Completion theorem for uniform entropy

Takashi Kimura

Abstract. Modifying Bowen's entropy, we introduce a new uniform entropy. We prove that the completion theorem for uniform entropy holds in the class of all metric spaces. However, the completion theorem for Bowen's entropy does not hold in the class of all totally bounded metric spaces.

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1. Introduction

Recently, several authors (see [A], [AM], [AO]) study topological dynamics from the point of view of General Topology. In this paper we study the uniform entropy. All spaces are assumed to be completely regular and T_1 unless otherwise stated. For standard results and notation in General Topology and Topological Dynamics we refer to [E], [DGS], [W].

In [AKM] Adler, Konheim and McAndrew introduced a topological entropy of a continuous self-mapping f on a compact space as an analogue of the measure theoretic entropy. We denote by $h_{\mathrm{AKM}}(f)$ this entropy. In [B] Bowen introduced another entropy of a uniformly continuous self-mapping f on a metric space (X, d); this will be denoted by $h_{B,d}(f)$. Bowen's entropy $h_{B,d}(f)$ is a uniform invariant. Since a continuous self-mapping f on a compact metrizable space X is uniformly continuous and since all metrics on X are uniformly equivalent, we can uniquely define its Bowen's entropy $h_{B,d}(f)$. It is known that $h_{\mathrm{AKM}}(f) = h_{B,d}(f)$ in this case.

In Section 3, modifying Bowen's entropy, we shall introduce a new uniform entropy $h(f)$ of a uniformly continuous self-mapping f on a uniform space (X, Φ).

In Section 4 we shall investigate the relationship among $h(f)$, $h_{B,d}(f)$ and $h_{\mathrm{AKM}}(f)$. Our uniform entropy $h(f)$ coincides with $h_{\mathrm{AKM}}(f)$ in the case when f is a continuous self-mapping on a compact space. Regarding a metric space (X, d) as the uniform space (X, Φ_d), where Φ_d is the uniformity induced by d, our uniform entropy $h(f)$ coincides with $h_{B,d}(f)$ in the case when (X, d) is complete. However, these two entropies do not coincide in general.

In Section 5 we shall investigate the completion theorem. For a uniformly continuous self-mapping f on a uniform space (X, Φ) we denote by $\widetilde{f}$ the uniformly continuous extension of f over the completion of (X, Φ). We shall prove that $h(\widetilde{f}) = h(f)$ in the case when Φ is totally bounded or Φ is induced by a metric on X. Thus in the class of all metric spaces the completion theorem for the uniform entropy holds, though it does not hold in general. However, the completion theorem for Bowen's entropy does not hold even in the class of all totally bounded metric spaces.

2. Preliminaries

In this section we record some terminology and some results that we shall need later.

For a set A we denote by $\#A$ the cardinality of A. For a mapping $f : X \to Y$ and for a subset A of X we denote by $f|_A$ the restriction of f to A. We denote by ω the first infinite ordinal number and by ω_1 the first uncountable ordinal number.

Let $\mathcal{A}$ and $\mathcal{B}$ be collections of subsets of a set X, $Y \subset X$ and $x, y \in X$. We set

$$\begin{aligned}
\mathrm{St}\,(x,\mathcal{A}) &= \bigcup\{A \in \mathcal{A} : x \in A\},\\
\mathrm{St}\,(Y,\mathcal{A}) &= \bigcup\{A \in \mathcal{A} : A \cap Y \neq \emptyset\},\\
\mathcal{A}^{\Delta} &= \{\mathrm{St}\,(x,\mathcal{A}) : x \in X\},\\
\mathcal{A}^{*} &= \{\mathrm{St}\,(A,\mathcal{A}) : A \in \mathcal{A}\},\\
\mathcal{A} \wedge \mathcal{B} &= \{A \cap B : A \in \mathcal{A} \ \text{ and } \ B \in \mathcal{B}\},\\
\bigcup \mathcal{A} &= \bigcup\{A : A \in \mathcal{A}\} \ \text{ and}\\
\bigcap \mathcal{A} &= \bigcap\{A : A \in \mathcal{A}\}.
\end{aligned}$$

We say that $\mathcal{A}$ *refines* $\mathcal{B}$, in symbol $\mathcal{A} < \mathcal{B}$ if for every $A \in \mathcal{A}$ there exists a $B \in \mathcal{B}$ such that $A \subset B$. If $x \in \mathrm{St}\,(y,\mathcal{A})$, then we write $d(x,y) < \mathcal{A}$. Otherwise, we write $d(x,y) > \mathcal{A}$.

It is obvious that if $\mathcal{A} < \mathcal{B}$ and $d(x,y) < \mathcal{A}$, then $d(x,y) < \mathcal{B}$.

