Compatible topologies and bornologies on modules

Dinamérico P. Pombo Jr.

Abstract. Compatible topologies and bornologies on modules are introduced and studied.

Keywords: module topologies, continuous linear mappings, module bornologies, bounded linear mappings, equicontinuous sets, equibounded sets, topological modules of linear mappings

Classification: 46H25, 46A17

Compatible topologies and bornologies on vector spaces have been considered by various authors; see [1], [2], [9], [15] and [16], for instance. In this paper we begin the study of compatible topologies and bornologies in the context of modules, taking as starting point the available results concerning module topologies and module bornologies. In the first part of the paper, the methods by which one constructs new compatible topologies and bornologies from given ones are discussed. In the second, the quasi-completeness of certain topological modules of linear mappings is obtained.

Throughout this work, A denotes a topological ring with an identity element and all modules under consideration are unitary left A-modules. Modt_A represents the category whose objects are A-modules endowed with A-module topologies and whose morphisms are continuous A-linear mappings, and Modb_A represents the category whose objects are A-modules endowed with A-module bornologies and whose morphisms are bounded A-linear mappings (see [13] for the precise definitions).

The following definition is already known in certain cases; see [1], [9] and [16].

Definition 1. Let E be an A-module. An A-module topology τ on E and an A-module bornology $\mathcal{B}$ on E are said to be *compatible* (or $(\tau, \mathcal{B})$ is said to be a *compatible pair* on E) if $\mathcal{B}$ is finer than $\mathbf{B}(\tau)$, where $\mathbf{B}(\tau)$ is the bornology consisting of all τ-bounded subsets of E.

We shall denote by Modtb_A the category whose objects are all triples $(E, \tau, \mathcal{B})$ formed by an A-module E and a compatible pair $(\tau, \mathcal{B})$ on E, where

$$\mathrm{Mor}_{\mathrm{Modtb}_A}((E,\tau,\mathcal{B}),(F,\tau',\mathcal{B}')) = \\ = \mathrm{Mor}_{\mathrm{Modt}_A}((E,\tau),(F,\tau')) \cap \mathrm{Mor}_{\mathrm{Modb}_A}((E,\mathcal{B}),(F,\mathcal{B}'))$$

if $(E,\tau,\mathcal{B}),(F,\tau',\mathcal{B}') \in \mathrm{Ob}(\mathrm{Modtb}_A)$. Then we have two covariant functors

$$\mathcal{O}\colon \mathrm{Modtb}_A \to \mathrm{Modt}_A \qquad \text{and} \qquad \mathcal{P}\colon \mathrm{Modtb}_A \to \mathrm{Modb}_A\,.$$

Example 1. If A is a discrete ring and E is an A-module, then $(E, \tau, \mathcal{B}) \in \mathrm{Ob}(\mathrm{Modtb}_A)$ for every A-module topology τ on E and for every A-module bornology $\mathcal{B}$ on E. It is enough to remember that $\mathbf{B}(\tau)$ is the set of all subsets of E is this case.

Example 2. If $(E, \tau) \in \mathrm{Ob}(\mathrm{Modt}_A)$, then $(E, \tau, \mathbf{B}(\tau)) \in \mathrm{Ob}(\mathrm{Modtb}_A)$ by Example 1 of [13]. Moreover, if $(F, \tau') \in \mathrm{Ob}(\mathrm{Modt}_A)$ and $u \in \mathrm{Mor}_{\mathrm{Modt}_A}((E, \tau), (F, \tau'))$, then $u \in \mathrm{Mor}_{\mathrm{Modtb}_A}((E, \tau, \mathbf{B}(\tau)), (F, \tau', \mathbf{B}(\tau')))$, so that we have a covariant functor

$$\widetilde{\mathcal{O}}\colon \mathrm{Modt}_A \longrightarrow \mathrm{Modtb}_A \ .$$

Proposition 1. *If $(E, \tau, \mathcal{B}) \in \mathrm{Ob}(\mathrm{Modtb}_A)$ and $(F, \tau') \in \mathrm{Ob}(\mathrm{Modt}_A)$, then*

$$\mathrm{Mor}_{\mathrm{Modtb}_A}((E, \tau, \mathcal{B}), (F, \tau', \mathbf{B}(\tau'))) = \mathrm{Mor}_{\mathrm{Modt}_A}((E, \tau), (F, \tau')).$$

Hence the functor $\mathcal{O}$ is left adjoint to the functor $\widetilde{\mathcal{O}}$.

