Paley-Wiener theorems for the Schrodinger operator on $\mathbb{R}$

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Abstract. In this paper we define and study generalized Fourier transforms associated with some class of Schrodinger operators on $\mathbb{R}$. Next, we establish Paley-Wiener type theorems which characterize some functional spaces by their generalized Fourier transforms.

Keywords: Schrodinger operator, generalized eigenfunctions, generalized Fourier transforms, Paley-Wiener theorems

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1. Introduction

We consider the symmetric differential operator (L, D_0) defined by

$$D_0 = \mathcal{D}(\mathbb{R}) \quad \text{and} \quad Lu(x) = -\frac{d^2u}{dx^2}(x) + q(x)u(x), \quad u \in \mathcal{D}(\mathbb{R}),$$

where $\mathcal{D}(\mathbb{R})$ is the space of C^∞-functions on $\mathbb{R}$ with compact support and q is a measurable function satisfying

$$\int_{-\infty}^{+\infty} (1+|x|)|q(x)|\, dx < +\infty.$$

The operator (L, D_0) has a unique self-adjoint extension (L, D_L), where (see [3])

$$D_L = \{f \in L^2(\mathbb{R}) : f, f' \text{ are absolutely continuous and } L(f) \in L^2(\mathbb{R})\}.$$

On the other hand, for $\mu \in \mathbb{C}_+ = \{\lambda \in \mathbb{C} : (\mathcal{I}m(\lambda) > 0) \text{ or } (\mathcal{I}m(\lambda) = 0 \text{ and } \mathcal{R}e(\lambda) \geq 0)\}$, the differential equation $Lu = \mu^2 u$ possesses two linear independent solutions $E_\pm(., \mu)$ satisfying

$$\lim_{x\to\pm\infty} e^{\mp i\mu x} E_\pm(x, \mu) = 1,$$

which are called generalized eigenfunctions.

We associate with the spectral decomposition of the self-adjoint operator (L, D_L) two generalized Fourier transforms defined by

$$\mathcal{F}_\pm(f)(\mu) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)\, E_\pm(x, \mp\mu)\, dx, \;\; \mu \in \mathbb{R}, \;\; f \in \mathcal{D}(\mathbb{R}).$$

In this paper we establish Paley-Wiener type theorems for the operator (L, D_L) which characterize some functional spaces by their generalized Fourier transforms. The main difficulty to prove these theorems is the study of the generalized eigenfunctions (existence, analicity, asymptotic behavior, ...).

This paper is organized as follows.

In the first section, we study the generalized eigenfunctions and we prove that there exist two kernels $K_\pm$ such that for all $\mu \in \mathbb{C}_+$, we have

$$E_+(x,\mu) = e^{i\mu x} + \int_x^{+\infty} K_+(x,s)\, e^{i\mu s}\, ds$$

and

$$E_-(x,\mu) = e^{-i\mu x} + \int_{-\infty}^{x} K_-(x,s)\, e^{-i\mu s}\, ds.$$

We establish, in the second section, that the generalized Fourier transforms $\mathcal{F}_\pm$ are related to the ordinary Fourier transforms $\mathcal{F}_0$ on $\mathbb{R}$ by

$$\mathcal{F}_\pm(f) = \mathcal{F}_0 \circ (I + {}^tK_\pm)(f),$$

where

$$\mathcal{F}_0(f)(\mu) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)\, e^{-i\mu x}\, dx, \;\; \mu \in \mathbb{R}, \;\; f \in \mathcal{D}(\mathbb{R})$$

and ${}^tK_\pm$ are the operators defined respectively by

$${}^tK_+(f)(x) = \int_{-\infty}^{x} K_+(u,x) f(u)\, du \;\; \text{and} \;\; {}^tK_-(f)(x) = \int_x^{\infty} K_-(u,x) f(u)\, du,$$

and we study the properties of the operators ${}^tK_\pm$.

