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Proper forcings and absoluteness in $L(\mathbb{R})$

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Abstract. We show that in the presence of large cardinals proper forcings do not change the theory of $L(\mathbb{R})$ with real and ordinal parameters and do not code any set of ordinals into the reals unless that set has already been so coded in the ground model.

Keywords: proper forcing, large cardinals
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0. Introduction

It is a well-established fact by now that in the presence of large cardinals the minimal model $L(\mathbb{R})$ of ZF set theory containing all reals and ordinals has strong canonicity properties — for example it satisfies the Axiom of Determinacy and its parameter-free theory is the same in all set generic extensions of the universe ([MS], [W1]). In this paper we give full proofs of three absoluteness theorems connecting the model $L(\mathbb{R})$ with the basic forcing-theoretic notion of properness ([Sh]).

Embedding Theorem. *Let δ be a weakly compact Woodin cardinal and P a proper forcing notion of size $< \delta$. Then in V^P there is an elementary embedding*

$$j : L(\mathbb{R}^V) \to L(\mathbb{R}^{V^P})$$

which fixes all ordinals.

This is related to the results of [FM, Theorem 3.4] and implies that in the presence of large cardinals proper forcings cannot change the ordinal parametrized theory of $L(\mathbb{R})$, in particular, the values of the projective ordinals or $\theta^{L(\mathbb{R})}$. On the other hand, it is known that semiproper forcings can increase the value of δ^1_2 ([W2]) and so the Embedding Theorem cannot be generalized to such posets.

Anticoding Theorem. *Let δ be a weakly compact Woodin cardinal, P a proper forcing notion of size $< \delta$ and $A \subset \mathrm{Ord}$. Then*

$$A \in L(\mathbb{R}) \text{ if an only if } P \Vdash \check{A} \in L(\mathbb{R}).$$

Thus while proper forcings can add many new reals to the universe no old sets of ordinals can be coded by these reals. This should be contrasted with [BJW]. Again, a generalization to semiproper forcings fails as shown in Section 7.

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Image Theorem. *Let δ be a weakly compact Woodin cardinal and A be a bounded subset of $\theta^{L(\mathbb{R})}$. Then*

$$A \in L(\mathbb{R}) \text{ just in case there is } B \text{ with } \mathbb{Q}_{<\delta} \Vdash j(\check{A}) = \check{B}.$$

This is mainly a technical tool used to establish the Anticoding Theorem.

In all the theorems quoted above the assumption on δ can be relaxed to "a supremum of Woodin cardinals with a measurable above it" (which is consistency-wise a weaker assumption) and the proofs will go through with only more complicated notation. All the three theorems have analogs for higher models of determinacy in place of $L(\mathbb{R})$.

The anatomy of the paper is the following. In Sections 1–3 the necessary technical background is presented, using mainly results of W. Hugh Woodin about HOD $^{L(\mathbb{R})}$ (Section 1), the nonstationary tower (Section 2) and the weakly homogeneous trees (Section 3). In the following four sections we handle the image theorem, the embedding theorem, the anticoding theorem and an example of coding in the presence of large cardinals one at a time.

Our notation follows the set theoretic standard set forth in [J]. The phrase "there is an external object x with a certain property" should be translated as "in some forcing extension there is x..." or "for a sufficiently large cardinal λ, $\mathrm{Coll}(\lambda) \Vdash \exists x \ldots$". This is done when the exact nature of the forcing extension is unimportant and the property in question is Δ_1 in x and the ground model. HOD $_x$ is the class of sets hereditarily ordinal definable from the parameter x. For a tree $T \subset (\omega \times Y)^{<\omega}$ the projection of T is the set $p[T] = \{x \in \omega^\omega : \exists z \in Y^\omega \ \langle x, y\rangle$ is an infinite branch through $T\}$. We use the letter $\mathbb{R}$ to denote "the reals" — the set of all functions from ω to ω. However, if some generic extensions of the universe are floating around, the symbols $\mathbb{R} \cap V$, $\mathbb{R} \cap V[G]$, $\mathbb{R} \cap V[H]$ denote the sets of reals in the respective models. No confusion should result.

The authors wish to thank W. Hugh Woodin for permission to include proofs of his results in the first three sections. A part of this paper was prepared during second author's stay at CRM, Universita Autónoma de Barcelona and thanks are due for the Center's hospitality. In [NZ] the reader can find an account of the proofs of the first two theorems using the quite different techniques of iteration trees and genericity iterations of inner models for large cardinals.

1. The theory of $L(\mathbb{R})$

In this section we prove the main technical result about the model $L(\mathbb{R})$ we will use later. The theorem is due to W. Hugh Woodin and our presentation owes much to the unpublished [S].

Theorem 1.1. *Suppose $L(\mathbb{R})$ satisfies the Axiom of Determinacy. Then $L(\mathbb{R})$ is a symmetric extension of its* HOD*.*

It must be said more precisely what is meant by a "symmetric extension". Work in $L(\mathbb{R})$. In HOD there is a regular chain $\mathbb{B}_0 \lessdot \mathbb{B}_1 \lessdot \ldots$ of complete boolean algebras with the direct limit $\mathbb{B}_\omega$ so that

(1) there are names $\dot{r}_i : i \in \omega$ such that $\dot{r}_i$ is a $\mathbb{B}_i$-name for a real and the algebra $\mathbb{B}_i$ is generated by $\dot{r}_j : j \leq i$. Let $\dot{\mathbb{R}}_{sym}$ be the $\mathbb{B}_\omega$-name for the set $\{\dot{r}_i : i \in \omega\}$;

(2) $\mathbb{B}_\omega \Vdash$"the reals of $L(\dot{\mathbb{R}}_{sym})$ are exactly $\dot{\mathbb{R}}_{sym}$"; moreover, for every formula ϕ, ordinal parameters $\vec{\alpha}$, real parameters $\vec{s} \in$ HOD and an integer i we have that $\mathbb{B}_i \Vdash$"the validity of $L(\dot{\mathbb{R}}_{sym}) \models \phi(\vec{\alpha}, \vec{s}, \dot{r}_j : j \leq i)$ is decided in the same way by every condition in $\mathbb{B}_\omega/\mathbb{B}_i$". In particular, for each $n \in \omega$ the Σ_n-theory of $L(\dot{\mathbb{R}}_{sym})$ with ordinal and real-in-HOD parameters is a definable class of HOD;

(3) whenever $\{r_i : i \in \omega\}$ is an $L(\mathbb{R})$-generic enumeration of $\mathbb{R}$ (via the poset of all finite sequences of reals ordered by endextension) then the equations $r_i = \dot{r}_i : i \in \omega$ determine a HOD-generic filter on $\mathbb{B}_\omega$. In particular, for every real r the equation $r = \dot{r}_0$ defines a HOD generic filter on $\mathbb{B}_0$.

Corollary 1.2. *Assume $V = L(\mathbb{R})$ and the Axiom of Determinacy holds. Then for every real x we have* $\mathrm{HOD}_x = \mathrm{HOD}[x]$.

PROOF: Obviously $\mathrm{HOD}[x] \subset \mathrm{HOD}_x$. Now suppose $x \in \mathbb{R}$ and $A \subset \mathrm{Ord}$ is definable from x and ordinal parameters $\vec{\alpha}$, say $A = \{\beta : \phi(\beta, \vec{\alpha}, x)\}$. We shall show that $A \in \mathrm{HOD}[x]$, proving $\mathrm{HOD}_x \subset \mathrm{HOD}[x]$.

In $\mathrm{HOD}[x]$, define $B = \{\beta :$ every condition in $\mathbb{B}_\omega/\mathbb{B}_0$ forces $L(\dot{\mathbb{R}}_{sym}) \models \phi(\beta, \vec{\alpha}, x)\}$ where the filter on $\mathbb{B}_0 \in$ HOD is given by the equation $\dot{r}_0 = x$. We claim that this filter is HOD-generic and $A = B$. But this follows immediately by inspection of (2) and (3) above. □

A set $X \subset \mathbb{R}$ is said to be ∞-Borel if it possesses an ∞-Borel code: a set A of ordinals and a formula ϕ such that

$$r \in X \text{ if and only if } L[A, r] \models \phi(A, r).$$

Corollary 1.3. *Suppose $V = L(\mathbb{R})$ and the Axiom of Determinacy holds. Every set of reals is ∞-Borel and every ordinal definable set of reals has an ordinal definable ∞-Borel code.*

PROOF: Choose a set $X \subset \mathbb{R}$. Fix a real s such that X is definable from s and ordinal parameters $\vec{\alpha}$, say $X = \{r : \phi(r, s, \vec{\alpha})\}$. The inductive definition of $L(\mathbb{R})$ guarantees the existence of such $s, \vec{\alpha}$. Choose a set $B \subset \mathrm{Ord}$ such that $B \in \mathrm{HOD}$, $\mathrm{Power}(\mathbb{B}_\omega) \cap \mathrm{HOD} \subset L[B]$ and an ordinal definable in s set A of ordinals — so $A \in \mathrm{HOD}_s$ — coding the tuple $(B, \mathbb{B}_\omega, s, \vec{\alpha})$. Then A is an ∞-Borel code for the set X:

$$r \in X \text{ iff } L[A, r] \models \mathbb{B}_\omega/\mathbb{B}_1 \Vdash L(\dot{\mathbb{R}}_{sym}) \models \phi(r, s, \vec{\alpha})$$

where the HOD generic filter on $\mathbb{B}_1$ is given by the equations $r = \dot{r}_0$, $s = \dot{r}_1$. □

In some sense, the above statements are more of a part of the proof of the Theorem than its consequences. At any rate, let us now turn to the proof of

Theorem 1.1. The main theme is the following fact due to Vopěnka [HV, Theorem 6322]. Let $\mathbb{A}$ be the algebra of ordinal definable sets of reals with operations of union and complementation; we shall freely confuse $\mathbb{A}$ with its HOD isomorph. Note that $\mathbb{A}$ is an ordinally definable structure on ordinally definable elements, and so there is ordinally definable isomorphism of $\mathbb{A}$ and some structure on the ordinals which then will be in HOD.