Proof: Since

$$\begin{aligned}\mathrm{Mor}_{\mathrm{Modtb}_A}&((E, \tau, \mathcal{B}), (F, \tau', \mathbf{B}(\tau'))) = \\ &= \mathrm{Mor}_{\mathrm{Modt}_A}((E, \tau), (F, \tau')) \cap \mathrm{Mor}_{\mathrm{Modb}_A}((E, \mathcal{B}), (F, \mathbf{B}(\tau'))),\end{aligned}$$

we have to show that

$$\mathrm{Mor}_{\mathrm{Modt}_A}((E, \tau), (F, \tau')) \subset \mathrm{Mor}_{\mathrm{Modb}_A}((E, \mathcal{B}), (F, \mathbf{B}(\tau'))).$$

But, if $u \in \mathrm{Mor}_{\mathrm{Modt}_A}((E, \tau), (F, \tau'))$, then $u \in \mathrm{Mor}_{\mathrm{Modb}_A}((E, \mathbf{B}(\tau)), (F, \mathbf{B}(\tau')))$. Therefore $u \in \mathrm{Mor}_{\mathrm{Modb}_A}((E, \mathcal{B}), (F, \mathbf{B}(\tau')))$ because $\mathcal{B}$ is finer than $\mathbf{B}(\tau)$. □

Example 3. If $(E, \mathcal{B}) \in \mathrm{Ob}(\mathrm{Modb}_A)$, then $(E, \mathbf{T}(\mathcal{B}), \mathcal{B}) \in \mathrm{Ob}(\mathrm{Modtb}_A)$ by Proposition 1 (a) of [13].

Example 4. Let $(E, \tau) \in \mathrm{Ob}(\mathrm{Modt}_A)$. If $\mathcal{B}$ is an A-module bornology on E such that $\tau = \mathbf{T}(\mathcal{B})$, then $(E, \tau, \mathcal{B}) \in \mathrm{Ob}(\mathrm{Modtb}_A)$ as we have observed in Example 3. Moreover, $\tau = \mathbf{T}(\mathbf{B}(\tau))$. In fact, τ is coarser than $\mathbf{T}(\mathbf{B}(\tau))$ by Remark 4 of [13]. Conversely, the fact that $\mathcal{B}$ is finer than $\mathbf{B}(\tau)$ implies that $\mathcal{B}$ is finer than $\mathbf{B}(\mathbf{T}(\mathbf{B}(\tau)))$, which implies that $\mathbf{T}(\mathbf{B}(\tau))$ is coarser than $\mathbf{T}(\mathcal{B}) = \tau$ by Proposition 1 (b) of [13].

We have seen in Example 5 of [13] that τ may be different from $\mathbf{T}(\mathbf{B}(\tau))$. In the next proposition we give a characterization of the $(E, \tau) \in \mathrm{Ob}(\mathrm{Modt}_A)$ for which $\tau = \mathbf{T}(\mathbf{B}(\tau))$.

Proposition 2. *Let $(E, \tau) \in \mathrm{Ob}(\mathrm{Modt}_A)$. In order that $\tau = \mathbf{T}(\mathbf{B}(\tau))$ it is necessary and sufficient that*

$$\mathrm{Mor}_{\mathrm{Modb}_A}((E, \mathbf{B}(\tau)), (G, \mathcal{D})) \subset \mathrm{Mor}_{\mathrm{Modt}_A}((E, \tau), (G, \mathbf{T}(\mathcal{D})))$$

for every $(G, \mathcal{D}) \in \mathrm{Ob}(\mathrm{Modb}_A)$.

Proof: To prove that the condition is necessary, let $(G, \mathcal{D}) \in \mathrm{Ob}(\mathrm{Modb}_A)$ and $u \in \mathrm{Mor}_{\mathrm{Modb}_A}((E, \mathbf{B}(\tau)), (G, \mathcal{D}))$. Then, by Proposition 2 of [13],

$$u \in \mathrm{Mor}_{\mathrm{Modt}_A}((E, \mathbf{T}(\mathbf{B}(\tau))), (G, \mathbf{T}(\mathcal{D}))) = \mathrm{Mor}_{\mathrm{Modt}_A}((E, \tau), (G, \mathbf{T}(\mathcal{D}))).$$

To prove that the condition is sufficient, take $(G, \mathcal{D}) = (E, \mathbf{B}(\tau))$. Then, by hypothesis, the identity mapping $1_E \in \mathrm{Mor}_{\mathrm{Modt}_A}((E, \tau), (G, \mathbf{T}(\mathbf{B}(\tau))))$, that is, $\mathbf{T}(\mathbf{B}(\tau))$ is coarser than τ. Therefore $\tau = \mathbf{T}(\mathbf{B}(\tau))$, as was to be shown. $\square$

Example 5. Let $(E, \mathcal{B}) \in \mathrm{Ob}(\mathrm{Modb}_A)$. If τ is an A-module topology on E such that $\mathcal{B} = \mathbf{B}(\tau)$, then $(E, \tau, \mathcal{B}) \in \mathrm{Ob}(\mathrm{Modtb}_A)$. Moreover, $\mathcal{B} = \mathbf{B}(\mathbf{T}(\mathcal{B}))$. In fact, since τ is coarser than $\mathbf{T}(\mathbf{B}(\tau))$, it follows that $\mathbf{B}(\mathbf{T}(\mathbf{B}(\tau))) = \mathbf{B}(\mathbf{T}(\mathcal{B}))$ is finer than $\mathbf{B}(\tau) = \mathcal{B}$. Hence $\mathcal{B} = \mathbf{B}(\mathbf{T}(\mathcal{B}))$.