By a *uniformity* Φ on a space X, we mean a collection of open covers of X satisfying:

(1) if $\mathcal{U} \in \Phi$ and $\mathcal{V}$ is an open cover of X with $\mathcal{U} < \mathcal{V}$, then $\mathcal{V} \in \Phi$,
(2) for $\mathcal{U}, \mathcal{V} \in \Phi$ there exists $\mathcal{W} \in \Phi$ such that $\mathcal{W} < \mathcal{U} \wedge \mathcal{V}$,
(3) for every $\mathcal{U} \in \Phi$ there exists $\mathcal{V} \in \Phi$ such that $\mathcal{V}^{*} < \mathcal{U}$,
(4) for every $x \in X$ the collection $\{\mathrm{St}\,(x,\mathcal{U}) : \mathcal{U} \in \Phi\}$ is a neighborhood base at x.

A *uniform space* is a pair (X, Φ) consisting of a space X and a uniformity Φ on X.

For a subset Y of a uniform space (X, Φ) we set $\Phi|_Y = \{\mathcal{U}|_Y : \mathcal{U} \in \Phi\}$, where $\mathcal{U}|_Y = \{U \cap Y : U \in \mathcal{U}\}$. Then $(Y, \Phi|_Y)$ is also a uniform space.

A uniform space (X, Φ) is *totally bounded* if every $\mathcal{U} \in \Phi$ has a finite subcover of X. Let $\mathcal{F}$ be a collection of subsets of a uniform space (X, Φ). We say that $\mathcal{F}$ *contains arbitrarily small sets* if for every $\mathcal{U} \in \Phi$ there exists $F \in \mathcal{F}$ such that $F \subset U$ for some $U \in \mathcal{U}$. A uniform space (X, Φ) is *complete* if every collection $\mathcal{F}$ of closed subsets of X which has the finite intersection property and contains arbitrarily small sets has non-empty intersection. It is well known ([E, 8.3.16]) that a uniform space X is compact if and only if X is totally bounded and complete. It is obvious that the unique uniformity on a compact space X is the collection of all open covers of X.

For a metric space (X, d) let Φ_d be the collection of all open covers of X which is refined by $\{B(x, \varepsilon) : x \in X\}$ for some $\varepsilon > 0$, where $B(x, \varepsilon)$ is the ε-neighborhood of x in X. Then (X, Φ_d) is a uniform space. We say that Φ_d is the uniformity which is *induced* by the metric d on X. It is well known ([E, 8.3.1 and 8.3.5]) that the metric space (X, d) is totally bounded (respectively complete) if and only if the uniform space (X, Φ_d) is totally bounded (respectively complete). Let Ψ be a subcollection of a uniformity Φ on a space X. We say that Ψ is a *base* for Φ if for every $\mathcal{U} \in \Phi$ there exists $\mathcal{V} \in \Psi$ such that $\mathcal{V} < \mathcal{U}$. The smallest cardinal number of the form $\#\Psi$, where Ψ is a base for Φ, is called the *weight of the uniformity* Φ. It is known ([E, 8.1.21]) that a uniformity Φ on a space X is induced by a metric on X if and only if the weight of Φ is at most countable.

We denote by $(\widetilde{X}, \widetilde{\Phi})$ the completion of a uniform space (X, Φ). The completion $(\widetilde{X}, \widetilde{\Phi})$ is the unique uniform space which is complete and contains (X, Φ) as a dense uniform subspace. For an open subset G of X we set $\gamma(G) = \widetilde{X} - \mathrm{Cl}_{\widetilde{X}}(X - G)$. Then it is known that $\{\gamma(\mathcal{U}) : \mathcal{U} \in \Phi\}$ is a base for $\widetilde{\Phi}$, where $\gamma(\mathcal{U}) = \{\gamma(U) : U \in \mathcal{U}\}$.

For more detailed information about uniform spaces, the reader is referred to the Engelking's book [E] and Morita's paper [M].

3. Definition of uniform entropy

In this section we shall define the uniform entropy $h(f)$ of a uniformly continuous self-mapping f on a uniform space.

The idea of this definition is essentially due to Adler, Konheim and McAndrew [AKM] and Bowen [B]. However, there is a slight difference between the uniform and Bowen's entropy.

For a while we shall fix a uniform space (X, Φ) and a uniform continuous self-mapping f on X.

Definition 3.1. Let Y be a totally bounded subset of X, $\mathcal{U} \in \Phi$ and $n > 0$.

