Multiple solutions of semilinear elliptic systems

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Abstract. We obtain in this paper a multiplicity result for strongly indefinite semilinear elliptic systems in bounded domains as well as in $\mathbb{R}^N$.

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1. Introduction

In this paper, we continue the study from [FY] of the semilinear elliptic systems

$$-\Delta u + u = \pm g(x, v), \tag{$\pm$1.1}$$

$$-\Delta v + v = \pm f(x, u), \tag{$\pm$1.2}$$

in $\mathbb{R}^N$. Our purpose is to establish a multiplicity result on the existence of solutions of the system (±1.1)–(±1.2). Problem (+1.1)–(+1.2) has been studied in [HV], [FF] and [FY] etc., where only one solution was obtained for systems in bounded domains and systems with radial coefficients in $\mathbb{R}^N$. There seem to be no existence results for problems similar to (−1.1)–(−1.2). We shall show that (±1.1)–(±1.2) possesses infinitely many solutions under the assumptions on the functions f and g precised below.

The special difficulties involved in the system (±1.1)–(±1.2), first, a lack of compactness due to the problem being considered in $\mathbb{R}^N$, and second, the type of growth of the functions f and g, require to work with fractional Sobolev spaces instead of the usual $H^1(\mathbb{R}^N)$. Third, since the functionals associated to the problem are strongly indefinite, a modified multiplicity critical points theorem will be used.

The way of regaining some sort of compactness here is based on working with special type of function spaces, such as radial symmetry function spaces and weighted function spaces. Although the compactness in these cases is retained for the spaces, there is no compactness for the linear differential operator ($-\Delta + id$ in $\mathbb{R}^N$). This contrasts with the class of $-\Delta$ in a bounded domain.

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Let real constants $p+1 \geq \alpha > p$ and $q+1 \geq \beta > q$ with $\alpha, \beta > 2$ and $p, q > 1$ satisfy

$$\Big\{2 - \Big(\frac{1}{p+1} + \frac{1}{q+1}\Big)\Big\} \max\Big\{\frac{p+1}{\alpha}, \frac{q+1}{\beta}\Big\} < 1 + \frac{2}{N}.$$

Our hypotheses on the functions f and g are as follows.

(H1) $f, g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are continuous functions and odd in second variable.

(H2) There are nonnegative functions $\kappa, \ell \in L^\infty(\mathbb{R}^N)$ and a constant $C > 0$ such that

$$|f(t,x)| \leq C\kappa(x)(1+|t|^p), |g(t,x)| \leq C\ell(x)(1+|t|^q), \quad \text{for all} \quad t,$$

where

$$\frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2}{N} \quad \text{for} \quad N \geq 3$$

and

$$p, q \leq \frac{N+4}{N-4} \quad \text{if} \quad N \geq 5.$$

(H3)

$$0 \leq \alpha F(x,t) \leq t f(x,t), \quad 0 \leq \beta G(x,t) \leq t g(x,t),$$

for all $|t| \geq 0$, where $F(x,t) = \int_0^t f(x,s)\,ds$ and $G(x,t) = \int_0^t g(x,s)\,ds$.

(H4) There are positive constants C and C_1 such that

$$C_1 \geq \lim_{t\to 0} |f(x,t)|/|t|^a \geq C\kappa(x), \quad C_1 \geq \lim_{t\to 0} |g(x,t)|/|t|^b \geq C\ell(x),$$

where $\alpha + 1 \geq a \geq 1$, $\beta + 1 \geq b \geq 1$.

Condition (H3) implies that both functions f and g are superlinear. Indeed, integrating the inequalities in (H3) and using (H4) we get

$$F(x,t) \geq C\kappa(x)|t|^\alpha, \ G(x,t) \geq C\ell(x)|t|^\beta, \tag{1.3}$$

and

$$|f(x,t)| \geq C\kappa(x)|t|^{\alpha-1}, \ |g(x,t)| \geq C\ell(x)|t|^{\beta-1}. \tag{1.4}$$

Let $\omega(\kappa) = \{x \in \mathbb{R}^N : \kappa(x) \neq 0\}$ and $\omega(\ell) = \{x \in \mathbb{R}^N : \ell(x) \neq 0\}$.

(H5) $\text{meas}\{\mathbb{R}^N \setminus \omega(\kappa)\} = 0$ and $\text{meas}\{\mathbb{R}^N \setminus \omega(\ell)\} = 0$.

Our main result is following.

Theorem 1. *Assume* (H1)–(H5).

(i) *If*

$$\lim_{R\to\infty} \operatorname*{ess\,sup}_{|x|\geq R} \kappa(x) = 0, \qquad \lim_{R\to\infty} \operatorname*{ess\,sup}_{|x|\geq R} \ell(x) = 0,$$

then problem (±1.1)–(±1.2) *possesses infinitely many pairs of strong solutions* $\pm(u,v)$.