We have seen in Example 4 of [13] that $\mathcal{B}$ may be different from $\mathbf{B}(\mathbf{T}(\mathcal{B}))$. In the next proposition we give a characterization of the $(E, \mathcal{B}) \in \mathrm{Ob}(\mathrm{Modb}_A)$ for which $\mathcal{B} = \mathbf{B}(\mathbf{T}(\mathcal{B}))$.

Proposition 3. *Let* $(E, \mathcal{B}) \in \mathrm{Ob}(\mathrm{Modb}_A)$. *In order that* $\mathcal{B} = \mathbf{B}(\mathbf{T}(\mathcal{B}))$ *it is necessary and sufficient that*

$$\mathrm{Mor}_{\mathrm{Modt}_A}((G, \theta), (E, \mathbf{T}(\mathcal{B}))) \subset \mathrm{Mor}_{\mathrm{Modb}_A}((G, \mathbf{B}(\theta)), (E, \mathcal{B}))$$

for every $(G, \theta) \in \mathrm{Ob}(\mathrm{Modt}_A)$.

Proof: To prove that the condition is necessary, let $(G, \theta) \in \mathrm{Ob}(\mathrm{Modt}_A)$ and $u \in \mathrm{Mor}_{\mathrm{Modt}_A}((G, \theta), (E, \mathbf{T}(\mathcal{B})))$. By Proposition 1,

$$\mathrm{Mor}_{\mathrm{Modt}_A}((G, \theta), (E, \mathbf{T}(\mathcal{B}))) = \mathrm{Mor}_{\mathrm{Modtb}_A}((G, \theta, \mathbf{B}(\theta)), (E, \mathbf{T}(\mathcal{B}), \mathbf{B}(\mathbf{T}(\mathcal{B})))),$$

so that $u \in \mathrm{Mor}_{\mathrm{Modb}_A}((G, \mathbf{B}(\theta)), (E, \mathbf{B}(\mathbf{T}(\mathcal{B})))) = \mathrm{Mor}_{\mathrm{Modb}_A}((G, \mathbf{B}(\theta)), (E, \mathcal{B}))$.

To prove that the condition is sufficient, take $(G, \theta) = (E, \mathbf{T}(\mathcal{B}))$. Then, by hypothesis, the identity mapping $1_E \in \mathrm{Mor}_{\mathrm{Modb}_A}((E, \mathbf{B}(\mathbf{T}(\mathcal{B}))), (E, \mathcal{B}))$, that is, $\mathbf{B}(\mathbf{T}(\mathcal{B}))$ is finer than $\mathcal{B}$. Therefore $\mathcal{B} = \mathbf{B}(\mathbf{T}(\mathcal{B}))$, as was to be shown. $\square$

Theorem 1. *Let* $((E_i, \tau_i, \mathcal{B}_i))_{i \in I}$ *be a family of objects in* Modtb_A. *Let E be an A-module and, for each $i \in I$, let $u_i \colon E \to E_i$ be an A-linear mapping. Let τ be the initial A-module topology on E for the family* $((E_i, \tau_i), u_i)_{i \in I}$ ([17, Theorem 12.5]), *and let $\mathcal{B}$ be the initial A-module bornology on E for the family* $((E_i, \mathcal{B}_i), u_i)_{i \in I}$ ([13, Theorem 1]). *Then $(\tau, \mathcal{B})$ is the unique compatible pair on E which is initial for the family* $((E_i, \tau_i, \mathcal{B}_i), u_i)_{i \in I}$.

Proof: First, we claim that τ and $\mathcal{B}$ are compatible. Indeed, let $\mathcal{B}'$ be the initial A-module bornology on E for the family $((E_i, \mathbf{B}(\tau_i)), u_i)_{i \in I}$. For each $i \in I$, the diagram of A-linear mappings

$$\begin{array}{ccc}(E_i,\mathcal{B}_i) & \xrightarrow{1_{E_i}} & (E_i,\mathbf{B}(\tau_i)) \\ u_i\uparrow & & \uparrow u_i \\ (E,\mathcal{B}) & \xrightarrow{1_E} & (E,\mathcal{B}')\end{array}$$

is commutative, where 1_{E_i} (respectively 1_E) is the identity mapping of E_i (respectively E). If we consider E_i and E endowed with the A-module bornologies indicated in the diagram, then $1_{E_i} \circ u_i$ is bounded, that is, $u_i \circ 1_E$ is bounded. By the arbitrariness of i, 1_E is bounded, that is, $\mathcal{B}$ is finer than $\mathcal{B}'$. On the other hand, Theorem 3 of [13] gives $\mathcal{B}' = \mathbf{B}(\tau)$, and so $\mathcal{B}$ is finer than $\mathbf{B}(\tau)$. Therefore τ and $\mathcal{B}$ are compatible, as asserted.