(1) We set

$$N(Y, \mathcal{U}) = \min\{\#\mathcal{V} : \mathcal{V} \subset \mathcal{U} \ \text{ and } \ Y \subset \bigcup \mathcal{V}\}.$$

(2) A subset Z of X is $(f, Y, \mathcal{U}, n)$-*spanning* if for every $y \in Y$ there exists $z \in Z$ such that $d(f^i(y), f^i(z)) < \mathcal{U}$ for each i, $0 \le i < n$.

(3) A subset Z of Y is $(f, Y, \mathcal{U}, n)$-*separated* if for any distinct points x and y of Z there exists i, $0 \le i < n$, such that $d(f^i(x), f^i(y)) > \mathcal{U}$.

For every $\mathcal{U} \in \Phi$ and any $n > 0$ we set

$$f^{-n}(\mathcal{U}) = \{f^{-n}(U) : U \in \mathcal{U}\},$$

where $f^{-n}(U)$ is the inverse image of U under the n-fold composition f^n of f.

Definition 3.2. Let Y be a totally bounded subset of X, $\mathcal{U} \in \Phi$ and $n > 0$. Then we set

$$\begin{aligned} c(f, Y, \mathcal{U}, n) &= N\big(Y, \mathcal{U} \wedge f^{-1}(\mathcal{U}) \wedge \cdots \wedge f^{-(n-1)}(\mathcal{U})\big), \\ r(f, Y, \mathcal{U}, n) &= \min\{\#Z : Z \text{ is } (f, Y, \mathcal{U}, n)\text{-spanning}\} \text{ and} \\ s(f, Y, \mathcal{U}, n) &= \max\{\#Z : Z \text{ is } (f, Y, \mathcal{U}, n)\text{-separated}\}. \end{aligned}$$

Since Y is totally bounded, the above $\sigma(f, Y, \mathcal{U}, n)$ is finite for $\sigma = c, r, s$.

Definition 3.3. Let Y be a totally bounded subset of X and $\mathcal{U} \in \Phi$. Then we set

$$\begin{aligned} \sigma(f, Y, \mathcal{U}) &= \limsup_{n \to \infty} \frac{1}{n} \log \sigma(f, Y, \mathcal{U}, n), \\ h_\sigma(f, Y) &= \sup\{\sigma(f, Y, \mathcal{U}) : \mathcal{U} \in \Phi\} \text{ and} \\ h_\sigma(f) &= \sup\{h_\sigma(f, Y) : Y \text{ is a totally bounded subset of } X\}, \end{aligned}$$

for $\sigma = c, r, s$.

Basic fact 3.4. *Let Y and Z be totally bounded subsets of $X, \mathcal{U}, \mathcal{V} \in \Phi$ and $n > 0$. Then*

(a) *if $\mathcal{V} < \mathcal{U}$, then $\sigma(f, Y, \mathcal{U}, n) \leq \sigma(f, Y, \mathcal{V}, n)$ and*
(b) *if $Y \subset Z$, then $\sigma(f, Y, \mathcal{U}, n) \leq \sigma(f, Z, \mathcal{U}, n)$,*

for $\sigma = c, r, s$.

Corollary 3.5. (a) *If Ψ is a base for Φ, then $h_\sigma(f, Y) = \sup\{\sigma(f, Y, \mathcal{V}) : \mathcal{V} \in \Psi\}$ for $\sigma = c, r, s$.*
(b) *If (X, Φ) is a totally bounded uniform space, then we have $h_\sigma(f) = h_\sigma(f, X)$ for $\sigma = c, r, s$.*

The proof of the following lemma is essentially the same as Bowen's proof [B, Lemma 1], so we omit the proof.

Lemma 3.6. *Let Y be a totally bounded subset of X, $\mathcal{U}, \mathcal{V} \in \Phi$ and $n > 0$. Then*

(a) *$r(f, Y, \mathcal{U}, n) \leq s(f, Y, \mathcal{U}, n)$,*
(b) *$s(f, Y, \mathcal{U}, n) \leq c(f, Y, \mathcal{U}, n)$ and*
(c) *if $\mathcal{V}^\Delta < \mathcal{U}$, then $c(f, Y, \mathcal{U}, n) \leq r(f, Y, \mathcal{U}, n)$.*

By Lemma 3.6, we obtain the following theorem.

Theorem 3.7. *For every uniformly continuous self-mapping f on a uniform space (X, Φ) the equalities $h_c(f) = h_r(f) = h_s(f)$ hold.*

Definition 3.8. By Theorem 3.7, we can write $h(f)$ instead of $h_c(f)$, $h_r(f)$ or $h_s(f)$. We say that $h(f)$ is the *uniform entropy* of f. We often write $h(f, \Phi)$ to specify the uniformity Φ.