(ii) *If* f, g *depend explicitly on* $r = |x|$, *the same conclusion as in* (i) *holds true.*

(iii) *If* $p, q < \frac{N+2}{N-2}$, *then the solutions* (u,v) *of* (±1.1)–(±1.2) *and* $(\nabla u, \nabla v)$ *have uniform limits zero at infinity.*

The multiplicity result for the problem

$$-\Delta u = \pm g(x,v) \quad \text{in} \quad \Omega, \tag{$\pm$1.5}$$

$$\begin{gathered} -\Delta v = \pm f(x,u) \quad \text{in} \quad \Omega, \\ u = v = 0 \quad \text{on} \quad \partial\Omega, \end{gathered} \tag{$\pm$1.6}$$

defined on a bounded domain $\Omega \subset \mathbb{R}^N$ seems not to have appeared in the literature either. In the same way we may obtain the following result.

Theorem 2. *Under the hypotheses* (H1)–(H5), *problem* (±1.5)–(±1.6) *possesses infinitely many strong solution pairs* $\pm(u,v)$.

We recall in Section 2 the framework developed in [FY], and then prove Theorem 1 in Section 3. Theorem 2 can be proved in the same way.

2. Abstract framework

Let H be a separable real Hilbert space with scalar product denoted by $\langle , \rangle$ and corresponding norm by $\|\cdot\|$. Let $T : D(T) \subset H \to H$ be a self-adjoint linear operator semibounded from below. That is, there is a constant δ such that

$$\langle Tu, u\rangle \geq \delta \|u\|^2 \quad \text{for} \quad u \in D(T). \tag{2.1}$$

For simplicity, we may take $\delta = 1$. So $0 \notin \sigma(T)$, where $\sigma(T)$ denotes the spectrum of T. Let $\{E(\lambda) : \lambda \in \mathbb{R}\}$ denote the unique right continuous spectral family associated with T. In view of (2.1) we have $E(\lambda) = 0$ for $\lambda < 1$.

It is well known that

$$D(T) = \Big\{u \in H : \int_1^\infty \lambda^2 \, d\langle E(\lambda)u, u\rangle < \infty\Big\}; \tag{2.2}$$

$$\langle Tu, v\rangle = \int_1^\infty \lambda \, d\langle E(\lambda)u, v\rangle \quad \text{for} \quad u \in D(T), \ v \in H; \tag{2.3}$$

$$\langle Tu, v\rangle \leq \lambda \|u\|^2 \quad \text{for} \quad u \in E(\lambda)H; \tag{2.4}$$

$$\lambda\|u\|^2 \leq \langle Tu, u\rangle \leq \mu\|u\|^2 \quad \text{for} \quad u \in E(\mu)H \ominus E(\lambda)H; \tag{2.5}$$

$$\langle Tu, u\rangle \geq \mu\|u\|^2 \quad \text{for} \quad u \in [E(\mu)H]^\perp \cap D(T). \tag{2.6}$$

Since T is a positive operator, it has a square root

$$T^{\frac{1}{2}} = \int_1^\infty \lambda^{\frac{1}{2}}\, dE(\lambda), \quad T^{\frac{1}{2}} : D(T^{\frac{1}{2}}) \to H$$

with

$$D(T^{\frac{1}{2}}) = \Big\{ u \in H : \int_1^\infty \lambda\, d\langle E(\lambda)u, u\rangle < \infty \Big\}.$$

We know that $T^{\frac{1}{2}}$ is self-adjoint, and from (2.1) we have

$$\langle T^{\frac{1}{2}}u, u\rangle \geq \|u\|^2 \ \text{ for } \ u \in D(T^{\frac{1}{2}}).$$

For each positive real s, we can define

$$T^{\frac{s}{2}} = \int_1^\infty \lambda^{\frac{s}{2}}\, dE(\lambda).$$

We use the notation $A = T^{\frac{1}{2}}$, $A^s = T^{\frac{s}{2}}$ and define the space E^s as

$$E^s := D(A^s) = \Big\{ u \in H : \int_1^\infty \lambda^s\, d\langle E(\lambda)u, u\rangle < \infty \Big\}. \tag{2.7}$$

Each E^s is a Hilbert space endowed with the graph norm

$$\langle u, v\rangle_{E^s} = \langle u, v\rangle + \langle A^s u, A^s v\rangle.$$

It follows from (2.1) that

$$\|A^s u\| \geq \|u\| \ \text{ for all } \ u \in E^s, \tag{2.8}$$

and as a consequence, $\|u\|_{E^s}$ and $\|A^s u\|$ are equivalent norms in E^s. So we write in E^s from now on that

$$\langle u, v\rangle_{E^s} = \langle A^s u, A^s v\rangle \ \text{ and } \ \|u\|_{E^s} = \|A^s u\|. \tag{2.9}$$

In view of (2.8), $A^s : E^s \to H$ is an isomorphism. We denote by A^{-s} the inverse of A^s.