Claim 1.4. *The algebra $\mathbb{A}$ is complete in* HOD*. Moreover, every real x determines a* HOD *generic filter $G_x \subset \mathbb{A}$ such that $x \in \mathrm{HOD}[G_x]$.*

PROOF: The completeness of $\mathbb{A}$ in HOD is nearly obvious. If $X \subset \mathbb{A}$ is an ordinal definable set, then its sum in $\mathbb{A}$ is the ordinal definable set $\bigcup X$. Now given $x \in \mathbb{R}$ let $G_x = \{b \in \mathbb{A} : x \in b\}$. This is obviously a filter; to prove its HOD-genericity let $A \subset \mathbb{A}$ be an ordinal definable maximal antichain. Then $\bigcup A = \mathbb{R}$, since otherwise $\mathbb{R} \setminus \bigcup A$ is a nonzero element of $\mathbb{A}$ incompatible with every element of A. This means that there is $b \in A$ with $x \in b$, so $b \in G_x$ and the filter is HOD-generic. To show that $x \in \mathrm{HOD}[G_x]$, let $b_n = \{r \in \mathbb{R} : n \in r\}$ for $n \in \omega$. The sets b_n as well as their sequence are ordinal definable , and one can define an $\mathbb{A}$-name $\dot{r} \in \mathrm{HOD}$ by setting $\check{n} \in \dot{r}$ iff b_n is in the generic filter. Then $x = \dot{r}/G_x$. $\square$

The question suggests itself: is $\mathrm{HOD}[x] = \mathrm{HOD}[G_x]$, in other words, does the term $\dot{r}$ generate the algebra $\mathbb{A}$ in HOD? In general, the answer is no; it can be shown that $\mathrm{HOD}[G_x] = \mathrm{HOD}_x$ and the latter model is frequently larger than $\mathrm{HOD}[x]$. We shall first identify the subalgebra of $\mathbb{A}$ generated by the term $\dot{r}$. Let $\mathbb{B}$ be the algebra of sets of reals which have an ordinal definable ∞-Borel code, with the operations of union and complementation. Obviously, $\mathbb{B} \subset \mathbb{A}$ since an ∞-Borel code provides a definition of the set it codes. Corollary 1.2 will eventually imply that under $V = L(\mathbb{R}) + AD$ these two algebras coincide, but there is a long way before we can prove that.

Claim 1.5. *The algebra $\mathbb{B}$ is a complete subalgebra of $\mathbb{A}$ in* HOD*. Moreover, every real x determines a* HOD*-generic filter $H_x \subset \mathbb{B}$ such that* $\mathrm{HOD}[x] = \mathrm{HOD}[H_x]$.

PROOF: For the completeness observe that if $X \subset \mathbb{B}$ is an ordinal definable collection of sets with ordinal definable Borel codes, then $\bigcup X$, which is the sum of X in A also has ordinal definable ∞-Borel code and so belongs to $\mathbb{B}$.

Now given $x \in \mathbb{R}$ let $H_x = \{b \in \mathbb{B} : x \in b\}$. As before, this is a HOD-generic filter and $x \in \mathrm{HOD}[H_x]$: in fact the name $\dot{r}$ described in the previous proof is a $\mathbb{B}$-name. We must show that $H_x \in \mathrm{HOD}[x]$. For every $b \in \mathbb{B}$ let A_b, ϕ_b be its ∞-Borel code which comes first in the natural wellordering of HOD. Then the correspondence $b \mapsto A_b, \phi_b$ is in HOD and H_x can be defined in $\mathrm{HOD}[x]$ as $\{b \in \mathbb{B} : L[A_b, x] \models \phi_b(A_b, x)\}$. $\square$

The above claims are easily seen to have been proved in ZF. Now we pass into the model $L(\mathbb{R})$ and make use of the determinacy assumption. For each integer

$n \in \omega$ define $\mathbb{B}_n$ to be the algebra of subsets of $\mathbb{R}^{n+1}$ with an ordinal definable ∞-Borel code, again confused with its HOD-isomorph. Obviously in HOD the algebras $\mathbb{B}_n$ are complete adding a sequence of reals of length $n+1$ — see the previous Claim.

Claim 1.6. *The maps $\pi_{mn} : \mathbb{B}_n \to \mathbb{B}_m, m \in n \in \omega$, defined by $\pi_{mn}(b) = \{x \in \mathbb{R}^{m+1} : \exists\, y\ x^\frown y \in b\}$ are projections.*

Proof: Fix $m \in n \in \omega$. Once we verify that the range of π_{mn} is included in $\mathbb{B}_m$ then the definitory properties of a projection easily follow: say for example that $c \in \mathbb{B}^m$, $c \leq \pi_{mn}(b)$. A condition $d \in \mathbb{B}_n$, $d \leq b$ must be produced such that $\pi_{mn}(d) = c$. But $d = \{z \in b : z = x^\frown y$ for some $x \in c\}$ is obviously such a condition.

So let $b \in \mathbb{B}_n$ and fix an ordinal definable ∞-Borel code A for the set b so that for some formula ϕ the equivalence $x \in b \leftrightarrow L[A,x] \models \phi(A,x)$ holds for all $x \in \mathbb{R}^{n+1}$. It must be proved that $a = \pi_{mn}(b)$ belongs to $\mathbb{B}_m$, that is, an ordinal definable ∞-Borel code for the set $a \subset \mathbb{R}^{m+1}$ must be found.

Fix a real r and work in $L[A,r]$. Let $M_r = \mathrm{HOD}_A$, and let $\mathbb{C}_r$ be the algebra of sets of reals with an ∞-Borel code in M_r, $\mathbb{C}_r$. Also let $\lambda_r = |\mathbb{C}_r|^{M_r}$. We have

(1) $M_r \models \mathbb{C}_r$ is a complete Boolean algebra,

(2) every real $x \in L[A,r]$ determines an M_r generic filter $G_x \subset \mathbb{C}_r$ such that $M_r[x] = M_r[G_x]$,

(3) λ_r is a countable ordinal in $L(\mathbb{R})$.

Here (1), (2) follow essentially from Claim 1.5 applied in $L[A,r]$ with HOD replaced with HOD_A. To see (3) note that $\lambda_r = |\mathbb{C}_x|^{M_r} \leq |\operatorname{Power}(\mathbb{R})|^{L[A,r]}$ and the latter is countable since $L[A,r]$ is a wellorderable model. Note that as we are working in the context of the Axiom of Determinacy, ω_1 is an inaccessible cardinal in every model of ZFC containing it. Now λ_r, $\mathbb{C}_r$, M_r as well as the canonical wellordering of the model M_r depend only on the Turing degree of the real r and we can form an ultrapower M of $M_r : r \in \mathbb{R}$ using the cone measure. There is enough choice to make Łos' theorem go through. To see this it is enough, for every function f on the reals such that $f(r)$ is a nonempty set in M_r depending only on the Turing degree of r, to produce a function g on the reals such that $g(r)$ depends only on the Turing degree of r and $g(r) \in f(r)$. Just let $g(r)$ be the least element of $f(r)$ in the canonical wellorder of M_r.

Let $\bar{\mathbb{C}} = [r \mapsto \mathbb{C}_r]$ be the equivalence class of the function $r \mapsto \mathbb{C}_r$, let $\lambda = [r \mapsto \lambda_r]$ and $\bar{A} = [r \mapsto A]$. So $M \models$"$\bar{\mathbb{C}}$ is a complete algebra of size λ and $\bar{A}$ is a set of ordinals", moreover, $M, \bar{\mathbb{C}}, \bar{A} \in \mathrm{HOD}$.

We claim that for every sequence $x \in \mathbb{R}^{m+1}$,

$$(*) \qquad x \in a \leftrightarrow M[x] \models \mathrm{Coll}(\lambda) \Vdash \exists\, y\ L[\bar{A}, x^\frown y] \models \phi(\bar{A}, x^\frown y).$$

This shows that any ordinal definable set coding a sufficiently large initial segment of M can serve as ∞-Borel code for the set a via the beefy formula on the right hand side of the above equivalence. The claim will follow.

So fix an arbitrary sequence $x \in \mathbb{R}^{m+1}$. Note that the model $M[x]$ is the ultrapower of models $M_r[x] : r \in \mathbb{R}$ using the cone measure.

Assume first that the right hand side of (*) is satisfied. By Łos' theorem there is a cone of reals r such that $M_r[x] \models \mathrm{Coll}(\lambda_r) \Vdash \exists y\ L[A, x^\frown y] \models \phi(A, x^\frown y)$. Since $|\lambda_r| = \aleph_0$ it is possible to choose an $M_r[x]$-generic filter $h \subset \mathrm{Coll}(\lambda_r)$ and in the model $M_r[x][h]$ to find a sequence y such that $L[A, x^\frown y] \models \phi(A, x^\frown y)$ meaning that $x^\frown y \in b$ and $x \in a$.

On the other hand, suppose $x \in a$; then there is a sequence y such that $x^\frown y \in b$. We shall show that for every real r coding x, y the model $M_r[x]$ satisfies $\mathrm{Coll}(\omega, < \lambda_r) \Vdash \exists y\ L[A, \check{x}^\frown y] \models \phi(A, x^\frown y)$. By Łos' theorem, this implies the right hand side of (*). So let $r \in R$ code x, y. There is an M_r-generic filter $H \subset \mathbb{C}_r$ such that $x, y \in M_r[r] = M_r[H]$. By basic forcing factoring facts applied in M_r, there is a poset $P \in M_r[x]$ of size $\leq |\mathbb{C}_r|^{M_r} = \lambda_r$ and an $M_r[x]$-generic filter $K \subset P$ such that $M_r[x][K] = M_r[H]$. So there must be a condition $p \in P$ so that $M_r[x] \models p \Vdash_P \exists y L[\check{A}, \check{x}^\frown y] \models \phi(\check{A}, \check{x}^\frown y)$. By Kripke's theorem in $M_r[x]$ the poset P regularly embeds into $\mathrm{Coll}(\lambda_r)$. By absoluteness $M_r[x] \models \mathrm{Coll}(\lambda_r) \Vdash \exists y L[\check{A}, \check{x}^\frown y] \models \phi(\check{A}, \check{x}^\frown y)$ as desired. $\square$

The sequence $\mathbb{B}_n : n \in \omega$ of algebras as well as the commutative system $\pi_{mn} : m \in n \in \omega$ of projections belongs to HOD. Making the appropriate identifications in HOD we get a regular chain $\mathbb{B}_0 \lessdot \mathbb{B}_1 \lessdot \dots$ of algebras with the direct limit $\mathbb{B}_\omega$. For an integer $n \in \omega$ let $\dot{r}_n$ be the $\mathbb{B}_n$-name for the last real of the sequence added by that algebra. Under the identifications $\dot{r}_m$ is a $\mathbb{B}_n$ name whenever $m \leq n$ and $\dot{r}_m : m \leq n$ is the $\mathbb{B}_n$ name for the sequence of reals added by $\mathbb{B}_n$, which generates $\mathbb{B}_n$ by Claim 1.5. This verifies the condition (1) after Theorem 1.1.

