite subfamily whose union is dense in X. It is worth noting that $|X| \leq 2^{\psi_c(X)}$ for every H-closed space X ([7]). Moreover there is an H-closed space X such that $|X| > 2^{2^{\psi(X)}}$ ([4]).

Question 7. *Let A be an H-set in the Urysohn space X. Is it true that $|A| \leq 2^{\psi_c(X)}$?*

Remark 8. Observe that it is not possible to obtain a bound for the cardinality of an ωH-set in terms of its character, in fact there are discrete H-sets of any cardinality (see, e.g., [1] and [4]).

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