Now, let $(G,\theta,\mathcal{D}) \in \mathrm{Ob}(\mathrm{Modtb}_A)$ and let $u\colon G \to E$ be an A-linear mapping. If $u \in \mathrm{Mor}_{\mathrm{Modtb}_A}((G,\theta,\mathcal{D}),(E,\tau,\mathcal{B}))$, then $u_i \circ u \in \mathrm{Mor}_{\mathrm{Modtb}_A}((G,\theta,\mathcal{D}),(E_i,\tau_i,\mathcal{B}_i))$ for all $i \in I$, since $u_i \in \mathrm{Mor}_{\mathrm{Modtb}_A}((E,\tau,\mathcal{B}),(E_i,\tau_i,\mathcal{B}_i))$ for all $i \in I$. Conversely, if $u_i \circ u \in \mathrm{Mor}_{\mathrm{Modtb}_A}((G,\theta,\mathcal{D}),(E_i,\tau_i,\mathcal{B}_i))$ for all $i \in I$, then $u_i \circ u \in \mathrm{Mor}_{\mathrm{Modt}_A}((G,\theta),(E_i,\tau_i))$ and $u_i \circ u \in \mathrm{Mor}_{\mathrm{Modb}_A}((G,\mathcal{D}),(E_i,\mathcal{B}_i))$ for all $i \in I$. Consequently,

$$\begin{aligned}u \in \mathrm{Mor}_{\mathrm{Modt}_A}((G,\theta),(E,\tau)) \cap \mathrm{Mor}_{\mathrm{Modb}_A}((G,\mathcal{D}),(E,\mathcal{B})) &= \\ = \mathrm{Mor}_{\mathrm{Modtb}_A}((G,\theta,\mathcal{D}),(E,\tau,\mathcal{B})).&\end{aligned}$$

Hence the compatible pair $(\tau,\mathcal{B})$ is initial for the family $((E_i,\tau_i,\mathcal{B}_i),u_i)_{i\in I}$.

Finally, it is clear that $(\tau,\mathcal{B})$ is the unique compatible pair on E which is initial for the family $((E_i,\tau_i,\mathcal{B}_i),u_i)_{i\in I}$, thereby concluding the proof of the theorem. $\square$

Corollary 1. (a) *Let $(E,\tau,\mathcal{B}) \in \mathrm{Ob}(\mathrm{Modtb}_A)$, M a submodule of E, τ_M the topology induced by τ on M, and $\mathcal{B}_M$ the bornology induced by $\mathcal{B}$ on M. Then $(M,\tau_M,\mathcal{B}_M) \in \mathrm{Ob}(\mathrm{Modtb}_A)$.*

(b) *Let E be an A-module, $(\tau_i)_{i\in I}$ a family of A-module topologies on E, and $(\mathcal{B}_i)_{i\in I}$ a family of A-module bornologies on E. Let $\tau = \sup\limits_{i\in I} \tau_i$ and $\mathcal{B} = \bigcap\limits_{i\in I} \mathcal{B}_i$. Then $(E,\tau,\mathcal{B}) \in \mathrm{Ob}(\mathrm{Modtb}_A)$.*

Proof: (a) Immediate from Theorem 1, since τ_M (respectively $\mathcal{B}_M$) is the initial A-module topology (respectively initial A-module bornology) for the pair $((E,\tau),i_M)$ (respectively $((E,\mathcal{B}),i_M)$), where $i_M\colon M \to E$ is the canonical injection.

(b) Immediate from Theorem 1, since τ (respectively $\mathcal{B}$) is the initial A-module topology (respectively initial A-module bornology) for the family $((E_i,\tau_i),u_i)_{i\in I}$

(respectively $((E_i, \mathcal{B}_i), u_i)_{i\in I}$), where $E_i = E$ and $u_i\colon E \to E_i$ is the identity mapping for all $i \in I$. $\square$

Corollary 2. *The category* Modtb_A *admits projective limits. The functors* $\mathcal{O}$ *and* $\mathcal{P}$ *commute with projective limits.*

Proof: Let (E_i, u_{ij}) $(i \in I)$ be a projective system in Modtb_A. Let $E = \varprojlim_{i\in I} E_i$ in the category of A-modules ([4, §6, no. 3]) and, for each $i \in I$, let $u_i\colon E \to E_i$ be the canonical A-linear mapping. Consider on E the initial compatible pair for the family $(E_i, u_i)_{i\in I}$ (Theorem 1). Let $F \in \mathrm{Ob}(\mathrm{Modtb}_A)$ and, for each $i \in I$, let $v_i \in \mathrm{Mor}_{\mathrm{Modtb}_A}(F, E_i)$ such that $v_i = u_{ij} \circ v_j$ if $i \le j$. By the algebraic case, there is a unique A-linear mapping $u\colon F \to E$ such that $v_i = u_i \circ u$ for all $i \in I$. Therefore $u \in \mathrm{Mor}_{\mathrm{Modtb}_A}(F, E)$, and hence $E = \varprojlim_{i\in I} E_i$ in Modtb_A. Finally, the second part of the corollary is clear. $\square$

Remark 1. By Proposition 1 and (1.5.7) of [8], the functor $\widetilde{\mathcal{O}}$ commutes with projective limits.