Remark 3.9. By the definition of Bowen's entropy, it is obvious that for a uniformly continuous self-mapping f on a metric space (X, d) we have

$$h_{B,d}(f) = \sup\{h_\sigma(f, Y) : Y \text{ is a compact subset of } X\},$$

where $\sigma = c$, r or s. Hence the uniform entropy $h(f)$ is one that is defined by replacing the condition of the compactness of Y in Bowen's entropy with the condition of the total boundedness of Y.

For the Bowen's entropy, the compactness of Y is needed to show that $\sigma(f, Y, \mathcal{U}, n)$ is finite. However, it is obvious that $\sigma(f, Y, \mathcal{U}, n)$ is finite for every $\mathcal{U} \in \Phi$ if and only if Y is totally bounded.

Bowen's entropy is a uniform invariant. However, the compactness is a topological property. Because the total boundedness is a uniform property, it is natural to define the uniform entropy by the total boundedness of Y.

4. Relationship between the uniform and Bowen's entropy

Let (X, d) be a metric space. Then a self-mapping f on X is uniformly continuous with respect to the metric d if and only if f is uniformly continuous with respect to the uniformity Φ_d induced by the metric d. Thus we can define the uniform entropy $h(f, \Phi_d)$ and Bowen's entropy $h_{B,d}(f)$. This section is concerned with the relationship between the uniform entropy $h(f)$ and Bowen's entropy $h_{B,d}(f)$.

Since every compact subset of a uniform space is totally bounded, we obtain the following theorem.

Theorem 4.1. *Let (X, d) be a metric space and f a uniformly continuous self-mapping on X. Then the inequality $h_{B,d}(f) \leq h(f, \Phi_d)$ holds.*

Theorem 4.2. *Let (X, d) be a complete metric space and f a uniformly continuous self-mapping on X. Then the equality $h_{B,d}(f) = h(f, \Phi_d)$ holds.*

PROOF: By Theorem 4.1, it suffices to show that $h_{B,d}(f) \geq h(f, \Phi_d)$. Let Y be a totally bounded subset of X. Then $\mathrm{Cl}_X Y$ is compact, because $\mathrm{Cl}_X Y$ is totally bounded and complete. Thus, by Basic fact 3.4(b) and Remark 3.9, we have

$$h(f, Y) \leq h(f, \mathrm{Cl}_X Y) \leq h_{B,d}(f).$$

This implies that $h_{B,d}(f) \geq h(f, \Phi_d)$. □

In [B] Bowen stated "An essential part of this paper is the computation of $h_d(T)$ for certain maps on noncompact spaces". The spaces which he considered are complete metric spaces. Thus, by Theorem 4.2, his computation is correct for the uniform entropy.

Example 4.3. *There exist a totally bounded metric space* (X, d) *and a uniformly continuous self-mapping* φ *on* X *such that* $h_{B,d}(f) < h(\varphi, \Phi_d)$.

Construction 4.4. Let 2^n be the set of all mappings from n to 2 and 2^ω the set of all mappings from ω to 2, where $2 = \{0, 1\}$, $n = \{0, 1, 2, \dots, n-1\}$ and $\omega = \{0, 1, 2, \dots\}$. Let us set

$$2^{\leq n} = \bigcup\{2^i : i \leq n\} \text{ for each } n < \omega, \text{ and}$$
$$Y = 2^{\leq \omega} = 2^\omega \cup \bigcup\{2^n : n < \omega\}.$$

For a point f of Y we write $f = (f_i)$, where $f_i = f(i)$. If we write f_i, then this means that f_i is definable, that is, $f \notin 2^{\leq i}$. For a pair of points f and g of Y we define

$$f \leq g \text{ if and only if } \begin{cases} f_i = g_i & \text{for each } i < n \text{ if } f \in 2^n \\ f = g & \text{if } f \in 2^\omega. \end{cases}$$

We write $f < g$ if $f \leq g$ and $f \neq g$. We shall define a metric d on Y. For a pair of points f and g of Y we define

$$d(f,g) = \begin{cases} 2 & \text{if } f_0 \neq g_0 \\ 1/\ell & \text{if } f_\ell \neq g_\ell \text{ and } f_i = g_i \text{ for each } i < \ell \\ 1/m & \text{if } (f \in 2^m \text{ and } f < g) \text{ or } (g \in 2^m \text{ and } g < f) \\ 0 & \text{if } f = g. \end{cases}$$

Lemma 4.5. *d is a metric on Y.*

Proof: It suffices to show that the triangle inequality

$$d(f,h) \leq d(f,g) + d(g,h)$$

holds for $f, g, h \in Y$. We may assume that $f \neq g$. There are three cases to consider.

Case 1. $d(f,h) = 0$ or 2.

In this case, obviously, the triangle inequality holds.