Now we show that the poset $\mathbb{R}^{<\omega}$ for adding a generic enumeration of reals of ordertype ω determines a HOD-generic filter on $\mathbb{B}_\omega$ as in (3) after Theorem 1.1. This is an elementary density argument: suppose $D \subset \bigcup_n \mathbb{B}_n$ is an open dense set in HOD and $\vec{r} = \langle r_m : m \leq n \rangle$ a sequence of reals — a condition in $\mathbb{R}^{<\omega}$. A prolongation $r_m : m \leq n'$ of this sequence will be found so that the HOD generic filter on $\mathbb{B}_{n'}$ determined by the equations $r_m = \dot{r}_m : m \leq n'$ contains a condition in D. This will be enough.

First note that the filter $H \subset \mathbb{B}_n$ given by the equations $r_m = \dot{r}_m : m \leq n$ is HOD-generic by virtue of Claim 1.5. Let $E = \{b \in \mathbb{B}_n : \exists c \in D, c \in \mathbb{B}_k\ \pi_{kn}(c) = b\}$. The set $E \subset \mathbb{B}_n$ is open dense in HOD, so $H \cap E \neq 0$. Pick a condition $b \in H \cap E$. It follows that $\vec{r} \in b$ and by the definition of the set E and the projections there is a sequence $\vec{s}$ of reals and a condition $c \in D$ such that $\vec{r}^\frown \vec{s} \in c$. Obviously the sequence $\vec{r}^\frown \vec{s}$ works as desired.

To prove the properties of $\mathbb{B}_\omega$ stated in (2) after the Theorem note that for every nonzero condition $b \in \mathbb{B}_\omega$ there is an external generic enumeration $r_n : n \in \omega$ of reals such that the HOD-generic filter $H \subset \mathbb{B}_\omega$ given by the equations $r_n = \dot{r}_n : n \in \omega$ meets the condition b: just pick a sequence $\vec{r} \in b$ and force the enumeration with the poset $\mathbb{R}^{<\omega}$ below the condition $\vec{r}$. As the last point, $\dot{\mathbb{R}}_{sym}/H = \mathbb{R} = L(\mathbb{R}) \cap \mathbb{R}$ proving that $\mathbb{B}_\omega \Vdash L(\dot{\mathbb{R}}_{sym}) \cap \mathbb{R} = \dot{\mathbb{R}}_{sym}$. The Theorem follows.

2. The nonstationary tower

Let δ be a cardinal. The *nonstationary tower forcing* $\mathbb{Q}_{<\delta}$ has been introduced in [W1] as the set of all stationary systems a of countable sets on $\bigcup a \in H_\delta$ ordered by $a \geq b$ if $\bigcup a \subset \bigcup b$ and $\forall x \in b\ x \cap \bigcup a \in a$. This poset introduces a natural generic ultrapower $j : \langle V, \in\rangle \to \langle M, E\rangle$ in the model $V[G]$, $G \subset \mathbb{Q}_{<\delta}$ generic as described in [W1], [FM]. The following facts were first proved in [W1] under the assumption of δ being supercompact. The reader may wish to consult [FM] for the more technical proofs using Woodinness of δ only. For every set $x \in H_\delta$ we have $j''x \in M$ and for every $a \in \mathbb{Q}_{<\delta}$ the equivalence $a \in G \leftrightarrow j''\bigcup a \in j(a)$ holds.

Fact 2.1 ([W1]). *Suppose δ is a Woodin cardinal. Then*

(1) $\mathbb{Q}_{<\delta} \Vdash M^\omega \subset M$, *in particular M is wellfounded and will be identified with its transitive isomorph,*
(2) $\mathbb{Q}_{<\delta} \Vdash \dot\omega_1 = \check\delta$, *in particular $j(\check\omega_1) = \check\delta$.*

The following definition is a key to constructing some interesting conditions in $\mathbb{Q}_{<\delta}$. Let $\delta \in \lambda$ be cardinals and $Z \prec H_\lambda$. We say that the model Z is selfgeneric at δ if $\delta \in Z$ and for every maximal antichain $A \subset \mathbb{Q}_{<\delta}$ in Z there is $a \in A \cap Z$ with $Z \cap \bigcup a \in a$.

Fact 2.2 ([W1]). *Let δ be a Woodin cardinal, $\delta \in \lambda$. For every countable elementary submodel Y of H_λ with $\delta \in Y$ and every $\kappa \in \delta \cap Y$ there is a selfgeneric at δ countable submodel $Z \prec H_\lambda$ with $Y \subset Z$ and $Y \cap H_\kappa = Z \cap H_\kappa$.*

Let $\delta \in \lambda \in \epsilon$ be cardinals and suppose a is a stationary set of countable selfgeneric at δ submodels of H_λ, $a \in \mathbb{Q}_{<\epsilon}$. The previous Fact shows that whenever δ is Woodin, there are plenty of such sets a. We wish to observe that $a \Vdash_{\mathbb{Q}_{<\epsilon}} \dot G \cap \mathbb{Q}_{<\delta}$ is a V-generic filter. And indeed, if $j : V \to M$ is the $\mathbb{Q}_{<\epsilon}$-term for the natural ultrapower embedding then $a \Vdash j''H^V_\lambda$ is selfgeneric at $j(\check\delta)$; that is, whenever $A \subset \mathbb{Q}_{<\delta}$ is a maximal antichain in V then there is $b \in A$ such that $j''H^V_\lambda \cap j(\bigcup b) = j''\bigcup b \in j(b)$, therefore $b \in \dot G$. So $a \Vdash_{\mathbb{Q}_{<\epsilon}}$ every maximal antichain $A \subset \mathbb{Q}_{<\delta}, A \in V$ has an element in $\dot G$ and $\dot G \cap \mathbb{Q}_{<\delta}$ is generic as desired.

Claim 2.3 ([W1]). *Let δ be a weakly compact Woodin cardinal and $G \subset \mathbb{Q}_{<\delta}$ be a generic filter. There exists an external V-generic filter $H \subset \mathrm{Coll}(\omega, <\delta)$ such that $\mathbb{R} \cap V[G] = \mathbb{R} \cap V[H]$.*

Proof: First observe that every real $r \in V[G]$ comes from a small generic extension — there is a V-Woodin cardinal $\kappa \in \delta$ such that $G \cap \mathbb{Q}_{<\kappa}$ is a V-generic filter and $r \in V[G \cap \mathbb{Q}_{<\kappa}]$. To see that, move back to V and choose an arbitrary condition $a \in \mathbb{Q}_{<\delta}$ and a $\mathbb{Q}_{<\delta}$-name $\dot r$ for a real. Then there are ω many maximal antichains $A_n : n \in \omega$ of $\mathbb{Q}_{<\delta}$ and functions $f_n : A_n \to \omega : n \in \omega$ making up the name $\dot r$. By Π^1_1 reflection at δ there is a Woodin cardinal $\kappa \in \delta$ such that $a \in \mathbb{Q}_{<\kappa}$ and all of $A_n \cap \mathbb{Q}_{<\kappa} : n \in \omega$ are maximal antichains of $\mathbb{Q}_{<\kappa}$. Let b consist of all countable elementary submodels $Z \prec H_{\kappa^+}$ which are selfgeneric at

κ and $Z \cap \bigcup a \in a$. Then $b \in \mathbb{Q}_{<\delta}, b \leq a$ and $b \Vdash_{\mathbb{Q}_{<\delta}} \dot{G} \cap \check{\mathbb{Q}}_{<\kappa}$ is a V-generic filter and $\dot{r} \in V[\dot{G} \cap \mathbb{Q}_{<\kappa}]$ as desired.

Working in $V[G]$ it is now possible to add the desired filter $H \subset \mathrm{Coll}(\omega, < \delta)$ by forcing it with initial segments. Let $R = \{h : h \subset \mathrm{Coll}(\omega, < \alpha)$ is a V-generic filter for some $\alpha \in \delta\}$ ordered by reverse inclusion. Suppose $K \subset R$ is a $V[G]$-generic filter and let $H = \bigcup K \subset \mathrm{Coll}(\omega, < \delta)$. Then

(1) H is a V-generic filter since each of its initial segments is V-generic and $\mathrm{Coll}(\omega, < \delta)$ has δ-c.c.,
(2) $\mathbb{R} \cap V[H] \subset \mathbb{R} \cap V[G]$ since first, $\mathbb{R} \cap V[H] = \bigcup_{\alpha \in \delta}(\mathbb{R} \cap V[H \cap \mathrm{Coll}(\omega, < \alpha)])$ by δ-c.c. of $\mathrm{Coll}(\omega, < \delta)$ and second, for every $\alpha \in \delta$ clearly $H \cap \mathrm{Coll}(\omega, < \alpha) \in K \subset V[G]$ and so $\mathbb{R} \cap V[H \cap \mathrm{Coll}(\omega, < \alpha)] \subset V[G]$,
(3) $\mathbb{R} \cap V[G] \subset \mathbb{R} \cap V[H]$. This is proved by a straightforward density argument, coding the reals of $V[G]$ into initial segments of H and using the first paragraph of this proof.

The Claim follows. □

It should be noted that the previous claim can fail at non-weakly compact Woodin cardinals, and that it may not be possible to find the required V-generic filter $H \subset \mathrm{Coll}(\omega, < \delta)$ in $V[G]$ even if δ has arbitrarily strong large cardinal properties.