Now let $s, t > 0$ with $s + t = 2$. We define the Hilbert space $E = E^s \times E^t$, with inner product $\langle , \rangle$ induced by inner products $\langle , \rangle_{E^s}$, $\langle , \rangle_{E^t}$ in the usual way. Next we define a bilinear form $B : E \times E \to \mathbb{R}$ by

$$B[(u, v), (\phi, \psi)] = \langle A^s u, A^t \psi\rangle + \langle A^s \phi, A^t v\rangle.$$

B is continuous and symmetric. Hence B induces a self-adjoint bounded linear operator $L : E \to E$ such that

$$B[z, \eta] = \langle Lz, \eta \rangle_E \quad \text{for} \quad z, \eta \in E.$$

It is easy to see that

$$Lz = (A^{-s}A^t v, A^{-t}A^s u) \quad \text{for} \quad z = (u, v) \in E.$$

We can then prove that L has two eigenvalues -1 and 1, whose corresponding eigenspaces are

$$E^- = \{(u, -A^{-t}A^s u) : u \in E^s\} \quad \text{for} \quad \lambda = -1, \tag{2.10}$$

$$E^+ = \{(u, A^{-t}A^s u) : u \in E^s\} \quad \text{for} \quad \lambda = +1. \tag{2.11}$$

We also have that

$$E = E^+ \oplus E^-$$

and

$$B[z^+, z^-] = 0 \quad \text{for} \quad z^+ \in E^+ \quad \text{and} \quad z^- \in E^-.$$

We consider

$$Q(z) = \frac{1}{2} B[z, z] = \langle A^s u, A^t v \rangle \tag{2.12}$$

for $z = (u, v) \in E$. It follows then that

$$\frac{1}{2} \|z\|_E^2 = Q(z^+) - Q(z^-),$$

where $z = z^+ + z^-$, $z^+ \in E^+$, $z^- \in E^-$. Particularly,

$$Q(z) = \frac{1}{2}\|z\|_E^2 \quad \text{for} \quad z \in E^+ \quad \text{and} \quad Q(z) = -\frac{1}{2}\|z\|_E^2 \quad \text{for} \quad z \in E^-. \tag{2.13}$$

If $z = (u, v) \in E^+$, i.e. $v = A^{-t}A^s u$, we have by (2.13) and the definition of the norm on E that

$$Q(z) = \frac{1}{2}\|z\|_E^2 = \frac{1}{2}\|(u, A^{-t}A^s u)\|_E^2 = \|A^s u\|^2. \tag{2.14}$$

Similarly

$$Q(z) = \|A^t v\|^2 = \|v\|_{E^t}^2 \tag{2.15}$$

for $z \in E^+$.

3. Multiplicity results

In this section, we shall prove (i) and (ii) of Theorem 1, respectively. First we consider the case (i) of Theorem 1. In the framework of Section 2, we take $H = L^2(\mathbb{R}^N)$ and $T = -\Delta + id$, with domain $D(T) = H^2(\mathbb{R}^N)$. For $0 \le s \le 2$, the space E^s, which is the domain $D(T^{\frac{s}{2}})$, is precisely the space obtained by interpolation between $H^2(\mathbb{R}^N)$ and $L^2(\mathbb{R}^N)$, namely

$$[H^2(\mathbb{R}^N), L^2(\mathbb{R}^N)]_{1-\frac{s}{2}}.$$

In this case E^s is the usual fractional Sobolev space $H^s(\mathbb{R}^N)$. Denoting by $A = (-\Delta + id)^{\frac{s}{2}}$, we have for all $0 \le s \le 2$

$$D(A^s) = H^s(\mathbb{R}^N) = [H^2(\mathbb{R}^N), L^2(\mathbb{R}^N)]_{1-\frac{s}{2}}.$$

Let κ be a nonnegative function. We denote by $L^\gamma(\kappa, \mathbb{R}^N)$ the weighted function spaces with norms $\|w\|_{L^\gamma(\kappa,\mathbb{R}^N)} = (\int_{\mathbb{R}^N} \kappa(x)|w|^\gamma)^{1/\gamma}$.

According to the properties of interpolation space, we have the following embedding theorem, see [AD], [PL].