Corollary 3. *The category* Modtb_A *admits products.*

Proof: Immediate from the first part of Corollary 2. $\square$

Remark 2. Corollary 3 also follows directly from Theorem 1. Indeed, let $(E_i)_{i\in I}$ be a family of objects in Modtb_A, E the A-module $\prod_{i\in I} E_i$, and $\mathrm{pr}_i\colon E \to E_i$ the projection on the i-th factor $(i \in I)$. Consider on E the initial compatible pair $(\tau, \mathcal{B})$ for the family $(E_i, \mathrm{pr}_i)_{i\in I}$ (Theorem 1); recall that τ (respectively $\mathcal{B}$) is the product topology (respectively product bornology). Then, for any $F \in \mathrm{Ob}(\mathrm{Modtb}_A)$, the mapping

$$u \in \mathrm{Mor}_{\mathrm{Modtb}_A}(F, E) \mapsto (\mathrm{pr}_i \circ u)_{i\in I} \in \prod_{i\in I} \mathrm{Mor}_{\mathrm{Modtb}_A}(F, E_i)$$

is bijective.

Theorem 2. *Let* $((E_i, \tau_i, \mathcal{B}_i))_{i\in I}$ *be a family of objects in* Modtb_A. *Let* E *be an* A*-module and, for each* $i \in I$, *let* $u_i\colon E_i \to E$ *be an* A*-linear mapping. Let* τ *be the final* A*-module topology on* E *for the family* $((E_i, \tau_i), u_i)_{i\in I}$ ([12, Proposition 2.2]), *and let* $\mathcal{B}$ *be the final* A*-module bornology on* E *for the family* $((E_i, \mathcal{B}_i), u_i)_{i\in I}$ ([13, Theorem 2]). *Then* $(\tau, \mathcal{B})$ *is the unique compatible pair on* E *which is final for the family* $((E_i, \tau_i, \mathcal{B}_i), u_i)_{i\in I}$.

Proof: First, we claim that τ and $\mathcal{B}$ are compatible. Indeed, let τ' be the final A-module topology on E for the family $((E_i, \mathbf{T}(\mathcal{B}_i)), u_i)_{i\in I}$. For each $i \in I$, the diagram of A-linear mappings

$$\begin{array}{ccc}
(E_i, \mathbf{T}(\mathcal{B}_i)) & \xrightarrow{1_{E_i}} & (E_i, \tau_i) \\
u_i \big\downarrow & & \big\downarrow u_i \\
(E, \tau') & \xrightarrow[1_E]{} & (E, \tau)
\end{array}$$

is commutative, where 1_{E_i} (respectively 1_E) is the identity mapping of E_i (respectively E). If we consider E_i and E endowed with the A-module topologies indicated in the diagram, then $u_i \circ 1_{E_i}$ is continuous (1_{E_i} is continuous since the fact that $\mathcal{B}_i$ is finer than $\mathbf{B}(\tau_i)$ implies that $\mathbf{T}(\mathbf{B}(\tau_i))$ is coarser than $\mathbf{T}(\mathcal{B}_i)$, and since τ_i is coarser than $\mathbf{T}(\mathbf{B}(\tau_i))$), that is, $1_E \circ u_i$ is continuous. By the arbitrariness of i, 1_E is continuous, that is, τ is coarser than τ'; hence $\mathbf{B}(\tau')$ is finer than $\mathbf{B}(\tau)$. On the other hand, Theorem 4 of [13] gives $\mathbf{T}(\mathcal{B}) = \tau'$. Consequently, $\mathcal{B}$ is finer than $\mathbf{B}(\tau)$ because $\mathcal{B}$ is finer than $\mathbf{B}(\mathbf{T}(\mathcal{B}))$. Therefore τ and $\mathcal{B}$ are compatible, as asserted.

Now, let $(G, \theta, \mathcal{D}) \in \mathrm{Ob}(\mathrm{Modtb}_A)$ and let $u\colon E \to G$ be an A-linear mapping. If $u \in \mathrm{Mor}_{\mathrm{Modtb}_A}((E, \tau, \mathcal{B}), (G, \theta, \mathcal{D}))$, then $u \circ u_i \in \mathrm{Mor}_{\mathrm{Modtb}_A}((E_i, \tau_i, \mathcal{B}_i), (G, \theta, \mathcal{D}))$ for all $i \in I$, since $u_i \in \mathrm{Mor}_{\mathrm{Modtb}_A}((E_i, \tau_i, \mathcal{B}_i), (E, \tau, \mathcal{B}))$ for all $i \in I$. Conversely, if $u \circ u_i \in \mathrm{Mor}_{\mathrm{Modtb}_A}((E_i, \tau_i, \mathcal{B}_i), (G, \theta, \mathcal{D}))$ for all $i \in I$, then $u \circ u_i \in \mathrm{Mor}_{\mathrm{Modt}_A}((E_i, \tau_i), (G, \theta))$ and $u \circ u_i \in \mathrm{Mor}_{\mathrm{Modb}_A}((E_i, \mathcal{B}_i), (G, \mathcal{D}))$ for all $i \in I$. Consequently,

$$u \in \mathrm{Mor}_{\mathrm{Modt}_A}((E, \tau), (G, \theta)) \cap \mathrm{Mor}_{\mathrm{Modb}_A}((E, \mathcal{B}), (G, \mathcal{D})) = \\ = \mathrm{Mor}_{\mathrm{Modtb}_A}((E, \tau, \mathcal{B}), (G, \theta, \mathcal{D})).$$

Hence the compatible pair $(\tau, \mathcal{B})$ is final for the family $((E_i, \tau_i, \mathcal{B}_i), u_i)_{i \in I}$.