Claim 2.4. *Let δ be a Woodin cardinal, $a \in \mathbb{Q}_{<\delta}$ and let P be a proper notion of forcing of size $< \delta$. If $G \subset P$ is a generic filter then there are an external V-generic filter $K \subset \mathbb{Q}_{<\delta}$ containing the condition a and external embeddings*

$$j : V \to M$$
$$j^* : V[G] \to N$$

such that j is the canonical K-ultrapower and $j \subset j^*$.

Proof: Let $a \in \mathbb{Q}_{<\delta}$, $p \in P$. By the standard genericity arguments it is enough to find external V-generic filters $K \subset \mathbb{Q}_{<\delta}$ with $a \in K$ and $G \subset P$ with $p \in G$ together with the required embeddings $j : V \to M$ and $j^* : V[G] \to N$ such that j is the K ultrapower and $j \subset j^*$.

Fix an inaccessible cardinal $\kappa \in \delta$ with $P \in H_\kappa$ and let $K \subset \mathbb{Q}_{<\delta}$ be a generic filter containing the condition a. By the properness of the forcing P and the elementarity of the K-ultrapower $j : V \to M$ it follows that $j''H_\kappa$ is in M a countable elementary submodel of $j(H_\kappa)$ which has a master condition $q \leq j(p)$ in the forcing $j(P)$. Let $H \subset j(P)$ be a $V[K]$-generic filter containing the condition q. Then $G = j^{-1}H \subset P$ is an H_κ-generic, that is, a V-generic filter containing the condition p and the embedding j naturally embeds to $j^* : V[G] \to M[H]$ by setting $j^*(\tau/G) = j(\tau)/H$ for every P-name $\tau \in V$. The claim follows. □

We do not have an explicit computation of the embedding j^* in terms of genericity over the model $V[G]$.

3. Weakly homogeneous trees

The following concept is central in the determinacy proofs. Let δ be a Woodin cardinal and Y be a set. A tree $T \subset (\omega \times Y)^{<\omega}$ is *$< \delta$-weakly homogeneous* if there are a set Z and a tree $T^* \subset (\omega \times Z)^{<\omega}$ such that $\mathrm{Coll}(\omega, < \delta) \Vdash p[\check{T}] = \dot{\mathbb{R}} \setminus p[\check{T}^*]$. The reader should be warned that this is a succinct equivalent due to Woodin [W1] of the real rather technical definition of $< \delta$-weak homogeneity. A set $A \subset \mathbb{R}$ is called $< \delta$-weakly homogeneously Souslin if it is a projection of a $< \delta$ weakly homogeneous tree. The importance of these notions is partially revealed in

Fact 3.1. *Suppose δ is a a weakly compact Woodin cardinal and $A \subset \mathbb{R}$ is a $< \delta$-weakly homogeneously Souslin set. Then the model $L(\mathbb{R}, A)$ satisfies the Axiom of Determinacy.*

Remark. The assumption of this Fact is not optimal.

Sketch of the proof: First argue as in [W1] that if A is $< \delta$-weakly homogeneously Souslin then so is $(\mathbb{R}, A)^{\#}$. Since every set of reals in $L(\mathbb{R}, A)$ is continuously reducible to $(\mathbb{R}, A)^{\#}$, every such set is $< \delta$-weakly homogeneously Souslin as well. By the results of [MS] all $< \delta$ weakly homogeneously Souslin sets are determined and the Fact follows. $\square$

The following is an abstract tree production lemma due to W. Hugh Woodin. Let x, y be sets and ϕ, ψ two-place formulas.

Theorem 3.2. *Suppose δ is a Woodin cardinal and*

$$\mathbb{Q}_{<\delta} \Vdash \forall r \in \mathbb{R}\ M \models \psi(r, j(\check{y})) \leftrightarrow V[r] \models \phi(r, x)$$

where $j : V \to M$ is the canonical ultrapower. Then the set $\{r \in \mathbb{R} : \psi(r, y)\}$ is $< \delta$-weakly homogeneously Souslin.

Fix a large cardinal λ such that ϕ, ψ reflect in H_λ and $cf(\lambda) > \delta$. A submodel $Z \prec H_\lambda$ is said to be *good* if it contains x, y, δ and writing $^- : Z \to \bar{Z}$ for the transitive collapse map, for every poset $P \in V_\delta \cap Z$, every $\bar{Z}$-generic filter $\bar{G} \subset \bar{P}$ and every real $r \in \bar{Z}[\bar{G}]$ we have

$$\psi(r, y) \leftrightarrow \bar{Z}[r] \models \phi(r, \bar{x}).$$

Note that this definition is internal meaning that the generic filters come from the universe we are working with. Not good models will be called *bad*; note that badness is witnessed by a poset, a filter on it and a real. One simple observation: suppose $\kappa \in \delta$ is an inaccessible cardinal, $Y \subset Z$ are submodels of H_λ with $H_\kappa \cap Y = H_\kappa \cap Z$ and $P \in H_\kappa \cap Y$. Then Y is a bad model as witnessed by P, $\bar{G}$, r if and only if Z is a bad model through the same witnesses.

Claim 3.3. *The set of all countable good submodels of H_λ contains a club in $[H_\lambda]^{\aleph_0}$.*

PROOF: Suppose for contradiction that the set a of all countable bad models is stationary. Stabilizing with respect to the poset witnessing badness we can find a forcing $P \in H_\kappa$ for some inaccessible cardinal $\kappa \in \delta$ and a stationary set $b \subset a$ of models whose badness is witnessed by P. By Fact 2.2 and the observation preceding this Claim the set c consisting of all countable models $Y \prec H_\lambda$ such that

(1) there is $Z \in b$ with $Z \subset Y$ and $Z \cap H_\kappa = Y \cap H_\kappa$,
(2) Y is self-generic at δ

is stationary and all models in Y are bad as witnessed by the poset P.

Now choose a large regular cardinal ϵ and a generic filter $H_1 \subset \mathbb{Q}_{<\epsilon}$ containing the condition c. It follows that the filter $H_0 = H_1 \cap \mathbb{Q}_{<\delta}$ is V-generic and the following diagram commutes,

$$\begin{array}{ccc} V & \xrightarrow{j_1} & M_1 \\ \| & & \uparrow k \\ V & \xrightarrow{j_0} & M_0 \end{array}$$

where j_0 is the H_0-ultrapower, j_1 the H_1-ultrapower and $k[f]_{H_0} = [f]_{H_1}$. The model M_1 is not necessarily wellfounded but certainly $j_1''H_\lambda$ is a bad submodel of $j_1 H_\lambda$ in M_1 as witnessed by the poset $j_1(P)$. Back in V choose an elementary submodel X of H_λ of size $< \delta$ containing all of H_κ. By the observation before the Claim the submodel $j_1''X \prec j_1''H_\lambda \prec j(H_\lambda)$ is bad in M_1 as witnessed by $j_1(P)$. Since $j_0''X \in M_0$ and $j_1''X = k''j_0''X = k(j_0''X)$ it follows from the elementarity of the embedding k that $j_0''X$ is a bad submodel of $j_0(H_\lambda)$ in M_0 as witnessed by the poset $j_0(P)$. Pick a real $r \in M_0$ witnessing this.

Writing $^- : X \to \bar{X}$ for the transitive collapse map we have

(1) $\bar{X}[r] \models \phi(r, \bar{x}) \leftrightarrow V[r] \models \phi(r, x)$ — this holds by the elementarity of X and $P \subset X$,
(2) $\bar{X}[r] \models \phi(r, \bar{x}) \not\leftrightarrow M_0 \models \psi(r, j_0(y))$ — by the badness of $j_0''X$ in M_0.

But the above two points contradict the assumption of the theorem that $M_0 \models \psi(r, j_0(y)) \leftrightarrow V[r] \models \phi(r, x)$. □

Fix a function $f : H_\lambda^{<\omega} \to H_\lambda$ such that all of its countable closure points are good submodels of H_λ. Define a tree T of triples of finite sequences so that

(1) $\langle s, t, u\rangle \in T$ just in case s is a finite sequence of integers, t is a finite sequence of finite subsets of H_λ and u is a finite sequence of elements of H_δ and s, t, u have the same length,
(2) $t(0) = \{P, \tau\}$ where $P \in H_\delta$ is a poset and τ is a P-name for a real,
(3) u is a decreasing sequence of elements of P such that $u(n)$ belongs to all open dense subsets of P which are in $t(n)$,

(4) for every integer n, $t(n+1) = f''(\mathrm{range}(u \restriction n+1) \cup \mathrm{range}(t \restriction n+1))^{<\omega}$,
(5) for every integer n, $u(n)$ decides the value of $\tau \restriction n$ and $s \restriction n$ is equal to this value,
(6) $u(0) \Vdash_P V[\tau] \models \phi(\tau, \check{x})$.

Obviously, T is closed under initial segment and whenever a triple s, t, u represents any infinite branch of T it gives rise to

(7) a good submodel $Z \prec H_\lambda$ defined by $Z = \bigcup \mathrm{range}(t)$ — this follows from (4) and the choice of the function f,
(8) a Z-generic filter $G \subset Z \cap P$ defined as the upwards closure of range(u) in the poset P, where $P \in Z \cap H_\delta$ is the poset indicated in $t(0)$ — see (3) above,
(9) a real r defined by $r = s$ or $r = \tau/G$

such that writing $^- : Z \to \bar{Z}$ for the transitive collapse map, we have — see (6) — $\bar{Z}[r] \models \phi(r, \bar{x})$ or $\psi(r, y)$ which amounts to the same thing due to the goodness of the model Z.

A tree T^* is defined in the same way replacing the requirement (6) by $u(0) \Vdash_P V[\tau] \models \neg\phi(\tau, \check{x})$. It is immediate to see that $p[T] = \{r \in \mathbb{R} : \psi(r, y)\} = \mathbb{R} \setminus p[T^*]$. The above observation shows that any real $r \in p[T]$ has $\psi(r, y)$; on the other hand, if $\psi(r, y)$ holds for a real r, it is possible to build a branch s, t, u through the tree T such that $t(0) = \{$the trivial poset and its name for $r\}$ and $s = r$, proving that $r \in p[T]$. The following claim shows that T is $< \delta$-weakly homogeneous and thus completes the proof of the Theorem.

Claim 3.4. $\mathrm{Coll}(\omega, < \delta) \Vdash p[\check{T}] = \mathbb{R} \setminus p[\check{T}^*]$.