Theorem 3.1. *Let $s > 0$. Then the inclusion of $H^s(\mathbb{R}^N)$ into $L^\gamma(\kappa, \mathbb{R}^N)$ is continuous if $2 \le \gamma \le 2N/(N-2s)$ and $\kappa \in L^\infty(\mathbb{R}^N)$. The inclusion is compact if $2 < \gamma < 2N/(N-2s)$ and κ satisfies the condition* (i) *of Theorem* 1.

Now if we choose $s, t > 0$, $s + t = 2$, such that

$$\begin{aligned} &\Big(1 - \frac{1}{p+1}\Big) \max\Big(\frac{p+1}{\alpha}, \frac{q+1}{\beta}\Big) < \frac{1}{2} + \frac{s}{N}, \\ &\Big(1 - \frac{1}{q+1}\Big) \max\Big(\frac{p+1}{\alpha}, \frac{q+1}{\beta}\Big) < \frac{1}{2} + \frac{t}{N}, \end{aligned} \tag{3.1}$$

then the inclusions $H^s(\mathbb{R}^N) \hookrightarrow L^{p+1}(\kappa, \mathbb{R}^N)$ and $H^t(\mathbb{R}^N) \hookrightarrow L^{q+1}(\ell, \mathbb{R}^N)$ are compact, where κ and ℓ are as in Theorem 1.

Let $E = H^s(\mathbb{R}^N) \times H^t(\mathbb{R}^N)$ and the bilinear form $B : E \times E \to \mathbb{R}$ be defined by

$$B[(u,v),(\phi,\psi)] = \int_{\mathbb{R}^N} A^s u A^t \psi + A^s \phi A^t v,$$

for $z = (u,v) \in E$ and $\eta = (\phi,\psi) \in E$. We have the corresponding quadratic form

$$Q(z) = \int_{\mathbb{R}^N} A^s u A^t v, \quad z = (u,v) \in E.$$

We consider the functional $\Phi^\pm : E \to \mathbb{R}^N$, defined by

$$\Phi^\pm(z) = \pm \int_{\mathbb{R}^N} A^s u A^t v - \int_{\mathbb{R}^N} F(x,u) - \int_{\mathbb{R}^N} G(x,v). \tag{3.2}$$

The critical points of $\Phi^{\pm}$ satisfy the equations

$$\pm\int_{\mathbb{R}^N} A^s u A^t \psi - \int_{\mathbb{R}^N} g(x,v)\psi = 0 \quad \text{for all } \psi \in H^t(\mathbb{R}^N), \tag{3.3}$$

$$\pm\int_{\mathbb{R}^N} A^s \phi A^t v - \int_{\mathbb{R}^N} f(x,u)\phi = 0 \quad \text{for all } \phi \in H^s(\mathbb{R}^N). \tag{3.4}$$

Equations (3.3)–(3.4) are the weak formulation of problem (±1.1)–(±1.2), and their weak solutions are actually strong solutions of (±1.1)–(±1.2), see [FF].

We shall use the generalized critical point theorem of Benci [B] in a version due to [He] to find critical points of $\Phi^{\pm}$. For completeness, we state the result from [He] here.

Theorem ([He])**.** *Let E be a real Hilbert space, and let $\Phi \in C^1(E,\mathbb{R})$ be a functional with the following properties:*

(i) *Φ has the form*

$$\Phi(z) = \frac{1}{2}(Lz,z) + \Psi(z) \quad \textit{for all} \quad z \in E, \tag{3.5}$$

where L is an invertible bounded self-adjoint linear operator in E and where $\Psi \in C^1(E,\mathbb{R})$ is such that $\Psi(0) = 0$ and the gradient $\nabla\Psi : E \to E$ is a compact operator;

(ii) *Φ is even, i.e. $\Phi(-z) = \Phi(z)\ \forall\, z \in E$;*

(iii) *Φ satisfies the Palais-Smale condition.*

Furthermore, let

$$E = E^+ \oplus E^-$$

be an orthogonal splitting into L-invariant subspaces E^+, E^- such that $\pm(Lz,z) \geq 0\ \forall\, z \in E^{\pm}$. Then:

(a) *suppose that there is an m-dimensional linear subspace E_m of E^+ ($m \in \mathbb{N}$) such that for the spaces*

$$V := E^+, \quad W := E^- \oplus E_m,$$

we have

(iv) *$\exists\, \rho_0 > 0$ such that $\inf\{\Phi(z) : z \in V, \|z\| = \rho\} > 0\ \forall\, \rho \in (0,\rho_0]$;*

(v) *$\exists\, c_\infty \in \mathbb{R}$ such that $\Phi(z) \leq c_\infty\ \forall\, z \in W$.*

Then there exist at least m pairs $(z_j, -z_j)$ of critical points of Φ such that $0 < \Phi(z_j) \leq c_\infty$ $(j = 1,\dots,m)$.