Finally, it is clear that $(\tau, \mathcal{B})$ is the unique compatible pair on E which is final for the family $((E_i, \tau_i, \mathcal{B}_i), u_i)_{i \in I}$, thereby concluding the proof of the theorem. □

Corollary 4. *Let $(E, \tau, \mathcal{B}) \in \mathrm{Ob}(\mathrm{Modtb}_A)$, F an A-module, and $u\colon E \to F$ a surjective A-linear mapping. Let τ' be the direct image under u of τ, and let $\mathcal{B}'$ be the direct image under u of $\mathcal{B}$. Then $(F, \tau', \mathcal{B}') \in \mathrm{Ob}(\mathrm{Modtb}_A)$.*

Proof: Immediate from Theorem 2, since τ' (respectively $\mathcal{B}'$) is the final A-module topology (respectively final A-module bornology) for the pair $((E, \tau), u)$ (respectively $((E, \mathcal{B}), u)$). □

Corollary 5. *The category* Modtb_A *admits inductive limits. The functors $\mathcal{O}$ and $\mathcal{P}$ commute with inductive limits.*

Proof: Analogous to that of Corollary 2. □

Remark 3. The fact that the functor $\mathcal{O}$ commutes with inductive limits also follows from Proposition 1 and (1.5.7) of [8].

Corollary 6. *The category* Modtb_A *admits direct sums.*

PROOF: Immediate from the first part of Corollary 5. □

Remark 4. Corollary 6 also follows directly from Theorem 2. Indeed, let $(E_i)_{i\in I}$ be a family of objects in Modtb_A, E the A-module $\bigoplus_{i\in I} E_i$, and $\lambda_i\colon E_i \to E$ the canonical injection ($i \in I$). Consider on E the final compatible pair for the family $(E_i, \lambda_i)_{i\in I}$ (Theorem 2). Then, for any $F \in \mathrm{Ob}(\mathrm{Modtb}_A)$, the mapping

$$u \in \mathrm{Mor}_{\mathrm{Modtb}_A}(E,F) \longmapsto (u \circ \lambda_i)_{i\in I} \in \prod_{i\in I} \mathrm{Mor}_{\mathrm{Modtb}_A}(E_i,F)$$

is bijective.

The following definition was suggested by Definition 3, p. 64 of [15].

Definition 2. Let $(E,\tau,\mathcal{B}) \in \mathrm{Ob}(\mathrm{Modtb}_A)$. $(E,\tau,\mathcal{B})$ is said to be *quasi-complete* if $\overline{B^\tau}$ is τ-complete for every $B \in \mathcal{B}$.

Example 6. If $(E,\tau,\mathcal{B}) \in \mathrm{Ob}(\mathrm{Modtb}_A)$ and (E,τ) is complete, then $(E,\tau,\mathcal{B})$ is quasi-complete by the first part of Proposition 8, p. 202 of [5].

Remark 5. Let $(E,\tau,\mathcal{B}) \in \mathrm{Ob}(\mathrm{Modtb}_A)$ such that $\overline{B^\tau} \in \mathcal{B}$ for every $B \in \mathcal{B}$ (for instance, for any $(E,\tau) \in \mathrm{Ob}(\mathrm{Modt}_A)$, $(E,\tau,\mathbf{B}(\tau))$ satisfies this property by Theorem 15.2 (1) of [17]). Then $(E,\tau,\mathcal{B})$ is quasi-complete if and only if every τ-closed element of $\mathcal{B}$ is τ-complete. In particular, a topological vector space (E,τ) over a non-trivially valued field is quasi-complete ([14, p. 27]) if and only if $(E,\tau,\mathbf{B}(\tau))$ is quasi-complete.

Remark 6. Let $(E,\tau) \in \mathrm{Ob}(\mathrm{Modt}_A)$, and suppose that $(E,\tau,\mathbf{B}(\tau))$ is quasi-complete. If $(x_n)_{n\in\mathbf{N}}$ is a Cauchy sequence in (E,τ), then $B = \{x_n; n \in \mathbf{N}\} \in \mathbf{B}(\tau)$ by Theorem 15.4 (2) of [17]. Therefore $\overline{B^\tau}$ is τ-complete, and so $(x_n)_{n\in\mathbf{N}}$ converges in (E,τ). In particular, (E,τ) is complete if (E,τ) is metrizable.