Proof: First observe that $p[T] \cap p[T^*] = 0$ and that this is absolute between models of ZFC containing T, T^* and all ordinals since it is a statement about wellfoundedness of the tree of attempts to build infinite branches through T, T^* with the same first coordinates.

Let $G \subset \mathrm{Coll}(\omega, < \delta)$ be a generic filter. We know that in $V[G]$, $p[T] \cap p[T^*] = 0$. It must be argued that for every real $r \in \mathbb{R} \cap V[G]$ either $r \in p[T]$ or $r \in p[T^*]$. Choose a cardinal $\kappa \in \delta$ and a V-generic filter $H \subset \mathrm{Coll}(\kappa)$, $H \in V[G]$ such that $r \in V[H]$. Now suppose for example that $V[r] \models \phi(r, x)$. It is easy working in $V[H]$ to produce a countable submodel Z of $H^V_\lambda[r]$, a $\mathrm{Coll}(\kappa)$-name $\tau \in V$ such that $\tau/H = r$ and an infinite branch s, t, u through the tree T so that

(1) $t(0) = \{\mathrm{Coll}(\kappa), \tau\}$,
(2) $\bigcup \mathrm{range}(t) = Z \cap V$,
(3) the upwards closure of $u(0)$ in $Z \cap \mathrm{Coll}(\kappa)$ is exactly $H \cap Z = H$,
(4) $s = \tau/H = r$ — this actually follows from (1) and (3).

Consequently, $r \in p[T]$. The claim follows. □

4. The Image Theorem

Suppose δ is a a weakly compact Woodin cardinal and A is a bounded subset of $\theta = \theta^{L(\mathbb{R})}$. The left to right direction of the Image Theorem is easier, follows essentially from Claim 2.3 and was known previously to the workers in the field, though we could not find a published reference. Here is the proof.

Claim 4.1. *Let $\alpha \in \theta+1$ be an ordinal. Then there is β such that $\mathbb{Q}_{<\delta} \Vdash j(\check{\alpha}) = \check{\beta}$.*

Proof: Let $\chi(\cdot,\cdot)$ be a two-place formula defining in $L(\mathbb{R}^{\#})$ a prewellordering of the reals of length $\theta^{L(\mathbb{R})}+1$. Let $\alpha \in \theta+1$ be an arbitrary ordinal and fix a real r such that

(*) $\quad L(\mathbb{R}^{\#}) \models r$ is in the α-th section of the χ-prewellorder

meaning that the unique map from the reals onto $\theta+1$ preserving the prewellorder assigns the ordinal α to r. By a homogeneity argument, there is an ordinal β such that

$$\mathrm{Coll}(\omega,<\delta) \Vdash L(\mathbb{R}^{\#}) \models \check{r} \text{ is in the } \check{\beta}\text{-th section of the } \chi\text{-prewellorder.}$$

We claim that β works, that is $\mathbb{Q}_{<\delta} \Vdash j(\check{\alpha}) = \check{\beta}$. To see that, note that for every V-generic filter $G \subset \mathbb{Q}_{<\delta}$ there is an external V-generic filter $H \subset \mathrm{Coll}(\omega,<\delta)$ such that $V[G]\cap\mathbb{R} = M\cap\mathbb{R} = V[H]\cap\mathbb{R}$, where $j: V \to M$ is the canonical G-ultrapower of the ground model, Claim 2.3. By the uniqueness of sharps, $\mathbb{R}^{\#V[H]} = \mathbb{R}^{\#M}$ and so by the choice of β,

(**) $\quad L(\mathbb{R}^{\#})^M \models r$ is in the β-th section of the χ-prewellorder.

Comparing the formulas (*) and (**), by the elementarity of the embedding j it follows that $j(\alpha)=\beta$ as desired. $\square$

It is easy to see that the above argument in fact shows that images of the lengths of $<\delta$-weakly homogeneously Souslin prewellorderings of the reals are determined by the largest condition in $\mathbb{Q}_{<\delta}$. It is not clear whether there is any ordinal whose image is not determined by the largest condition in $\mathbb{Q}_{<\delta}$ and if so, what is the least such ordinal.

So suppose now that $A \in L(\mathbb{R})$ is a bounded subset of $\theta^{L(\mathbb{R})}$. We shall produce a set B which is outright forced to be the image of A under the $\mathbb{Q}_{<\delta}$-ultrapower. Let $\alpha = \sup(A) \in \theta$. Our assumptions imply that $L(\mathbb{R})$ satisfies the Axiom of Determinacy and thus by the Coding Lemma ([M]) the set $A \subset \alpha$ is definable in $L(\mathbb{R})$ from some real r and the ordinal α, say

$$L(\mathbb{R}) \models A = \{\xi : \chi(\xi,\alpha,r)\}.$$

Using the previous Claim find an ordinal β such that $\mathbb{Q}_{<\delta} \Vdash j(\check{\alpha}) = \check{\beta}$. Let $B = \{\xi \in \beta : \mathrm{Coll}(\omega,<\delta) \Vdash L(\mathbb{R}) \models \chi(\check{\xi},\check{\beta},\check{r})\}$. Arguing much like in the previous Claim it follows from Claim 2.3 that $\mathbb{Q}_{<\delta} \models j(\check{A}) = \check{B}$ and we are done.

To prove the opposite direction of the Image Theorem, suppose B is a set such that $\mathbb{Q}_{<\delta} \models j(\check{A}) = \check{B}$. We wish to conclude that $A \in L(\mathbb{R})$. Let $\alpha = \sup(A) \in \theta$ and choose a formula $\chi(\alpha, \cdot, \cdot)$ defining in $L(\mathbb{R})$ a prewellordering of the reals of length α. Let $A^* \subset \mathbb{R}$ be the set of all reals whose rank in this prewellordering belongs to A. We shall prove that A^* is $< \delta$-weakly homogeneously Souslin. Then by Fact 3.1, the model $L(\mathbb{R}, A^*)$ satisfies the Axiom of Determinacy and also $A \in L(\mathbb{R}, A^*)$. By an application of the coding lemma in $L(\mathbb{R}, A^*)$, we have $A \in L(\mathbb{R})$ as desired.

Let $\beta = \sup(B)$; so $\mathbb{Q}_{<\delta} \Vdash j(\check{\alpha}) = \check{\beta}$. We claim that the assumptions of Theorem 3.2 are satisfied with $y = A, \psi(r, y) =$"the rank of the real r in the prewellorder defined in $L(\mathbb{R})$ by the formula $\chi(\sup(y), \cdot, \cdot)$ belongs to y" and $x = \langle \delta, B \rangle, \phi(r, \langle x_0, x_1 \rangle) =$ "$\mathrm{Coll}(\omega, < x_0) \Vdash$ the rank of the real $\check{r}$ in the prewellorder defined in $L(\mathbb{R})$ by the formula $\chi(\sup(\check{x}_1), \cdot, \cdot)$ belongs to $\check{x}_1$". To see this suppose $G \subset \mathbb{Q}_{<\delta}$ is a generic filter and $j : V \to M$ the corresponding embedding, and $r \in \mathbb{R} \cap V[G]$. We must prove that $M \models$"the rank of r in the prewellorder ... belongs to $j(A) = B$" if and only if $V[r] \models \mathrm{Coll}(\omega, < \delta) \Vdash$"the rank of $\check{r}$ in the prewellorder ... belongs to $\check{B}$". Using Claim 2.3 choose an external V-generic filter $H \subset \mathrm{Coll}(\omega, < \delta)$ such that $\mathbb{R} \cap V[G] = \mathbb{R} \cap V[H]$. By factoring facts about $\mathrm{Coll}(\omega, < \delta)$ [J, Exercise 25.11] there is a $V[r]$ generic filter $K \subset \mathrm{Coll}(\omega, < \delta)$ such that $V[H] = V[r][K]$, in particular $\mathbb{R} \cap V[r][K] = \mathbb{R} \cap V[H] = \mathbb{R} \cap V[G] = \mathbb{R} \cap M$. Thus $V[r] \models \mathrm{Coll}(\omega, < \delta) \Vdash$ the rank of $\check{r}$ in the prewellorder ... is in $\check{B}$" if and only if $V[r][K] \models$"the rank of r in the prewellorder ... is in B" if and only if $M \models$"the rank of r in the prewellorder ... is in B", where the first equivalence follows from the forcing theorem and the second from the fact that $V[r][K]$ and M have the same reals and both contain the set B.

Therefore the assumptions of Theorem 3.2 are satisfied and it applies to show that the set $A^* = \{r \in \mathbb{R} : \psi(r, A)\}$ is $< \delta$-weakly homogeneously Souslin. The Image Theorem follows.

5. The Embedding Theorem

Suppose $\mathbb{R} \subset \mathbb{R}^*$ are sets of reals, possibly $\mathbb{R}$ are the reals of V and R^* are the reals of some of the generic extensions of V. If there is an elementary embedding $i : L(\mathbb{R}) \to L(\mathbb{R}^*)$ fixing all ordinals, this embedding must be unique: every set $x \in L(\mathbb{R})$ is definable from some real r and an ordinal α, say as the unique solution of the condition $\phi(\cdot, r, \alpha)$. Then necessarily $i(x)$ is the unique solution of the condition $\phi(\cdot, r, \alpha)$ in $L(\mathbb{R}^*)$ since the reals and ordinals are fixed by i. To confirm an existence of such an embedding we must prove that the above correspondence is well-defined, and for that it is enough to show that for every formula ϕ, every real $r \in \mathbb{R}$ and every ordinal α

(*) $$L(\mathbb{R}) \models \phi(\alpha, r) \text{ if and only if } L(\mathbb{R}^*) \models \phi(\alpha, r).$$

Since $\mathrm{HOD}^{L(\mathbb{R})}$ can be coded by a set of ordinals and such sets are fixed by i it must be the case that $\mathrm{HOD}^{L(\mathbb{R}^*)} = i(\mathrm{HOD}^{L(\mathbb{R})}) = \mathrm{HOD}^{L(\mathbb{R})}$. It follows that if $L(\mathbb{R})$ satisfies the Axiom of Determinacy then $L(\mathbb{R}^*)$ is a symmetric extension of HOD $^{L(\mathbb{R})}$ using the algebra $\mathbb{B}_\omega$ described in Section 1. This is our route of proof of the Embedding Theorem.