(b) *A similar result holds when $E_m \subset E^-$ and we take $V = E^-$, $W = E^+ \oplus E_m$.*

It is known from Section 2 that the operator L induced by the bilinear form B is an invertible bounded self-adjoint linear operator satisfying $\pm\langle Lz,z\rangle \geq 0$

$\forall\, z \in E^{\pm}$. Now we introduce some finite dimensional subspaces of E. Let (e_j), $j = 1, 2, \dots$, be a complete orthogonal system in $H^s(\mathbb{R}^N)$. Let H_n denote the finite dimensional subspaces of $H^s(\mathbb{R}^N)$ generated by (e_j), $j = 1, \dots, n$. Since $A^s : H^s(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ and $A^t : H^t(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ are isomorphisms, we know that $\widehat{e}_j = A^{-t}A^s e_j$, $j = 1, 2, \dots$, is a complete orthogonal system in $H^t(\mathbb{R}^N)$. Let $\widehat{H}_n$ denote the finite dimensional subspace of $H^t(\mathbb{R}^N)$ generated by $(\widehat{e}_j)$, $j = 1, \dots, n$. For each $n \in \mathbb{N}$, we introduce the following subspaces of E^+ and E^-:

$$\begin{aligned} E_n^+ &= \text{ subspace of } E^+ \text{ generated by } \ (e_j, \widehat{e}_j),\ j = 1, \dots, n, \\ E_n^- &= \text{ subspace of } E^- \text{ generated by } \ (e_j, -\widehat{e}_j),\ j = 1, \dots, n. \end{aligned}$$

Lemma 3.2. *Let the assumptions of Theorem* 1 *hold, then the functional* $\Phi^{\pm}$ *defined in* (3.2) *satisfies conditions* (ii), (iv) *and* (v) *of Theorem* [HE].

Proof: Condition (ii) is an immediate consequence of the definition of $\Phi^{\pm}$ and assumptions of functions f and g. For condition (iv), we use (2.14) and assumptions (H2) and (H4) to deduce that for $z \in V := E^{\pm}$

$$\Phi^{\pm}(z) \geq \frac{1}{2}\|z\|_E^2 - C\int |u|^{p+1} - C\int |u|^{a+1} - C\int |v|^{q+1} - C\int |v|^{b+1}.$$

Using Theorem 3.1 we get

$$\Phi^{\pm}(z) \geq \frac{1}{2}\|z\|_E^2 - C\|z\|_E^{a+1} - C\|z\|_E^{b+1}$$

for small $\|z\|$. And since $a, b > 1$ we conclude that $\Phi^{\pm}(z) > 0$ for $z \in E^{\pm}$ with $\|z\|$ small.

Next, let us prove condition (v). Let $n \in \mathbb{N}$ be fixed and let $z \in W = E_n^{\pm} \oplus E^{\mp}$, write $z = (u, v)$ and $z = z^+ + z^-$. We have by assumption (H5) and (1.3)

$$\begin{aligned} \Phi^{\pm}(z) &= \pm[Q(z^+) + Q(z^-)] - \int F(x,u) - \int G(x,v) \\ &\leq -\frac{1}{2}\|z^{\mp}\|_E^2 + \frac{1}{2}\|z^{\pm}\|_E^2 - C\int \kappa(x)|u|^{\alpha} - C\int \ell(x)|v|^{\beta}. \end{aligned} \tag{3.6}$$

Let $z^+ = (u^+, v^+) \in E^+$ and $z^- = (u^-, v^-) \in E^-$. Then we have $v^+ = A^{-t}A^s u^+$ and $v^- = -A^{-t}A^s u^-$. Furthermore, we may write $u^{\mp} = \gamma u^{\pm} + \widehat{u}$, where $\widehat{u}$ is orthogonal to $u^{\pm}$ in $L^2(\kappa, \mathbb{R}^N)$. We also have $v^{\mp} = \tau v^{\pm} + \widehat{v}$, where $\widehat{v}$ is orthogonal to $v^{\pm}$ in $L^2(\ell, \mathbb{R}^N)$. It is easy to see that either γ or τ is positive. Suppose $\gamma > 0$. Then we have

$$(1+\gamma)\int \kappa(x)|u^{\pm}|^2 = \int \kappa(x)[(1+\gamma)u^{\pm} + \widehat{u}]u^{\pm} \leq \|u\|_{L^{\alpha}(\kappa,\mathbb{R}^N)}\|u^{\pm}\|_{L^{\alpha'}(\kappa,\mathbb{R}^N)}.$$