Case 2. $d(f,h) = 1/\ell$ ($f_\ell \neq h_\ell$ and $f_i = h_i$ for each $i < \ell$).

If $g \in 2^{\leq \ell}$, then $d(f,g) \geq 1/\ell$. Suppose $g \notin 2^{\leq \ell}$. If $f_i \neq g_i$ for some $i < \ell$, then $d(f,g) \geq 1/i > 1/\ell$. If $f_i = g_i$ for each $i < \ell$, then $g_i = h_i$ for each $i < \ell$. Since $f_\ell \neq h_\ell$, we have $f_\ell \neq g_\ell$ or $g_\ell \neq h_\ell$. Thus $d(f,g) = 1/\ell$ or $d(g,h) = 1/\ell$.

Case 3. $d(f,g) = 1/m$ ($f \in 2^m$ and $f < h$).

If $g \in 2^{\leq m}$, then $d(f,g) \geq 1/m$. Suppose $g \notin 2^{\leq m}$. If $f < g$, then $d(f,g) = 1/m$. If $f \not< g$, then $f_i \neq g_i$ for some $i < m$, therefore $d(f,g) \geq 1/i > 1/m$.

In any case the triangle inequality holds. Hence d is a metric on Y. □

Lemma 4.6. *The metric space (Y,d) is compact.*

PROOF: Let F be an infinite subset of Y. We shall show that F has an accumulation point. Inductively, for each $i < \omega$, we can take $g_i \in 2$ such that $\{f \in F : (g_0, g_1, \dots, g_n) \leq f\}$ is infinite. Then $g = (g_0, g_1, \dots) \in 2^\omega$ is an accumulation point of F. Hence (Y,d) is compact.

□

Construction 4.7. Let $\psi : Y \to Y$ be the mapping defined by

$$\psi(f) = \begin{cases} (f_1, f_2, \dots, f_{n-1}) \in 2^{n-1} & \text{if } f = (f_0, f_1, \dots, f_{n-1}) \in 2^n \\ (f_1, f_2, \dots) \in 2^\omega & \text{if } f = (f_0, f_1, \dots) \in 2^\omega \\ \phi \in 2^0 & \text{if } f = \phi \in 2^0. \end{cases}$$

Note that, by this definition, $\psi(f) = \phi \in 2^0$ for each $f = (f_0) \in 2^1$. Then it is easy to see that ψ is continuous and $\psi(\bigcup\{2^n : n < \omega\}) = \bigcup\{2^n : n < \omega\}$. Let us set

$$X = \bigcup\{2^n : n < \omega\}, \quad \varphi = \psi|_X \text{ and } d_X = d|_{X \times X}.$$

Then, obviously, φ is a uniformly continuous self-mapping on (X, d_X). By Lemma 4.6, (X, d_X) is totally bounded.

Lemma 4.8. $h_{B,d}(\varphi) = 0$.

PROOF: Since X is discrete, for every compact subset K of X we have $\#K = n$ for some $n < \omega$. By the definition of Bowen's entropy, we have $\overline{s}_\varphi(\varepsilon, K)$ ($= s(\varphi, K, \mathcal{U}_\varepsilon)$, where $\mathcal{U}_\varepsilon$ is the collection of all open subsets U of X with $\operatorname{diam} U < \varepsilon$) $= 0$ (see [B]), therefore $h_{B,d}(\varphi) = 0$. □

Lemma 4.9. $h(\varphi, \Phi_{d_X}) \neq 0$.

PROOF: Let us set

$$Z = Y - X = 2^\omega, \quad \sigma = \psi|_Z \text{ and } d_Z = d|_{Z \times Z}.$$

Then $\sigma : 2^\omega \to 2^\omega$ is the shift. It is well known that $h_{B,d_Z}(\sigma) = \log 2$ (see [W, p. 182]). Since X is dense in Y and Y is compact, (Y, Φ_d) is the completion of (X, Φ_{d_X}). By Theorem 5.3 (see below), we have $h(\varphi, \Phi_{d_X}) = h(\psi, \Phi_d)$. By Lemma 5.1 (see below), we have $h(\psi, \Phi_d) \geq h(\sigma, \Phi_{d_Z})$. Hence we have

$$h(\varphi, \Phi_{d_X}) \geq h(\sigma, \Phi_{d_Z}) = h_{B,d_Z}(\sigma) \neq 0.$$

□

It is well known that for a compact space X there exists exactly one uniformity Φ on X and every continuous self-mapping on X is uniformly continuous with respect to Φ. Thus we can define two entropies $h(f)$ and $h_{\mathrm{AKM}}(f)$.