In the third section we study the analyticity of the generalized eigenfunctions and the Fourier-Plancherel transforms associated with the operator (L, D_L).

The proof of the Paley-Wiener type theorem is given in the last section.

2. Generalized eigenfunctions and generalized Fourier transforms associated with the operator (L, D_L)

We consider the symmetric differential operator L defined on $\mathbb{R}$ by

$$Lu(x) = -\frac{d^2u}{dx^2}(x) + q(x)u(x), \quad u \in \mathcal{D}(\mathbb{R}),$$

where $\mathcal{D}(\mathbb{R})$ is the space of C^∞-functions on $\mathbb{R}$, with compact support and q is a measurable function satisfying

$$\int_{-\infty}^{+\infty} (1+|x|)|q(x)|\, dx < +\infty.$$

For all $\mu \in \mathbb{C}_+ = \{\lambda \in \mathbb{C} : (\mathcal{I}m(\lambda) > 0) \text{ or } (\mathcal{I}m(\lambda) = 0 \text{ and } \mathcal{R}e(\lambda) \geq 0)\}$, the differential equation

$$Lu = \mu^2 u \tag{2.1}$$

possesses two linear independent solutions $E_\pm(., \mu)$ satisfying (see [1], [3] and [4])

$$\lim_{x \to \pm\infty} e^{\mp i\mu x} E_\pm(x, \mu) = 1, \tag{2.2}$$

which we call generalized eigenfunctions associated with the differential operator L.

Proposition 2.1. *For all $\mu \in \mathbb{C}_+$ and $t \in \mathbb{R}$, we have*

$$E_+(t, \mu) = e^{i\mu t} + \int_t^{+\infty} \frac{\sin \mu(s-t)}{\mu} q(s) E_+(s, \mu)\, ds \tag{2.3}$$

and

$$E_-(t, \mu) = e^{-i\mu t} + \int_{-\infty}^{t} \frac{\sin \mu(s-t)}{\mu} q(s) E_-(s, \mu)\, ds. \tag{2.4}$$

In particular, there exist constants $C_\pm$, independent of μ, such that

$$|E_\pm(t, \mu)| \leq C_\pm\, e^{\mp \mathcal{I}m(\mu) t}. \tag{2.5}$$

Proof: Let μ be in $\mathbb{C}_+$. By using the method of the variations of constants and relations (2.1) and (2.2), we deduce relations (2.3) and (2.4). On the other hand, we can see that there exists a constant c_1 independent of μ such that

$$|E_+(t, \mu)| \leq e^{-\mathcal{I}m(\mu) t} \Big[1 + c_1 \int_t^{+\infty} (1 + |s|) |q(s)|\, e^{+\mathcal{I}m(\mu) s} E_+(s, \mu)\, ds\Big].$$

We put $f(t) = e^{\mathcal{I}m(\mu) t} E_+(t, \mu)$ and $g(t) = c_1(1 + |t|)|q(t)|$, then we have

$$f(t) \leq 1 + \int_t^{+\infty} f(s) g(s)\, ds.$$

Using the Gromwell lemma (see [6]), we obtain relation (2.5) for the function $E_+(., \mu)$ with

$$C_+ = \exp\Big(\int_{-\infty}^{+\infty} g(s)\, ds\Big).$$

In the same way, we prove relation (2.5) for the function $E_-(., \mu)$. $\square$

Theorem 2.2. *There exist kernels $K_{\pm}(t,s)$ with support respectively in $\{(t,s)\in\mathbb{R}^2 : t\le s\}$ and $\{(t,s)\in\mathbb{R}^2 : t\ge s\}$ such that*

$$E_+(t,\mu)=e^{i\mu t}+\int_t^{+\infty}K_+(t,s)e^{i\mu s}\,ds \tag{2.6}$$

and

$$E_-(t,\mu)=e^{-i\mu t}+\int_{-\infty}^{t}K_-(t,s)e^{-i\mu s}\,ds. \tag{2.7}$$

Furthermore these kernels are respectively the unique solution of the following integral equations:

$$K_+(t,s)=\frac{1}{2}\int_{\frac{t+s}{2}}^{+\infty}q(u)\,du-\int_{\frac{t+s}{2}}^{+\infty}\Big[\int_0^{\frac{s-t}{2}}q(x-y)K_+(x-y,x+y)\,dy\Big]\,dx \tag{2.8}$$

and

$$K_-(t,s)=\frac{1}{2}\int_{-\infty}^{\frac{t+s}{2}}q(u)\,du-\int_{-\infty}^{\frac{t+s}{2}}\Big[\int_{\frac{s-t}{2}}^{0}q(x-y)K_-(x-y,x+y)\,dy\Big]\,dx. \tag{2.9}$$

Proof: The proof is a consequence of relations (2.1), (2.2), (2.3) and (2.4), the assumptions on q, the method of the successive approximations, the Fubini theorem and the injectivity of the ordinary Fourier transform on $\mathbb{R}$. (See [1] and [3] for more details.)

We put

$$\sigma_+(t)=\int_t^{+\infty}|q(u)|\,du,\quad \sigma_-(t)=\int_{-\infty}^{t}|q(u)|\,du$$

and

$$\epsilon_+(t)=\int_t^{+\infty}(1+|u|)|q(u)|\,du,\quad \epsilon_-(t)=\int_{-\infty}^{t}(1+|u|)|q(u)|\,du.$$

□

Corollary 2.3. *For all t and s in $\mathbb{R}$, we have*

$$|K_{\pm}(t,s)|\le\frac{1}{2}\sigma_{\pm}\Big(\frac{t+s}{2}\Big)\exp(\epsilon_{\pm}(t)).$$

Corollary 2.4. *If q is a C^n-function, n in $\mathbb{N}$, (respectively C^∞-function) on $\mathbb{R}$, then the kernels $K_{\pm}$ are C^{n+1}-functions (respectively C^∞-functions) on $\mathbb{R}^2$.*

Corollary 2.5. *Let a be in $\mathbb{R}$. We have*

(1) *if the support of q is in $]-\infty, a]$, then $\frac{t+s}{2} \geq a \Rightarrow K_+(t,s) = 0$,*

(2) *if the support of q is in $[a, +\infty[$, then $\frac{t+s}{2} \leq a \Rightarrow K_-(t,s) = 0$.*

Corollary 2.6. (1) *If the support of q is in $]-\infty, a]$, $a \in \mathbb{R}$, (respectively in $[a, +\infty[$), then for all t in $\mathbb{R}$, the solution $\mu \to E_+(t,\mu)$ (respectively $\mu \to E_-(t,\mu)$) is analytic on $\mathbb{C}$.*

(2) *if the support of q is compact, then for all t in $\mathbb{R}$, the solutions $\mu \to E_\pm(t,\mu)$ are analytic on $\mathbb{C}$.*

Definition 2.7. *The generalized Fourier transforms $\mathcal{F}_\pm$ associated with the operator (L, D_L) are defined on $\mathcal{D}(\mathbb{R})$ by*

$$\mathcal{F}_\pm(f)(\mu) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)\, E_\pm(x, \mp\mu)\, dx,\ \ \mu \in \mathbb{R}. \tag{2.10}$$

The generalized Fourier transforms $\mathcal{F}_\pm$ are injective (see [4] and [2]) and are related to the ordinary Fourier transform $\mathcal{F}_0$ on $\mathbb{R}$ by the relation

$$\mathcal{F}_\pm(f) = \mathcal{F}_0 \circ (I + {}^tK_\pm)(f),\ \ f \in \mathcal{D}(\mathbb{R}), \tag{2.11}$$

where

$$\mathcal{F}_0(f)(\mu) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)\, e^{-i\mu x}\, dx,\ \ \mu \in \mathbb{R},\ \ f \in \mathcal{D}(\mathbb{R})$$

and ${}^tK_\pm$ are the operators defined respectively by

$${}^tK_+(f)(x) = \int_{-\infty}^{x} K_+(u,x) f(u)\, du,\ \ f \in \mathcal{D}(\mathbb{R}) \tag{2.12}$$

and

$${}^tK_-(f)(x) = \int_{x}^{\infty} K_-(u,x) f(u)\, du,\ \ f \in \mathcal{D}(\mathbb{R}). \tag{2.13}$$

3. The study of the operators ${}^tK_\pm$

In the following we state theorems which characterize some functional spaces on which the operators $I + {}^tK_\pm$ are bijective.