Using the fact that the norms in $E_n^{\pm}$ are equivalent we obtain

$$(1+\gamma)\|u^{\pm}\|_{L^\alpha(\kappa,\mathbb{R}^N)} \le C\|u\|_{L^\alpha(\kappa,\mathbb{R}^N)}$$

with constant $C > 0$ independent of u. So from (3.6) and (2.14) we obtain

$$\begin{aligned}\Phi^{\pm}(z) &\le -\frac{1}{2}\|z^{\mp}\|_E^2 + \frac{1}{2}\|z^{\pm}\|_E^2 - C\|u^{\pm}\|^{\alpha}_{L^\alpha(\kappa,\mathbb{R}^N)}\\ &= -\frac{1}{2}\|z^{\mp}\|_E^2 + \|u^{\pm}\|^2_{E^s} - C\|u^{\pm}\|^{\alpha}_{L^\alpha(\kappa,\mathbb{R}^N)}.\end{aligned} \tag{3.7}$$

The same arguments can be applied if $\tau > 0$. So the result follows from (3.7). □

Lemma 3.3. *Let the assumptions of Theorem* 1 *hold. Then the functional* $\Phi^{\pm}$ *satisfies the* (PS) *condition.*

PROOF: Let $(z_n) = (u_n, v_n) \in E$ be a sequence such that

$$|\Phi^{\pm}(z_n)| \le c = const, \tag{3.8}$$

$$|\langle \nabla\Phi^{\pm}(z_n), \eta\rangle| \le \varepsilon_n\|\eta\|_E, \quad \text{with} \quad \varepsilon_n \to 0 \quad \text{and} \quad \eta \in E. \tag{3.9}$$

Taking $\eta = z_n$ in (3.9), we obtain from (3.8) and (3.9) that

$$c + \varepsilon_n\|z_n\|_E \ge -\int F(x,u_n) - \int G(x,v_n) + \frac{1}{2}\int f(x,u_n) + \frac{1}{2}\int g(x,v_n)v_n.$$

Now it follows from (H3) that

$$c + \varepsilon_n\|z_n\|_E \ge \Big(\frac{\alpha}{2} - 1\Big)\int F(x,u_n) + \Big(\frac{\beta}{2} - 1\Big)\int G(x,v_n),$$

and then, in view of (1.3),

$$C + \varepsilon_n\|z_n\|_E \ge C\Big(\int \kappa(x)|u_n|^{\alpha} + \int \lambda(x)|v_n|^{\beta}\Big). \tag{3.10}$$

Next, we estimate $\|u_n\|_{H^s}$ and $\|v_n\|_{H^t}$. It follows from (H2) and (H4) that, given $\varepsilon > 0$, there is a $c_\varepsilon > 0$

$$|f(x,u)| \le \kappa(\varepsilon|u| + c_\varepsilon|u|^p) \quad \text{for all} \quad u. \tag{3.11}$$

From (3.9) with $\psi = 0$ we have

$$\Big|\int A^s\phi A^t v_n\Big| \le \Big|\int f(x,u_n)\phi\Big| + \varepsilon_n\|\phi\|_{H^s} \quad \text{for all} \quad \phi \in H^s.$$

Using (3.11) and Hölder's inequality, we obtain

$$\left|\int A^s\phi A^t v_n\right| \leq \tag{3.12}$$
$$\leq \varepsilon\|u_n\|_{L^2(\kappa,\mathbb{R}^N)}\|\phi\|_{L^2} + c_\varepsilon\|u_n\|^p_{L^\alpha(\kappa,\mathbb{R}^N)}\|\phi\|_{L^{\frac{\alpha}{\alpha-p}}} + \varepsilon_n\|\phi\|_{H^s}.$$

Since $2 \leq \alpha/(\alpha-p) \leq 2N/(N-2s)$, we get from (3.12)

$$\left|\int A^s\phi A^t v_n\right| \leq \left[\varepsilon\|u_n\|_{H^s} + c_\varepsilon\|u_n\|^p_{L^\alpha(\kappa,\mathbb{R}^N)} + C\right]\|\phi\|_{H^s},\ \forall\,\phi\in H^s,$$

which implies that

$$\|v_n\|_{H^t} \leq \varepsilon\|u_n\|_{H^s} + C_\varepsilon\|u_n\|^p_{L^\alpha(\kappa,\mathbb{R}^N)} + C. \tag{3.13}$$

Similarly, we prove that

$$\|u_n\|_{H^s} \leq \varepsilon\|v_n\|_{H^t} + C_\varepsilon\|v_n\|^q_{L^\beta(\ell,\mathbb{R}^N)} + C. \tag{3.14}$$

Adding (3.13) and (3.14) we conclude that

$$\|u_n\|_{H^s} + \|v_n\|_{H^t} \leq C\left[\|u_n\|^p_{L^\alpha(\kappa,\mathbb{R}^N)} + \|v_n\|^q_{L^\beta(\ell,\mathbb{R}^N)} + 1\right]. \tag{3.15}$$