Theorem 4.10. *Let f be a continuous self-mapping on a compact space X. Then the equality $h_{\mathrm{AKM}}(f) = h(f)$ holds.*

PROOF: Since X is compact, the unique uniformity Φ coincides with the set of all open covers of X. Thus we have $h_{\mathrm{AKM}}(f) = h_c(f, X) = h(f)$. □

5. Completion theorem for the uniform entropy

This section is concerned with the completion theorem for the uniform entropy.

Let f be a self-mapping on a set X. Then a subset Y of X is f-*invariant* if $f(Y) \subset Y$.

Lemma 5.1. *Let f be a uniformly continuous self-mapping on a uniform space (X, Φ) and let Y be an f-invariant subset of X. Then the inequality $h(f|_Y, \Phi|_Y) \le h(f, \Phi)$ holds.*

Proof: Let Z be a totally bounded subset of Y. Then, obviously, Z is a totally bounded subset of X. Since Y is f-invariant, we have $c(f|_Y, Z, \mathcal{U}|_Y, n) = c(f, Z, \mathcal{U}, n)$, therefore, $h_c(f|_Y, Z) = h_c(f, Z)$. This implies that $h(f|_Y, \Phi|_Y) \le h(f, \Phi)$. □

Let (X, Φ) be a uniform space and f a uniformly continuous self-mapping on X. We denote by $(\widetilde{X}, \widetilde{\Phi})$ the completion of (X, Φ) and by $\widetilde{f}$ the uniformly continuous extension of f over $\widetilde{X}$. We concern ourselves with the relationship between $h(f, \Phi)$ and $h(\widetilde{f}, \widetilde{\Phi})$.

Theorem 5.2. *Let f be a uniformly continuous self-mapping on a uniform space (X, Φ). Then the inequality $h(f, \Phi) \le h(\widetilde{f}, \widetilde{\Phi})$ holds.*

Proof: This follows from Lemma 5.1. □

Theorem 5.3. *Let f be a uniformly continuous self-mapping on a totally bounded uniform space (X, Φ). Then the equality $h(f, \Phi) = h(\widetilde{f}, \widetilde{\Phi})$ holds.*

Proof: By Theorem 5.2, it suffices to show that $h(f, \Phi) \ge h(\widetilde{f}, \widetilde{\Phi})$. For every $\mathcal{U} \in \widetilde{\Phi}$ we take $V \in \Phi$ such that $\gamma(V)^\Delta < \mathcal{U}$. Let F be a $(f, X, \mathcal{V}, n)$-spanning set such that $\#F = r(f, X, \mathcal{V}, n)$. Let us set

$$\mathcal{W} = \bigwedge\{\widetilde{f}^{-i}(\gamma(\mathcal{V})) : 0 \le i < n\}.$$

For every $y \in \widetilde{X}$ there exists $y' \in X$ such that $d(y, y') < \mathcal{W}$. Then we have

$$d(\widetilde{f}^i(y), \widetilde{f}^i(y')) < \gamma(\mathcal{V})$$

for each $i,\ 0 \le i < n$. Since F is $(f, X, \mathcal{V}, n)$-spanning, we can take $x \in F$ such that

$$d(f^i(x), f^i(y')) < \mathcal{V}$$

for each $i,\ 0 \le i < n$. Then, obviously, we have

$$d(\widetilde{f}^i(x), \widetilde{f}^i(y)) < \gamma(\mathcal{V})^\Delta$$

for each $i,\ 0 \le i < n$. Since $\gamma(\mathcal{V})^\Delta < \mathcal{U}$, we have

$$d(\widetilde{f}^i(x), \widetilde{f}^i(y)) < \mathcal{U}$$

for each $i,\ 0 \le i < n$. This implies that F is $(\widetilde{f}, \widetilde{X}, \mathcal{U}, n)$-spanning. Thus we have

$$r(\widetilde{f}, \widetilde{X}, \mathcal{U}, n) \le \#F = r(f, X, \mathcal{V}, n).$$

Hence we have $h(f, \Phi) \ge h(\widetilde{f}, \widetilde{\Phi})$. Theorem 5.3 has been proved. □

Theorem 5.4. *Let f be a uniformly continuous self-mapping on a uniform space (X,Φ) such that the weight of the uniformity Φ is countable* (*i.e.* Φ *is induced by a metric on* X), *Then the equality* $h(f,\Phi) = h(\widetilde{f},\widetilde{\Phi})$ *holds.*