As in the theory of topological vector spaces, we have:

Proposition 4. (a) *Let* $(E,\tau,\mathcal{B}) \in \mathrm{Ob}(\mathrm{Modtb}_A)$, *and suppose that* $(E,\tau,\mathcal{B})$ *is quasi-complete. If* M *is a* τ*-closed submodule of* E, τ_M *is the topology induced by* τ *on* M *and* $\mathcal{B}_M$ *is the bornology induced by* $\mathcal{B}$ *on* M, *then* $(M,\tau_M,\mathcal{B}_M)$ *is quasi-complete.*

(b) *Let* $((E_i,\tau_i,\mathcal{B}_i))_{i\in I}$ *be a family of objects in* Modtb_A *such that* $(E_i,\tau_i,\mathcal{B}_i)$ *is quasi-complete and* (E_i,τ_i) *is separated for all* $i \in I$. *Let* E *be the* A*-module* $\prod_{i\in I} E_i$, τ *the product topology on* E *and* $\mathcal{B}$ *the product bornology on* E. *Then* $(E,\tau,\mathcal{B})$ *is quasi-complete.*

Proof: (a) First, recall that $(M,\tau_M,\mathcal{B}_M)\in \mathrm{Ob}(\mathrm{Modtb}_A)$ by Corollary 1 (a). Let $B\in\mathcal{B}_M$. Then $B\in\mathcal{B}$, and so $\overline{B}^{\tau}$ is τ-complete. Therefore $\overline{B}^{\tau_M}$ is τ_M-complete since $\overline{B}^{\tau_M}=\overline{B}^{\tau}\cap M$.

(b) First, recall that $(E,\tau,\mathcal{B})\in\mathrm{Ob}(\mathrm{Modtb}_A)$ by Remark 2. For each $i\in I$, let $\mathrm{pr}_i\colon E\to E_i$ be the projection on the i-th factor. Let B be an arbitrary element of $\mathcal{B}$. Then $B_i=\mathrm{pr}_i(B)\in\mathcal{B}_i$ for all $i\in I$, and hence $\overline{B_i}^{\tau_i}$ is τ_i-complete for all $i\in I$. Since $B\subset\prod\limits_{i\in I}\overline{B_i}^{\tau_i}$ and since $\prod\limits_{i\in I}\overline{B_i}^{\tau_i}$ is τ-complete ([5, p. 203, Proposition 10]), it follows from the first and second parts of Proposition 8, p. 202 of [5] that $\overline{B}^{\tau}$ is τ-complete. Therefore $(E,\tau,\mathcal{B})$ is quasi-complete, as asserted. □

For the rest of this paper we shall assume that A is commutative. In this case, if $(E,\tau,\mathcal{B}),(F,\tau',\mathcal{B}')\in\mathrm{Ob}(\mathrm{Modtb}_A)$, then

$$\mathrm{Mor}_{\mathrm{Modtb}_A}((E,\tau,\mathcal{B}),(F,\tau',\mathcal{B}'))$$

is a submodule of the A-module $\mathrm{Mor}_{\mathrm{Modt}_A}((E,\tau),(F,\tau'))$, and is also a submodule of the A-module $\mathrm{Mor}_{\mathrm{Modb}_A}((E,\mathcal{B}),(F,\mathcal{B}'))$. If $u\in\mathrm{Mor}_{\mathrm{Modtb}_A}((E,\tau,\mathcal{B}),(F,\tau',\mathcal{B}'))$ and $B\in\mathcal{B}$, then $u(B)\in\mathcal{B}'$, and so $u(B)\in\mathbf{B}(\tau')$ because $\mathcal{B}'$ is finer than $\mathbf{B}(\tau')$. By Proposition (a) of [11], the topology $\Upsilon_{\mathcal{B}}^{\tau'}$ of $\mathcal{B}$-convergence on $\mathrm{Mor}_{\mathrm{Modtb}_A}((E,\tau,\mathcal{B}),(F,\tau',\mathcal{B}'))$ is an A-module topology, which is separated if τ' is separated.

Definition 3 [15], [16]**.** Let $(E,\mathcal{B}),(F,\mathcal{B}')\in\mathrm{Ob}(\mathrm{Modb}_A)$. A set H of mappings from E into F is said to be *equibounded* if $H(B)=\{u(x);u\in H,x\in B\}\in\mathcal{B}'$ for all $B\in\mathcal{B}$. The bornology $\mathcal{E}_{\mathcal{B},\mathcal{B}'}$ on $\mathrm{Mor}_{\mathrm{Modb}_A}((E,\mathcal{B}),(F,\mathcal{B}'))$ whose elements are the equibounded subsets of $\mathrm{Mor}_{\mathrm{Modb}_A}((E,\mathcal{B}),(F,\mathcal{B}'))$ is clearly an A-module bornology.