Using (3.10), (3.15) and (H5) we obtain

$$\|u_n\|^\alpha_{L^\alpha(\kappa,\mathbb{R}^N)} + \|v_n\|^\beta_{L^\beta(\ell,\mathbb{R}^N)} \leq C\left[\|u_n\|^p_{L^\alpha(\kappa,\mathbb{R}^N)} + \|v_n\|^q_{L^\beta(\ell,\mathbb{R}^N)}\right] + C.$$

Since $\alpha > p$ and $\beta > q$, we conclude that both $\|u_n\|_{L^\alpha(\kappa,\mathbb{R}^N)}$ and $\|v_n\|_{L^\beta(\ell,\mathbb{R}^N)}$ are bounded, and consequently $\|u_n\|_{H^s}$, $\|v_n\|_{H^t}$ are also bounded in terms of (3.15).

Last, we show that (z_n) contains a strongly convergent subsequence. It follows from Theorem 3.1 that (z_n) contains a subsequence, denoted again by $(z_n) = ((u_n, v_n))$, such that

$$u_n \rightharpoonup u \ \text{ in } \ H^s,\ v_n \rightharpoonup v \ \text{ in } \ H^t, \tag{3.16}$$
$$u_n \to u \ \text{ in } \ L^\gamma(\kappa,\mathbb{R}^N),\ 2<\gamma<2N/(N-2s), \tag{3.17}$$
$$v_n \to v \ \text{ in } \ L^\gamma(\ell,\mathbb{R}^N),\ 2<\gamma<2N/(N-2t). \tag{3.18}$$

It follows then from (3.9) and (3.16) that

$$\int[A^s u A^t\psi + A^s\phi A^t v] = \lim\int[\phi f(x,u_n) + \psi g(x,v_n)] \ \text{ for all } \ (\phi,\psi)\in E. \tag{3.19}$$

Now we claim that

$$\lim \int \phi f(x,u_n) = \int \phi f(x,u), \quad \text{and} \quad \lim \int \psi g(x,v_n) = \int \psi g(x,v). \tag{3.20}$$

Actually for $R>0$ we have

$$\begin{aligned} I_1 &= \int_{B_R} |\phi[f(x,u_n) - f(x,u)]| \\ &\le \|\phi\|_{L^{\theta_1'}(B_R)} \|f(x,u_n) - f(x,u)\|_{L^{\theta_1}(B_R)} \\ &\le \|\phi\|_{H^s} \|f(x,u_n) - f(x,u)\|_{L^{\theta_1}(B_R)}, \end{aligned} \tag{3.21}$$

where $1/\theta_1 + 1/\theta_1' = 1$, $1 < \theta_1 < \gamma/p$. It is easy to verify that for each $R > 0$

$$\|f(x,u_n) - f(x,u)\|_{L^{\theta_1}(B_R)} \to 0. \tag{3.22}$$

Next we deduce from (H2) and (H4) that

$$I_2 := \Big| \int_{B_R^c} \phi[f(x,u_n) - f(x,u)] \Big| \le c \int_{B_R^c} \kappa(x)|\phi| \left[|u_n|^a + |u|^a + |u_n|^p + |u|^p \right],$$

where $B_R^c := \mathbb{R}^N \setminus B_R$. Using Hölder's inequality we have

$$\begin{aligned} I_2 \le\ & c\|\phi\|_{L^{\theta_2'}(B_R^c)} \Big\{ \|u_n\|^a_{L^{a\theta_2}(\kappa,B_R^c)} + \|u\|^a_{L^{a\theta_2}(\kappa,B_R^c)} \Big\} \\ &+ c\|\phi\|_{L^{\theta_3'}(B_R^c)} \Big\{ \|u_n\|^p_{L^{p\theta_3}(\kappa,B_R^c)} + \|u\|^p_{L^{p\theta_3}(B_R^c)} \Big\}. \end{aligned}$$

One can choose θ_2, θ_3 in such a way that $2 \le \theta_2', \theta_2' \le 2N/(N-2s)$ and $2 < a\theta_2, p\theta_3 < 2N/(N-2s)$. Then

$$\begin{aligned} I_2 \le\ & \|\phi\|_{H^s(\mathbb{R}^N)} \Big\{ \|u_n - u\|^a_{L^{a\theta_2}(\kappa,\mathbb{R}^N)} + \|u\|^a_{L^{a\theta_2}(\kappa,B_R^c)} \\ &+ \|u_n - u\|^p_{L^{p\theta_3}(\kappa,\mathbb{R}^N)} + \|u\|^p_{L^{p\theta_3}(\kappa,B_R^c)} \Big\}. \end{aligned} \tag{3.23}$$