Let a be in $\mathbb{R}$, n in $\mathbb{N}$ and $R > 0$. We denote by

- $C^n_{R,a}$ the space of C^n-functions on $\mathbb{R}$, with support in $[-R+a, R+a]$,
- $\mathcal{D}_{R,a}(\mathbb{R})$ the space of C^∞-functions on $\mathbb{R}$, with support in $[-R+a, R+a]$.

Theorem 3.1. *We suppose that the support of the function q is in $]-\infty, a]$. Then the operator $I + {}^tK_+$ is bijective*

(i) *from $C^1_{R,a}$ onto itself,*

(ii) *from $C^{n+1}_{R,a}$ onto itself if q is C^n on $\mathbb{R}$,*

(iii) *from $\mathcal{D}_{R,a}(\mathbb{R})$ onto itself if q is C^∞ on $\mathbb{R}$.*

The proof of the previous theorem is a consequence of the following propositions.

Proposition 3.2. *We suppose that the support of the function q is in $]-\infty, a]$. Then we have*

(i) $(I + {}^tK_+)(C^1_{R,a}) \subset C^1_{R,a}$;

(ii) *if q is C^n on $\mathbb{R}$, then* $(I + {}^tK_+)(C^{n+1}_{R,a}) \subset C^{n+1}_{R,a}$;

(iii) *if q is C^∞ on $\mathbb{R}$, then* $(I + {}^tK_+)(\mathcal{D}_{R,a}(\mathbb{R})) \subset \mathcal{D}_{R,a}(\mathbb{R})$.

Proof: The proof is a consequence of Corollary 2.4 and the fact that $({}^tK_+)(f)(t) = 0$, for all $t \notin [-R+a, R+a]$, and for all f in $C^n_{R,a}$, $n \in \mathbb{N}$. □

Notation. We put

$$N^R_+(s,u) = \begin{cases} K_+(u,s) & \text{if } \ -R+a \le u \le s \le +\infty, \\ 0 & \text{elsewhere.} \end{cases}$$

We consider the following integral equations:

$$h(s) = f(s) + \int_{-\infty}^{s} K_+(u,s) f(u)\, du, \tag{3.1}$$

and

$$h(s) = f(s) + \int_{-\infty}^{s} N^R_+(u,s) f(u)\, du, \tag{3.2}$$

where h is a given function and f is an unknown function.

Proposition 3.3. *We suppose that the function h is in $C^1_{R,a}$. Then*

(i) *the support of every solution f of (3.2) is in $[-R+a, R+a]$;*

(ii) *let f be a function with support in $[-R+a, R+a]$, then f is a solution of (3.1) if and only if f is a solution of (3.2).*

Proof: Let h be a function in $C^1_{R,a}$.

(i) It is clear that

$$\forall\, u \in \mathbb{R},\ \forall\, s, \quad s > R+a \Rightarrow N^R_+(u,s) = 0,$$

hence

$$\forall\, u \in \mathbb{R},\ \forall\, s, \quad s \notin [-R+a, R+a] \Rightarrow N^R_+(u,s) = 0.$$

Then we deduce that the support of every solution f of the equation (3.2) is in $[-R+a, R+a]$.