Let δ be a weakly compact Woodin cardinal , P a proper forcing notion of size $< \delta$, and let $G \subset P$ be a generic filter. We shall show that $L(\mathbb{R} \cap V[G])$ is a symmetric extension of HOD $^{L(\mathbb{R}\cap V)}$: if $r_i : i \in \omega$ is a $V[G]$-generic enumeration of $\mathbb{R} \cap V[G]$ then the filter on the algebra $\mathbb{B}_\omega$ computed in $L(\mathbb{R} \cap V)$ given by the equations $\dot{r}_i = r_i : i \in \omega$ will be proved HOD $^{L(\mathbb{R}\cap V)}$-generic. Then (*) follows: for every formula ϕ, real $r \in \mathbb{R} \cap V$ and an ordinal α

$$L(\mathbb{R} \cap V) \models \phi(\alpha, r)$$

iff

$$\mathrm{HOD}^{L(\mathbb{R}\cap V)}[r] \models \mathbb{B}_\omega/\mathbb{B}_0 \Vdash L(\dot{\mathbb{R}}_{sym}) \models \phi(\check{\alpha}, \check{r})$$

iff

$$L(\mathbb{R} \cap V[G]) \models \phi(\alpha, r).$$

Here the HOD $^{L(\mathbb{R}\cap V)}$-generic filter on $\mathbb{B}_0$ is given by the equation $r = \dot{r}_0$. Above, the first equivalence is due to the symmetricity of $\mathbb{B}_\omega$ as described after Theorem 1.1 and the second equivalence comes from the forcing theorem.

Now suppose $r_k : k \leq i$ is a finite sequence of reals in $\mathbb{R} \cap V[G]$. We shall prove that the following holds in $V[G]$:

(1) the equations $r_k = \dot{r}_k : k \leq i$ define a HOD $^{L(\mathbb{R}\cap V)}$-generic filter on $\mathbb{B}_i$ as computed in $L(\mathbb{R} \cap V)$,
(2) for every open dense set $D \subset \mathbb{B}_\omega$ in HOD $^{L(\mathbb{R}\cap V)}$ there is a prolongation $r_k : k \leq i^*$ of the original sequence such that the filter on $\mathbb{B}_{i^*}$ as computed in $L(\mathbb{R}\cap V)$ given by the equation $\dot{r}_k = r_k : k \leq i^*$ contains some condition in D.

An elementary density argument then shows that for any $V[G]$-generic enumeration $r_k : k \in \omega$ of the $V[G]$ reals the filter on the algebra $\mathbb{B}_\omega$ — as computed in $L(\mathbb{R} \cap V)$ — defined by the equations $\dot{r}_k = r_k : k \in \omega$ is HOD $^{L(\mathbb{R}\cap V)}$-generic. Therefore $L(\mathbb{R} \cap V[G])$ is a symmetric extension of HOD $^{L(\mathbb{R}\cap V)}$ and as above (*) and the Embedding Theorem follow.

So fix a sequence $r_k : k \leq i$ of $V[G]$ reals. We shall need a couple of external objects. By Claim 2.4 there are external embeddings

$$j : V \to M$$
$$j^* : V[G] \to M[H]$$

so that j is the $\mathbb{Q}_{<\delta}$-generic ultrapower of V and $j \subset j^*$. While we know that the reals of the model M come from a $\mathrm{Coll}(\omega, < \delta)$ generic extension of V — Claim 2.3 — it is not clear whether the same holds of the reals of $M[H]$. However, a weaker property of $M[H]$ will be sufficient:

Claim 5.1. *There is an external V-generic filter $K \subset \operatorname{Coll}(\omega, < \delta)$ such that $\{r_k : k \leq i\} \subset \mathbb{R} \cap V[K] \subset \mathbb{R} \cap M[H]$.*

PROOF: Force K with initial segments which belong to $M[H]$ and code over V the reals $r_k : k \leq i$. Note that these reals are generic over V using the poset P whose size is $< \delta$. The density arguments are virtually trivial noting that $V_\delta \cap V$ is a collection of sets hereditarily countable in $M[H]$. In the end, $\mathbb{R} \cap V[K] = \bigcup_{\alpha\in\delta}(\mathbb{R} \cap V[K \restriction \alpha])$ by the δ-c.c. of $\operatorname{Coll}(\omega, < \delta)$ and each of $\mathbb{R} \cap V[K \restriction \alpha] : \alpha \in \delta$ is a member of $M[H]$ since $K \restriction \alpha$ as well as big chunks of V belong to $M[H]$. It follows that $\mathbb{R} \cap V[K] \subset \mathbb{R} \cap M[H]$ as desired. □

We shall show that (1) and (2) above hold of $r_k : k \leq i = j^*(r_k) : k \leq i$ in the model $M[H]$, replacing $\mathbb{R} \cap V$ with $\mathbb{R} \cap M$ and $\mathbb{R} \cap V[G]$ with $\mathbb{R} \cap M[H]$. By elementarity of j^* this will complete the proof.

Claim 5.2. *In V there is a class model N such that $\operatorname{Coll}(\omega, < \delta) \Vdash \check{N} = \mathrm{HOD}^{L(\mathbb{R})}$. Moreover, there are algebras $\mathbb{A}_0 \lessdot \mathbb{A}_1 \lessdot \cdots \lessdot \mathbb{A}_\omega$ in N such that $\operatorname{Coll}(\omega, < \delta) \Vdash \check{\mathbb{A}}_\omega \in \check{N}$ has the same definition in $L(\mathbb{R})$ as the algebra $\mathbb{B}_\omega$ from Theorem* 1.1.

PROOF: Since the forcing $\operatorname{Coll}(\omega, < \delta)$ is homogeneous, every ordinal definable in $L(\mathbb{R}^{\operatorname{Coll}(\omega,<\delta)})$ set of ordinals belongs to the ground model V. The claim follows. □

Note that $N = \mathrm{HOD}$ of $L(\mathbb{R} \cap M) = j(\mathrm{HOD}$ of $L(\mathbb{R} \cap V))$ and $\mathbb{A}_\omega = j(\mathbb{B}_\omega$ as computed in $L(\mathbb{R} \cap V))$ by Claim 2.3. Also $N = \mathrm{HOD}$ of $L(\mathbb{R} \cap V[K])$. Now the analysis of Section 1 can be applied in the model $L(\mathbb{R} \cap V[K])$: there the equations $\dot{r}_k = r_k : j \leq i$ determine an N-generic filter on $\mathbb{A}_i$ and for an arbitrary open dense set $D \subset \mathbb{A}_\omega$ in N there is a prolongation $r_k : k \leq i^*$ of the sequence $r_k : k \leq i$ such that the filter on $\mathbb{A}_{i^*}$ defined by the equations $\dot{r}_k = r_k : j \leq i^*$ is N-generic and contains some condition from D. But then the same must hold in $M[H]$ which contains N and all the reals of $V[K]$. The Embedding Theorem follows.

W. Hugh Woodin pointed out to us that the Embedding Theorem can be derived from Theorem 3.4 of [FM], which in turn follows from the Embedding Theorem for higher models of determinacy of the form $L(\mathbb{R}, A)$, $A \subset \mathbb{R}$ weakly homogeneously Souslin.

6. The Anticoding Theorem

Any set A of ordinals can be coded into a real in a generic extension — just collapse the size of $\sup(A)$ onto $\aleph_0$. The Anticoding Theorem says that such cheap tricks are impossible if the forcing in question is to be proper.

Let δ be a weakly compact Woodin cardinal and P a proper forcing notion of size $< \delta$. Choose a set A of ordinals. Obviously if $A \in L(\mathbb{R})$ then $P \Vdash \check{A} \in L(\mathbb{R})$, namely, $A = i(A)$ where i is the ordinal-fixing elementary embedding described in the Embedding Theorem. To prove the Anticoding Theorem we must show that if $A \notin L(\mathbb{R})$ then $P \Vdash \check{A} \notin L(\mathbb{R})$. This will be done in two stages: first, under the assumption that A is a bounded subset of $\theta^{L(\mathbb{R})}$ and then in the general case.

So suppose for now that $A \subset \theta^{L(\mathbb{R})}$ is bounded and not in $L(\mathbb{R})$. The Image Theorem provides an ordinal ξ such that both $\check{\xi} \in j(\check{A})$ and $\check{\xi} \notin j(\check{A})$ have nonzero boolean value in $\mathbb{Q}_{<\delta}$; set such a ξ aside. Suppose for contradiction that some condition $p \in P$ forces A into $L(\mathbb{R})$. By strengthening p if necessary it is possible to find a formula ϕ, an ordinal $\alpha \in \theta^{L(\mathbb{R})}$ and a P-name τ for a real such that

$$p \Vdash \check{A} = \{\beta : L(\mathbb{R}) \models \phi(\check{\beta}, \check{\alpha}, \tau)\}.$$

As in Claim 5.2, let N be a class model such that $\mathrm{Coll}(\omega, <\delta) \Vdash \check{N} = \mathrm{HOD}^{\,L(\mathbb{R})}$ and let $\mathbb{B}_0, \mathbb{B}_\omega \in N$ be the algebras such that $\mathrm{Coll}(\omega, <\delta) \Vdash \check{\mathbb{B}}_0 \subset \check{\mathbb{B}}_\omega$ are the algebras defined in $L(\mathbb{R})$ by the analysis of Section 1. As in Claim 4.1, let γ be an ordinal such that $\mathbb{Q}_{<\delta} \Vdash j(\check{\alpha}) = \check{\gamma}$. By strengthening the condition p again we may assume that it decides the statement

$$(*) \qquad N[\tau] \models \mathbb{B}_\omega/\mathbb{B}_0 \Vdash L(\dot{\mathbb{R}}_{sym}) \models \phi(\check{\xi}, \check{\gamma}, \tau).$$

Here, the N-generic filter on $\mathbb{B}_0$ is given by the equation $\tau = \dot{r}_0$ — see Section 1 for the definition of the $\mathbb{B}_0$-name $\dot{r}_0$. Note that $p \Vdash$"such a filter is N-generic" since P can be embedded into $\mathrm{Coll}(\omega, <\delta)$ and $\mathrm{Coll}(\omega, <\delta) \Vdash$"for every real r the equation $r = \dot{r}_0$ determines an $\check{N}$ generic filter on $\mathbb{B}_0$."