PROOF: By Theorem 5.2, it suffices to show that $h(f,\Phi) \geq h(\widetilde{f},\widetilde{\Phi})$. Since the weight of the uniformity Φ is countable, we can take a countable base $\{\mathcal{U}_n : n < \omega\}$ for Φ. Let Y be a totally bounded subset of $\widetilde{X}$. Then for each $n < \omega$ we can take a finite subcollection $\mathcal{W}_n$ of $\mathcal{U}_n$ such that $\gamma(\mathcal{W}_n)$ covers Y. Note that $\widetilde{X}$ is metrizable, because the weight of $\widetilde{\Phi}$ is countable. Since Y is a totally bounded subset of a metrizable space $\widetilde{X}$, Y is separable and, therefore, we can take a countable dense subset $D = \{d_i : i < \omega\}$ of Y. For $i,n < \omega$ with $i \leq n$ we take $x_i^n \in X$ such that

$$x_i^n \in \bigcap\{W : d_i \in W \in \mathcal{W}_j \ \text{ and } \ j \leq n\}.$$

Let us set

$$Z = \{x_i^n : i,n < \omega \ \text{ with } \ i \leq n\}.$$

Then it is easy to see that Z is totally bounded and $Y \subset \mathrm{Cl}_{\widetilde{X}}\, Z$. For every $\mathcal{U} \in \widetilde{\Phi}$ there exists $\mathcal{V} \in \Phi$ such that $\gamma(\mathcal{V})^{\Delta} < \mathcal{U}$. Let F be $(f,Z,\mathcal{V},n)$-spanning such that $\#F = r(f,Z,\mathcal{V},n)$. We shall show that F is $(\widetilde{f},Y,\mathcal{U},n)$-spanning. To end this, let $y \in Y$. Then there exists $z \in Z$ such that

$$d(\widetilde{f}^{\,i}(z),\widetilde{f}^{\,i}(y)) < \gamma(\mathcal{V})$$

for each $i,\ 0 \leq i < n$, because $Y \subset \mathrm{Cl}_{\widetilde{X}}\, Z$. Then we can take $x \in F$ such that

$$d(f^i(x),f^i(z)) < \mathcal{V}$$

for each $i,\ 0 \leq i < n$, because F is $(f,Z,\mathcal{V},n)$-spanning. Then, obviously, we have

$$d(\widetilde{f}^{\,i}(x),\widetilde{f}^{\,i}(y)) < \gamma(\mathcal{V})^{\Delta}$$

for each $i,\ 0 \leq i < n$. Since $\gamma(\mathcal{V})^{\Delta} < \mathcal{U}$, we have

$$d(\widetilde{f}^{\,i}(x),\widetilde{f}^{\,i}(y)) < \mathcal{U}$$

for each $i,\ 0 \leq i < n$. This implies that F is $(\widetilde{f},Y,\mathcal{U},n)$-spanning. Thus we have

$$r(\widetilde{f},Y,\mathcal{U},n) \leq \#F = r(f,Z,\mathcal{V},n).$$

Hence we have $h(f,\Phi) \geq h(\widetilde{f},\widetilde{\Phi})$. Theorem 5.4 has been proved. □

In the class of all spaces, the completion theorem for the uniform entropy does not hold.

Recall that a homeomorphism f on a space X is *topologically transitive* if the orbit $\mathrm{Orb}(x) = \{f^n(x) : n \in \mathbb{Z}\}$ of x is dense in X for some $x \in X$. It is known that the shift $\sigma : k^{\mathbb{Z}} \to k^{\mathbb{Z}}$ defined by $(\sigma(x))_i = x_{i+1}$ is topologically transitive (see [W, p. 122]), where $k = \{0,1,2,\dots,k-1\}$.

Example 5.5. *There exist a uniform space (X,Φ) and a uniform isomorphism f on X such that $h(f,\Phi) < h(\widetilde{f},\widetilde{\Phi})$.*

Let $D(\omega_1)$ be the discrete space of cardinality ω_1 and $A(\omega_1) = D(\omega_1) \cup \{x_*\}$, where $x_* \notin D(\omega_1)$. For every $x \in A(\omega_1)$ we put

$$\mathcal{B}(x) = \begin{cases} \{\{x\}\} & \text{if } x \in D(\omega_1) \\ \{U : x_* \in U \ \text{ and } \ \#(A(\omega_1) - U) \le \omega\} & \text{if } x = x_*. \end{cases}$$

We endow $A(\omega_1)$ with a topology by taking $\{\mathcal{B}(x) : x \in A(\omega_1)\}$ as a neighborhood system. We can take a pairwise disjoint cover $\{A_\alpha : \alpha < \omega_1\}$ of $D(\omega_1)$ such that $\#A_\alpha = \omega$ for every $\alpha < \omega_1$.