On the other hand, by (3.9) we obtain

$$\Big| \pm \int A^s\phi A^t v_n - \int \phi f(x,u_n) \Big| \le \varepsilon_n \|\phi\|_{H^s}, \ \phi \in E^s. \tag{3.24}$$

Therefore, using (3.21), (3.23) and (3.24) we obtain

$$\begin{aligned} \frac{|\int A^s\phi A^t(v_n - v)|}{\|\phi\|_{H^s}} \le\ & C\Big\{ \|f(x,u_n) - f(x,u)\|_{L^{\theta_1}(B_R)} \\ &+ \|u_n - u\|^a_{L^{a\theta_2}(\kappa,\mathbb{R}^N)} + \|u\|^a_{L^{a\theta_2}(\kappa,B_R^c)} \\ &+ \|u_n - u\|^p_{L^{p\theta_3}(\kappa,\mathbb{R}^N)} + \|u\|^p_{L^{p\theta_3}(\kappa,B_R^c)} \Big\}, \ \phi \in E^s. \end{aligned} \tag{3.25}$$

Since the supremum of the left hand side of (3.25) is $\|v_n - v\|_{H^t}$, we conclude that $v_n \to v$ strongly in E^t. In a similar way, we may prove that $u_n \to u$ strongly in E^t. Thus the proof is completed. $\square$

Remark 3.4. *Taking in Section 2 $H = L^2_\gamma(\mathbb{R}^N)$ the space of radially symmetric L^2-functions in $\mathbb{R}^N$ and $T = -\Delta + id$ with domain $D(T) = H^2_\gamma(\mathbb{R}^N)$ the space of radially symmetric functions in L^2 having second derivatives in L^2, we get the following imbedding theorem due to* [FY].

Theorem ([FY]). *Let $s > 0$. Then, the restriction to $H^s_\gamma(\mathbb{R}^N)$ of the Sobolev imbedding of $W^{s,2}(\mathbb{R}^N)$ into $L^\gamma(\mathbb{R}^N)$ is continuous if $2 \le \gamma \le 2N/(N-2s)$, and it is compact if $2 < \gamma < 2N/(N-2s)$.*

Therefore, the same argument allow us to establish consequences of Lemmas 3.2 *and* 3.3 *for the case when f and g depend explicitly on $r = |x|$.*

Proof of Theorem 1: (i) is an immediate consequence of Lemma 3.2, Lemma 3.3 and Theorem [He]. (ii) follows by Remark 3.4 and the same approach. (iii) is a result of Theorem 2.1 of [FY]. □

References

[A] Adams R.A., *Sobolev Spaces*, Academic Press, 1975.

[BL] Berestycki H., Lions P.L., *Nonlinear scalar field equations, I and II*, Arch. Rational Mech. Anal. **82** (1983), 313–376.

[CGM] Coleman S., Glazer V., Martin A., *Action minima among solutions to a class of Euclidean scalar field equations*, Comment. Math. Phys. **58** (1978), 211–221.

[FF] de Figueiredo D.G., Felmer P.L., *On superquadratic elliptic systems*, Trans. Amer. Math. Soc. **343** (1994), 99–116.

[FY] de Figueiredo D.G., Yang Jianfu, *Decay, symmetry and existence of solutions of semilinear elliptic systems*, to appear in Nonlinear Anal. TMA.

[GT] Gilbarg D., Trudinger N.S., *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, 1983.

[HE] Heinz H.P., *Existence and gap-bifurcation of multiple solutions to certain nonlinear eigenvalue problems*, Nonlinear Anal. TMA **21** (1993), 457–484.

[HV] Hulshof J., van der Vorst R., *Differential systems with strongly indefinite variational structure*, J. Funct. Anal. **114** (1993), 32–58.

[JW] Weidmann J., *Linear Operators in Hilbert Spaces*, GMT 68, Springer-Verlag, 1980.

[KA] Kato T., *Perturbation Theory for Linear Operators*, Springer-Verlag, 1966.

[LM] Lions J.L., Magenes E., *Non-homogeneous Boundary Value Problems and Applications, I*, Springer-Verlag, 1972.

[PA] Palais R., *The principle of symmetric critically*, Comment. Math. Phys. **69** (1979), 19–30.

[PL] Lions P.L., *Symétrie et compacité dans les espaces de Sobolev*, J. Funct. Anal. **49** (1982), 315–334.

[R] Rabinowitz P., *Minimax methods in critical point theory with applications to differential equations*, CBMS Regional Conf. Ser. in Math., No. 65, Amer. Math. Soc., Providence, R.I., 1986.

[S] Strauss W., *Existence of solitary waves in higher dimensions*, Comment. Math. Phys. **55** (1977), 149–162.

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