(ii) Let f be a function with support in $[-R+a, R+a]$. We obtain

$$\forall\, s,\ s \le -R+a \Rightarrow {}^tK_+(f)(s) = 0,$$

so that

$$\forall\, s \in \mathbb{R},\ {}^tK_+(f)(s) = \int_{-\infty}^{s} N^R_+(u,s) f(u)\, du.$$

□

Proposition 3.4. *Let h be a function in $C^1_{R,a}$. Then the integral equation* (3.2) *possesses a unique solution f in $C^1_{R,a}$.*

Proof: The equation (3.2) is a Volterra integral equation, we resolve it by using the method of successive approximations. We put

$$N_0(s,u) = N_+^R(s,u), \text{ and for all } n \geq 1,\ N_n(s,u) = \int_u^s N_{n-1}(s,v) N_+^R(v,u)\, dv.$$

We put

$$\begin{aligned} h_n(s) &= (-1)^{n+1} \int_{-R+a}^s N_n(s,u) h(u)\, du, \\ m &= \sup_{s\in\mathbb{R}} |h(s)|, \\ D &= [-R+a, R+a] \times [-R+a, R+a], \\ M &= \sup \big\{ |N_+^R(s,u)|;\ (s,u) \in D \big\}. \end{aligned}$$

Hence, for all $(s,u) \in D$, we have $|N_0(s,u)| \leq M$, and for all $n \geq 1$,

$$|N_n(s,u)| \leq M^{n+1} \frac{(s-u)^n}{n!} \leq M^{n+1} \frac{[2(R+a)]^n}{n!}.$$

Since the support of the kernel N_n, $n \geq 1$, is in D we deduce that the series of general term $N_n(s,u)$ is absolutely and uniformly convergent on $\mathbb{R}^2$; and its sum denoted by H_+^R is with support in D and satisfies

$$|H_+^R(s,u)| \leq M \exp[2(R+a)M].$$

In the same way we prove that the series of general term $h_n(s)$ is absolutely and uniformly convergent on $\mathbb{R}$; and its sum denoted by $\sum_{n=0}^\infty h_n(s)$ has support in $[-R+a, R+a]$ and satisfies

$$\Big| \sum_{n=0}^\infty h_n(s) \Big| \leq m \exp[2(R+a)M].$$

We put

$$\begin{aligned} f(s) &= h(s) + \sum_{n=0}^\infty h_n(s) \\ &= h(s) + \int_{-R+a}^s H_+^R(s,u) h(u)\, du, \end{aligned}$$

so that f is supported in $[-R+a, R+a]$ and it is a solution of equation (3.2).

The uniqueness of the solution is a consequence of relation (2.10) and the fact that $\mathcal{F}_+$ is injective.

The derivability of the solution is a consequence of the derivability of the kernel K and relation (3.2). □

The proof of the following theorem is analogous to that one given for Theorem 3.1.

Theorem 3.5. *We suppose that the support of the function q is in $[a,+\infty[$. Then the operator $I+{}^tK_-$ is bijective*

(i) *from $C^1_{R,a}$ onto itself,*
(ii) *from $C^{n+1}_{R,a}$ onto itself if q is C^n on $\mathbb{R}$,*
(iii) *from $\mathcal{D}_{R,a}(\mathbb{R})$ onto itself if q is C^∞ on $\mathbb{R}$.*

4. Paley-Wiener type theorems

For all n in $\mathbb{N}$ and $R>0$, we denote by

– $\mathcal{H}^{n+1}_R$ the space of analytic functions ψ on $\mathbb{C}$ such that

$$\begin{aligned}&\forall\, m\in\{0,1,\dots,n+1\},\exists\, c_m>0 \ \text{ such that}\\ &\forall\,\mu\in\mathbb{C},\ |\psi(\mu)|\le c_m(1+|\mu|)^{-m}e^{|\mathcal{I}m(\mu)|R};\end{aligned} \tag{4.1}$$

– $\mathcal{H}_R$ the space of functions in $\mathcal{H}^{n+1}_R$, for all n in $\mathbb{N}$.