We denote by $2^{\mathbb{Z}}$ the countably infinite product space $\prod\{2_i : i \in \mathbb{Z}\}$, where $2_i = 2 = \{0,1\}$ for every $i \in \mathbb{Z}$, endowed with the product topology. Let $\sigma : 2^{\mathbb{Z}} \to 2^{\mathbb{Z}}$ be the shift on $2^{\mathbb{Z}}$. Since σ is topologically transitive, we can take a point $y \in 2^{\mathbb{Z}}$ such that $\mathrm{Orb}(y) = \{\sigma^i(y) : i \in \mathbb{Z}\}$ is dense in $2^{\mathbb{Z}}$.

Since $\#A_\alpha = \omega$ for every $\alpha < \omega_1$, we take a bijection $f_\alpha : A_\alpha \to \mathrm{Orb}(y)$. Let us set

$$X = \{(x, f_\alpha(x)) : x \in A_\alpha \ \text{ and } \ \alpha < \omega_1\}, \quad \text{and}$$
$$Y = X \cup (\{x_*\} \times 2^{\mathbb{Z}}).$$

We regard Y as a subspace of $A(\omega_1) \times 2^{\mathbb{Z}}$. Let $g : Y \to Y$ be the mapping defined by $g(x, f_\alpha(x)) = (f_\alpha^{-1}(\sigma(f_\alpha(x))), \sigma(f_\alpha(x)))$ for every $(x, f_\alpha(x)) \in X$ and $g(x_*, z) = (x_*, \sigma(z))$ for every $(x_*, z) \in \{x_*\} \times 2^{\mathbb{Z}}$. It is easy to see that g is a homeomorphism. Since every point of X is isolated and $2^{\mathbb{Z}}$ is compact, Y is paracompact. Thus (Y,Ψ) is a complete uniform space, where Ψ is the collection of all open covers of Y (see [E, 8.5.13]). Obviously, g is uniformly continuous with respect to Ψ and X is g-invariant. Let us set

$$f = g|_X \ \text{ and } \ \Phi = \Psi|_X.$$

Then we have

$$\widetilde{f} = g \ \text{ and } \ \widetilde{\Phi} = \Psi.$$

Since $g|_{\{x_*\}\times 2^{\mathbb{Z}}}$ is the shift on $\{x_*\} \times 2^{\mathbb{Z}}$, by Theorem 5.5(b), we have

$$h(\widetilde{f},\widetilde{\Phi}) = h(g,\Psi) \ge h(g|_{\{x_*\}\times 2^{\mathbb{Z}}}, \Psi|_{\{x_*\}\times 2^{\mathbb{Z}}}) = \log 2.$$

We shall prove that $h(f,\Phi) = 0$. To end this, it suffices to show that every totally bounded subset of X is finite. Let Z be a totally bounded subset of X. Assume that $\mathrm{Cl}_Y \cap (\{x_*\} \times 2^{\mathbb{Z}}) \ne \emptyset$. We take a countably infinite subset Z' of Z. Then we can take an open subset U of Y such that $Z' \cap U = \emptyset$ and $\{x_*\} \times 2^{\mathbb{Z}} \subset U$. Then $\mathcal{U} = \{U\} \cup \{\{x\} : x \in X\} \in \Psi$ has no finite subcollection which covers Z. This contradicts the total boundedness of Z. Thus $\mathrm{Cl}_Y Z \subset X$. Since X is discrete and $\mathrm{Cl}_Y Z$ is compact, Z is finite. Hence we have $h(f,\Phi) = 0$.

By Theorem 5.4, in the class of all metric spaces the completion theorem for the uniform entropy holds. However, the completion theorem for Bowen's entropy does not hold even in the class of all totally bounded metric spaces.

Example 5.6. *There exist a totally bounded metric space* (X, d_X) *and a uniformly continuous self-mapping* φ *on* X *such that* $h_{B,d_X}(\varphi) < h_{B,\widetilde{d}_X}(\varphi)$, *where* $(\widetilde{X}, \widetilde{d}_X)$ *is the completion of* (X, d_X).

Let (X, d_X), (Y, d), φ and ψ be as in Constructions 4.4 and 4.7. It is obvious that $(Y, d) = (\widetilde{X}, \widetilde{d}_X)$ and $\psi = \widetilde{\varphi}$. In Lemma 4.8 we have shown that $h_{B,d}(\varphi) = 0$. On the other hand, by Theorems 4.2, 5.4 and Lemma 4.9, we have

$$h_{B,\widetilde{d}_X}(\widetilde{\varphi}) = h_{B,d}(\psi) = h(\psi, \Phi_d) = h(\widetilde{\varphi}, \widetilde{\Phi}_{d_X}) = h(\varphi, \Phi_{d_X}) \neq 0.$$

Hence we have $h_{B,d}(\varphi) < h_{B,\widetilde{d}_X}(\widetilde{\varphi})$.

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Department of Mathematics, Faculty of Education, Saitama University, Urawa, Saitama, 338, Japan

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