Suppose for example that the condition p forces (*) to hold. By Claim 2.4 and the choice of ξ it is possible to find external filters G, H and elementary embeddings

$$j : V \to M$$
$$j^* : V[G] \to M[H]$$

so that $G \subset P$ is a V-generic filter containing the condition p, j is a $\mathbb{Q}_{<\delta}$ generic ultrapower of V such that $\xi \notin j(A)$, $H \subset j(P)$ is an M-generic filter extending $j''G$ and $j \subset j^*$. Let $r = \tau/G = j(\tau)/H$.

By Claim 2.3, $N = \mathrm{HOD}^{\,L(\mathbb{R} \cap M)}$. By the Embedding Theorem applied in M to $j(P)$, $N = \mathrm{HOD}^{\,L(\mathbb{R} \cap M[H])}$. By the elementarity of j^*,

$$j^*(A) = \{\beta : L(\mathbb{R} \cap M[H]) \models \phi(\beta, j^*(\alpha) = j(\alpha) = \gamma, r)\}.$$

By the results of Section 1 applied in $L(\mathbb{R} \cap M[H])$,

$$j^*(A) = \{\beta : N[r] \models \mathbb{B}_\omega/\mathbb{B}_0 \Vdash L(\dot{\mathbb{R}}_{sym}) \models \phi(\beta, \gamma, r)\}.$$

Since (*) was forced to hold, from the last equality it follows that $\xi \in j^*(A)$. But $j^*(A) = j(A)$ and $\xi \notin j(A)$ by the choice of the embedding j, a contradiction proving the special case of the Anticoding Theorem.

Let now A be an arbitrary set of ordinals, $A \notin L(\mathbb{R})$ and suppose for contradiction that in a generic extension $V[G]$ using the forcing P it so happens that

$A \in L(\mathbb{R} \cap V[G])$. By the minimality properties of that model one can choose a formula ϕ, an ordinal η and a real $r \in V[G]$ so that

$$L(\mathbb{R} \cap V[G]) \models A = \{\beta : \phi(\beta, \eta, r)\}.$$

Let $\theta = \theta^{L(\mathbb{R}\cap V)} = \theta^{L(\mathbb{R}\cap V[G])}$ and let N be the common HOD of $L(\mathbb{R}\cap V)$ and $L(\mathbb{R} \cap V[G])$. (Note that the two models have the same HOD by the Embedding Theorem.) Choose a large regular cardinal λ such that $\eta \in \lambda$ and ϕ reflects in $V_\lambda \cap L(\mathbb{R} \cap V[G])$. Move into N and construct an inclusion increasing sequence $Z_\alpha : \alpha \in \theta$ of elementary submodels of $V_\lambda \cap N$ such that

(1) $|Z_\alpha| < \theta$,
(2) $\alpha \subset Z_\alpha$,
(3) $\eta, \mathbb{B}_1, \mathbb{B}_\omega \in Z_0$, where $\mathbb{B}_1, \mathbb{B}_\omega$ are the algebras from Section 1 calculated in $L(\mathbb{R} \cap V)$ or $L(\mathbb{R} \cap V[G])$ — by the Embedding Theorem both of these calculations give the same algebra.

This is easily done since θ is a regular cardinal in the model N. Note that all of these models and their transitive collapses belong to N and therefore to all of the other four class models named so far.

Claim 6.1. *For each $\alpha \in \theta$ there is a real $s \in V$ such that $L(\mathbb{R} \cap V[G]) \models A \cap Z_\alpha = \{\beta \in Z_\alpha : \phi(\beta, \alpha, s)\}$.*

Proof: This follows immediately from the Embedding Theorem once we prove that $A \cap Z_\alpha \in L(\mathbb{R} \cap V)$. For then, there must be a real $s \in V$ such that $L(\mathbb{R} \cap V) \models A \cap Z_\alpha = \{\beta \in Z_\alpha : \phi(\beta, \eta, s)\}$ since in $V[G]$ there is such a real, namely r. The Embedding Theorem applied once again shows that this real $s \in V$ works as desired in the Claim.

To see that $A \cap Z_\alpha \in L(\mathbb{R} \cap V)$ we use the first part of the proof of the Anticoding Theorem. Let $^- : Z_\alpha \to \bar{Z}_\alpha$ be the transitive collapse and let $\bar{A}$ be the image of $A \cap Z_\alpha$under the bar map. From the cardinality requirement (2) on Z_α it follows that $\bar{Z}_\alpha \cap \mathrm{Ord} \in \theta$ and so $\bar{A}$ is a bounded subset of θ. Since $\bar{A} \in V \cap L(\mathbb{R} \cap V[G])$ the first part of the proof of the Theorem applied in V to P and $\bar{A}$ implies that $\bar{A} \in L(\mathbb{R} \cap V)$. But the bar map belongs to N and $L(\mathbb{R} \cap V)$ as well and so $A \cap Z_\alpha \in L(\mathbb{R} \cap V)$. □

Claim 6.2. *For every real $s \in V$ there is $\alpha \in \theta$ such that $L(\mathbb{R}\cap V[G]) \models A \cap Z_\alpha \neq \{\beta \in Z_\alpha : \phi(\beta, \eta, s)\}$.*

Proof: Fix a real $s \in V$. There must be an ordinal $\beta \in \lambda$ such that

$$L(\mathbb{R} \cap V) \models \phi(\beta, \eta, s) \not\leftrightarrow L(\mathbb{R} \cap V[G]) \models \phi(\beta, \eta, r)$$

since otherwise the set $A = \{\beta : L(\mathbb{R}\cap V) \models \phi(\beta, \eta, s)\}$ would belong to $L(\mathbb{R}\cap V)$ contradicting our assumption on it. For each β as above, from the Embedding Theorem it is the case that

$$(**) \qquad L(\mathbb{R} \cap V[G]) \models \phi(\beta, \eta, s) \not\leftrightarrow L(\mathbb{R} \cap V[G]) \models \phi(\beta, \eta, r).$$

We need to find such an ordinal in the model $Z = \bigcup_{\alpha\in\theta} Z_\alpha$ since then any ordinal $\alpha \in \theta$ with $\beta \in Z_\alpha$ will work as required in the Claim.

Let $H \subset \mathbb{B}_1$ be the N-generic filter given by the equations $\dot{r}_0 = r, \dot{r}_1 = s$. Applying the analysis of Section 1 to $L(\mathbb{R} \cap V[G])$ for each ordinal β as above we get

$$N[H] \models \mathbb{B}_\omega/\mathbb{B}_1 \Vdash L(\dot{\mathbb{R}}_{sym}) \models \phi(\beta, \eta, s) \not\leftrightarrow \phi(\beta, \eta, r).$$

Let $Z[H] = \{\tau/H : \tau \text{ is a } \mathbb{B}_1\text{-name in } Z\}$. As usual, $Z[H]$ is an elementary submodel of $V_\lambda \cap N[H]$ and moreover $Z[H] \cap N = Z$. The latter assertion follows from the fact that $\mathbb{B}_1 \in Z$, $|\mathbb{B}_1| = \theta$, $\theta \subset Z$ and so $\mathbb{B}_1 \subset Z$. Now by the elementarity of the submodel $Z[H] \prec V_\lambda \cap N[H]$ there must be an ordinal $\beta \in Z[H]$ as in (**). But such an ordinal lies in Z as desired. $\square$

In $L(\mathbb{R} \cap V[G])$ define a function $f : \mathbb{R} \cap V[G] \to \theta$ by setting $f(s)$ =the least α such that there is $\beta \in Z_\alpha$ with $\phi(\beta, \eta, s) \not\leftrightarrow \phi(\beta, \eta, r)$ if such α exists, and $f(s) = 0$ otherwise. The previous two claims show that the range of f is cofinal in θ contradicting the definition of θ in $L(\mathbb{R})$. The Anticoding Theorem has been demonstrated.

7. Examples of coding

The Anticoding Theorem cannot be generalized to semiproper forcings. A simple argument for that was pointed out to us by W. Hugh Woodin. Let $\delta \in \kappa$ be a Woodin and a measurable cardinal respectively and $A \subset \delta$ a countable subset of δ which does not belong to $L(\mathbb{R})$ — for example an infinite set of $L(\mathbb{R})$-indiscernibles. By a semiproper forcing it is possible to make the nonstationary ideal on ω_1 saturated and $\omega_2 = \delta = \delta^1_2$ — [W2]. In the resulting model A is a countable subset of δ^1_2 and therefore belongs to $L(\mathbb{R})$. In this section we handle the much finer problem of coding subsets of ω_1 into reals.

Theorem 7.1. *It is consistent with large cardinals to have a set $A \subset \omega_1$, $A \notin L(\mathbb{R})$ and a forcing preserving stationary subsets of ω_1 such that $P \Vdash \check{A} \in L(\mathbb{R})$.*

It follows from the results of [W2] that in the context of Martin's Maximum no $\aleph_1$-preserving forcing can code a set $A \subset \omega_1$, $A \notin L(\mathbb{R})$ into a real and therefore one has to resort to a mere consistency result in Theorem 7.1.

For the proof of Theorem 7.1 a generalization of the nonstationary tower forcing will be needed. Given a cardinal δ, the *full nonstationary tower forcing* ([W1]) $\mathbb{P}_{<\delta}$ is the set $\{a : a \text{ is a stationary system of subsets of } \bigcup a \in H_\delta\}$ ordered by $a \geq b$ if $\bigcup a \subset \bigcup b$ and $\forall x \in b\ b \cap \bigcup a \in a$ holds. The natural $\mathbb{P}_{<\delta}$-generic ultrapower $j : V \to M$ has similar properties as the one introduced by $\mathbb{Q}_{<\delta}$. An exposition can be found in [FM]. We shall use the fact due to Woodin that if δ is Woodin then M is wellfounded, closed under $< \delta$ sequences in $V[G]$ and $a \in G \leftrightarrow j'' \bigcup a \in j(a)$ whenever $a \in \mathbb{P}_{<\delta}$ and $G \subset \mathbb{P}_{<\delta}$ is the generic filter.