Theorem 4.1. *Let q be a C^∞-function and b in $\mathbb{R}$.*

(i) *If the support of q is in $]-\infty,b]$, then the transform $\mathcal{F}_+$ is bijective from $\mathcal{D}_{R,b}$ onto $e^{-i\mu b}\mathcal{H}_R$.*
(ii) *If the support of q is in $[b,+\infty[$, then the transform $\mathcal{F}_-$ is bijective from $\mathcal{D}_{R,b}$ onto $e^{+i\mu b}\mathcal{H}_R$.*
(iii) *If the support of q is in $[-|b|,|b|]$, then the transform $\mathcal{F}_\pm$ is bijective from $\mathcal{D}_{R,b}$ onto $e^{\mp i\mu|b|}\mathcal{H}_R$.*

The proof of the previous theorem is a consequence of the following proposition.

Proposition 4.2. *Let n be in $\mathbb{N}$, q a C^n-function and b in $\mathbb{R}$.*

(i) *If the support of q is in $]-\infty,b]$, then*

$$\mathcal{F}_+(\mathcal{D}_{R,b})\subset e^{-i\mu b}\mathcal{H}^{n+1}_R \tag{4.2}$$

and

$$\mathcal{F}_+^{-1}(e^{-i\mu b}\mathcal{H}^{n+1}_R)\subset\mathcal{C}^{n+1}_{R,b}. \tag{4.3}$$

(ii) *If the support of q is in $[b,+\infty[$, then*

$$\mathcal{F}_-(\mathcal{D}_{R,b})\subset e^{i\mu b}\mathcal{H}^{n+1}_R \ \text{ and } \ \mathcal{F}_-^{-1}(e^{i\mu b}\mathcal{H}^{n+1}_R)\subset\mathcal{C}^{n+1}_{R,b}.$$

(iii) *If the support of q is in $[-|b|,|b|]$, then*

$$\mathcal{F}_\pm(\mathcal{D}_{R,\pm|b|})\subset e^{\mp i\mu|b|}\mathcal{H}^{n+1}_R \ \text{ and } \ \mathcal{F}_\pm^{-1}(e^{\mp i\mu|b|}\mathcal{H}^{n+1}_R)\subset\mathcal{C}^{n+1}_{R,\pm|b|}.$$

PROOF: (i) We begin to prove that $\mathcal{F}_+(\mathcal{D}_{R,b}) \subset e^{-i\mu b}\mathcal{H}_R^{n+1}$. From (2.10) and Theorem 3.1 we see that it is sufficient to prove that $\mathcal{F}_0(\mathcal{C}_{R,b}^{n+1}) \subset e^{-i\mu b}\mathcal{H}_R^{n+1}$. Let f be in $\mathcal{C}_{R,b}^{n+1}$, then the function

$$\mu \to \mathcal{F}_0(f)(\mu) = \frac{1}{\sqrt{2\pi}} \int_{-R+b}^{R+b} e^{-i\mu t} f(t)\, dt$$

is analytic on $\mathbb{C}$. On the other hand, we have

$$\mathcal{F}_0(f)(\mu) = e^{-i\mu b}\Big\{\frac{1}{\sqrt{2\pi}} \int_{-R}^{R} e^{-i\mu t} f_b(t)\, dt\Big\} = e^{-i\mu b}\mathcal{F}_0(f_b)(\mu),$$

where $f_b(t) = f(t+b)$. It is clear that the function $\mathcal{F}_0(f_b)$ is analytic on $\mathbb{C}$. Furthermore, by integrating by parts, we deduce that the function $\mathcal{F}_0(f_b)$ satisfies relation (4.1).

The proof of the relation (4.3) is a consequence of Theorem 3.1 and the fact that

$$\mathcal{F}_0^{-1}(e^{-i\mu b}\mathcal{H}_R) = \mathcal{D}_{R,b} \subset \mathcal{C}_{R,b}^{n+1}\,.$$

In the same way we obtain the proof of (ii). The proof of (iii) is a consequence of (i) and (ii). □

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