Let $\kappa \in \delta$ be a measurable and a Woodin cardinal respectively and fix a set $A \subset \kappa$ such that $V_\kappa^\# \in L[A]$. Consider Magidor's forcing $\mathbb{M}$ for making $\kappa = \aleph_1$

and the nonstationary ideal on ω_1 precipitous [JMMP] and the full nonstationary tower forcing $\mathbb{P}_{<\delta}$ on δ. We shall find a condition $a \in \mathbb{P}_{<\delta}$ and a complete embedding of the completion of the poset $\mathbb{M}$ into the completion of the poset $\mathbb{P}_{<\delta} \restriction a$ such that

(1) $\mathbb{M} \Vdash \check{A} \notin L(\mathbb{R})$ — this is of course true regardless of the embedding,
(2) $\mathbb{M} \Vdash \mathbb{P}_{<\delta} \restriction a/\mathbb{M}$ preserves stationary subsets of $\omega_1 = \check{\kappa}$,
(3) $\mathbb{P}_{<\delta} \restriction a \Vdash \check{A}$ is constructible from a real.

So the generic extension of the universe using the poset $\mathbb{M}$ is the model needed for Theorem 7.1. There the stationary preserving forcing $\mathbb{P}_{<\delta} \restriction a/\mathbb{M}$ nontrivially codes the set A into a real.

The construction of $\mathbb{M}$ is somewhat convoluted and its exact form is immaterial for our purposes. The definition has as parameters a normal measure U on κ with the associated ultrapower embedding $j : V \to M$, and a certain simple bookkeeping tool which we shall neglect in the sequel. The following two key properties of the poset $\mathbb{M}$ can be found in [JMMP]:

(1) in the generic extension by $\mathbb{M}$, the nonstationary ideal on ω_1 is precipitous and the algebra Power(ω_1) modulo NS_{ω_1} forces the canonical generic ultrapower to extend the embedding j. In fact this is how the precipitousness of NS_{ω_1} is proved;
(2) the reals of the $\mathbb{M}$ generic extension are exactly the reals of some Coll$(\omega, < \kappa)$ generic extension. Indeed, $\mathbb{M}$ is an iteration of Coll$(\omega, < \kappa)$ and an $\aleph_0$ distributive forcing.

Claim 7.2. *Suppose $G \subset \mathbb{M}$ is a generic filter and $S \in V[G]$ is in $V[G]$ a stationary subset of $\omega_1 = \kappa$. Then there is an external M-generic filter $H \subset j(\mathbb{M})$ such that*

(1) $j''G \subset H$,
(2) if $j^* : V[G] \to M[H]$ is the unique extension of the embedding j then $\kappa \in j^*(S)$.

Proof: Fix G, S as in the statement of the claim and force over $V[G]$ with the algebra Power(ω_1) modulo NS_{ω_1} below the equivalence class of the stationary set S. Let $j^* : V \to N$ be the generic ultrapower embedding. Obviously $\kappa \in j^*(S)$ and since $j \subset j^*$, by elementarity of j^* the model N is of the form $M[H]$ for some M-generic filter $H \subset j(\mathbb{M})$ such that $j''G \subset H$. This filter obviously works. $\square$

Claim 7.3. *Let λ be an inaccessible cardinal between κ and δ. There is an elementary submodel $Z \prec H_\lambda$ such that*

(1) $A, U, \kappa, \mathbb{M} \in Z$, the ordertype of $Z \cap \kappa$ is ω_1,
(2) writing $^-: Z \to \bar{Z}$ for the transitive collapse, the model $\bar{Z}$ is constructible from a real,
(3) there is an $\bar{Z}$-generic filter $G \subset \bar{\mathbb{M}}$ such that the model $\bar{Z}[G]$ is correct about stationary sets: if $\bar{Z}[G] \models$ "$S \subset \omega_1$ is stationary" then S is a stationary subset of ω_1 in V.

Proof: Choose a countable elementary submodel $Z_0 \prec H_\lambda$ with $A, U, \kappa, \mathbb{M} \in Z_0$, let $X = \bigcap(U \cap Z_0)$ and choose a strictly increasing sequence $\xi_\alpha : \alpha \in \omega_1$ of ordinals in the set $X \in U$.

First, some notation. Let Z_α be the Skolem hull of the set $Z_0 \cup \{\xi_\beta : \beta \in \alpha\}$ in H_λ for $\alpha \in \omega_1 + 1$. For all such αs let $c_\alpha : Z_\alpha \to \bar{Z}_\alpha$ be the transitive collapse maps, let $\kappa_\alpha = c_\alpha(\kappa), \mathbb{M}_\alpha = c_\alpha(\mathbb{M})$ and for $\alpha \in \beta \in \omega_1 + 1$ let $j_{\alpha\beta} : \bar{Z}_\alpha \to \bar{Z}_\beta$ be the elementary embedding lifting the inclusion map $Z_\alpha \subset Z_\beta$. It is well-known and easy to verify that the sequence $\bar{Z}_\alpha : \alpha \in \omega_1 + 1$ together with the commutative system of maps $j_{\alpha\beta}$ is just the iteration of the model $\bar{Z}_0$ using the measure $c_0(U)$ ω_1 many times. The continuous increasing sequence $\kappa_\alpha : \alpha \in \omega_1$ of countable ordinals is exactly the sequence of the critical points of the iteration.

We claim that $Z = Z_{\omega_1}$ is the desired model. By its construction, the property (1) is satisfied. The transitive collapse $\bar{Z} = \bar{Z}_{\omega_1}$ of Z is just an iterand of the countable model $\bar{Z}_0$ and is therefore constructible from any real coding that model; so (2) holds true as well. We must produce an $\bar{Z}$-generic filter as in (3).

Let $x_\beta : \beta \in \omega_1$ be an enumeration of $\bar{Z}_{\omega_1}$ and fix a partition $S_\beta : \beta \in \omega_1$ of ω_1 into stationary sets. By an induction on $\alpha \in \omega_1 + 1$ we shall build a sequence $G_\alpha \subset \mathbb{M}_\alpha$ of $\bar{Z}_\alpha$-generic filters such that

(1) $\gamma \in \alpha$ implies $j''_{\gamma\alpha} G_\gamma \subset G_\alpha$,

(2) if $\alpha = \gamma + 1$, $\gamma \in S_\beta$ for some unique ordinal $\beta \in \omega_1$, $x_\beta = j_{\gamma\omega_1}(y)$ for some unique $y \in \bar{Z}_\gamma$ and $\bar{Z}_\gamma \models$"y is an $\mathbb{M}_\gamma$-name for a stationary subset of ω_1" then G_α contains a condition forcing in $\mathbb{M}_\alpha$ that $\check{\kappa}_\gamma \in j_{\gamma\alpha}(y)$.

This is rather easily done: at $\alpha = 0$, any $\bar{Z}_0$-generic filter $G_0 \subset \mathbb{M}_0$ will do. At limit ordinals α let $G_\alpha = \bigcup_{\gamma \in \alpha} j''_{\gamma\alpha} G_\gamma$. Since $\bar{Z}_\alpha$ is a direct limit of the previous models this will be an $\bar{Z}_\alpha$-generic filter, (1) holds by its definition and (2) is vacuously true. At a successor stage $\alpha = \gamma + 1$ use the previous claim in $\bar{Z}_\gamma$ with $S = y/G_\gamma$. Note that $\bar{Z}_\alpha$ is a class in $\bar{Z}_\gamma$, namely it is the ultrapower of its universe by the measure $c_\gamma(U)$.

We claim that $G = G_{\omega_1}$ is the $\bar{Z}$-generic filter desired. And indeed, suppose $\bar{Z}[G] \models$"$S \subset \omega_1$ is a stationary set". Pick some $\mathbb{M}_{\omega_1}$-name $\tau \in \bar{Z}$ for a stationary subset of ω_1 and countable ordinals α, β such that $\tau/G = S$, $\tau = x_\beta$ and $\tau \in$range(j_{α,ω_1}). The induction hypothesis (2) then ensures that S includes the set $\{\kappa_\gamma : \gamma \in S_\beta, \alpha \in \gamma\}$. Now the latter set is stationary being an image of the stationary set $S_\beta \setminus \alpha$ under the continuous increasing map $\gamma \mapsto \kappa_\gamma$. Thus $S \subset \omega_1$ itself must be stationary and the claim follows. $\square$

The rest of the proof of the Theorem is a rather routine argument. Fix an inaccessible cardinal λ between κ and δ and let a be the set of all elementary submodels of H_λ as in the previous claim. Claim 7.3 of course essentially shows that the set a is stationary. Now $a \Vdash_{\mathbb{P}_{<\delta}} i''\check{H}_\lambda \in i(\check{a})$, where $i : V \to N$ is the $\mathbb{P}_{<\delta}$-generic ultrapower. It follows that whenever $H \subset \mathbb{P}_{<\delta}$ is a generic filter containing the condition a and $i : V \to N$ is the ultrapower, in the model N we have $\omega_1 = \kappa$ and there is an H_κ-generic, that is a V-generic filter $G \subset \mathbb{M}$ such that $V[G]$ is correct about stationary subsets of ω_1 in N.

Let $\mathbb{M} \lessdot \mathbb{P}_{<\delta} \restriction a$ be an embedding given by a name for some such filter G. We claim that (1)–(3) after the statement of the Theorem hold. And indeed,

(1) holds since $L(\mathbb{R} \cap V[G]) \subset L(V_\kappa \cap V, K)$ for some V-generic filter $K \subset \mathrm{Coll}(\omega, <\kappa)$ as follows from the second property of the forcing $\mathbb{M}$. Now the latter model is a generic extension of $L(V_\kappa \cap V)$ and so does not contain $V_\kappa^\#$ or the set A;

(2) holds since $V[G]$ is correct about stationary subsets of ω_1 in the model N — as follows from the requirement (3) in the Claim — and $N^{<\delta} \subset N$ in $V[H]$; consequently $V[G]$ is correct about such sets even in $V[H]$;

(3) holds since the set H_λ^V is constructible from a real in N — the requirement (2) of the claim. So $A \in H_\lambda^V$ is constructible from some real in $V[H]$.

The Theorem follows.

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