Let $(E,\tau,\mathcal{B}),(F,\tau',\mathcal{B}')\in\mathrm{Ob}(\mathrm{Modtb}_A)$. We shall also denote by $\mathcal{E}_{\mathcal{B},\mathcal{B}'}$ the A-module bornology induced by $\mathcal{E}_{\mathcal{B},\mathcal{B}'}$ on $\mathrm{Mor}_{\mathrm{Modtb}_A}((E,\tau,\mathcal{B}),(F,\tau',\mathcal{B}'))$. Assume that the product of any neighborhood of zero in A by any neighborhood of zero in (E,τ) is a neighborhood of zero in (E,τ). In this case, Example 3 of [13] ensures that the equicontinuous bornology $\mathcal{E}_{\tau,\tau'}$ on $\mathrm{Mor}_{\mathrm{Modt}_A}((E,\tau),(F,\tau'))$ is an A-module bornology. We shall also denote by $\mathcal{E}_{\tau,\tau'}$ the A-module bornology induced by $\mathcal{E}_{\tau,\tau'}$ on $\mathrm{Mor}_{\mathrm{Modtb}_A}((E,\tau,\mathcal{B}),(F,\tau',\mathcal{B}'))$. The A-module bornology $\mathcal{E}_{\tau,\tau'}\cap\mathcal{E}_{\mathcal{B},\mathcal{B}'}$ on $\mathrm{Mor}_{\mathrm{Modtb}_A}((E,\tau,\mathcal{B}),(F,\tau',\mathcal{B}'))$ shall be represented by $\mathcal{E}_{\tau,\tau'}^{\mathcal{B},\mathcal{B}'}$.

Proposition 5. *Let $(E,\tau,\mathcal{B}),(F,\tau',\mathcal{B}')\in\mathrm{Ob}(\mathrm{Modtb}_A)$, and suppose that the product of any neighborhood of zero in A by any neighborhood of zero in (E,τ) is a neighborhood of zero in (E,τ). Then*

$$\left(\mathrm{Mor}_{\mathrm{Modtb}_A}((E,\tau,\mathcal{B}),(F,\tau',\mathcal{B}')),\Upsilon_{\mathcal{B}}^{\tau'},\mathcal{E}_{\tau,\tau'}^{\mathcal{B},\mathcal{B}'}\right)\in\mathrm{Ob}(\mathrm{Modtb}_A).$$

Proof: Let $H \in \mathcal{E}^{\mathcal{B},\mathcal{B}'}_{\tau,\tau'}$. We have to show that $H \in \mathbf{B}(\Upsilon^{\tau'}_{\mathcal{B}})$. For this purpose, let $B \in \mathcal{B}$. Then $H(B) \in \mathcal{B}'$, and so $H(B) \in \mathbf{B}(\tau')$ because $\mathcal{B}'$ is finer than $\mathbf{B}(\tau')$. Therefore $H \in \mathbf{B}(\Upsilon^{\tau'}_{\mathcal{B}})$, which concludes the proof. $\square$

Theorem 3. *Let $(E,\tau,\mathcal{B}),(F,\tau',\mathcal{B}') \in \mathrm{Ob}(\mathrm{Modtb}_A)$. Suppose that the product of any neighborhood of zero in A by any neighborhood of zero in (E,τ) is a neighborhood of zero in (E,τ), and that $\overline{B'^{\tau'}} \in \mathcal{B}'$ for all $B' \in \mathcal{B}'$. If $(F,\tau',\mathcal{B}')$ is quasi-complete and (F,τ') is separated, then*

$$\left(\mathrm{Mor}_{\mathrm{Modtb}_A}((E,\tau,\mathcal{B}),(F,\tau',\mathcal{B}')), \Upsilon^{\tau'}_{\mathcal{B}}, \mathcal{E}^{\mathcal{B},\mathcal{B}'}_{\tau,\tau'}\right)$$

is quasi-complete.

In order to prove the theorem, we shall need the following

Lemma. *Let E be an A-module, $(F,\tau') \in \mathrm{Ob}(\mathrm{Modt}_A)$ with (F,τ') separated, and consider the A-module $\mathcal{F}(E;F)$ of all mappings from E into F endowed with the A-module topology $\Upsilon^{\tau'}_s$ of pointwise convergence. Then the submodule $Ł(E;F)$ of $\mathcal{F}(E;F)$ consisting of all A-linear mappings from E into F is $\Upsilon^{\tau'}_s$-closed in $\mathcal{F}(E;F)$.*

Proof: For each $z \in E$, the A-linear mapping

$$\delta_z : f \in \mathcal{F}(E;F) \longmapsto f(z) \in F$$

is $\Upsilon^{\tau'}_s$-continuous. Since

$$\begin{aligned} Ł(E;F) &= \bigcap_{x,y\in E, a,b\in A} \{f \in \mathcal{F}(E;F); f(ax+by) - af(x) - bf(y) = 0\} \\ &= \bigcap_{x,y\in E, a,b\in A} \mathrm{Ker}(\delta_{ax+by} - a\,\delta_x - b\,\delta_y), \end{aligned}$$

and since $\mathrm{Ker}(\delta_{ax+by} - a\,\delta_x - b\,\delta_y)$ is $\Upsilon^{\tau'}_s$-closed in $\mathcal{F}(E;F)$ for all $x,y \in E$, $a,b \in A$ ((F,τ') is separated), the result follows. $\square$

Proof of Theorem 3: Let $H \in \mathcal{E}^{\mathcal{B},\mathcal{B}'}_{\tau,\tau'}$. We have to show that H_1, the closure of H in

$$\left(\mathrm{Mor}_{\mathrm{Modtb}_A}((E,\tau,\mathcal{B}),(F,\tau',\mathcal{B}')), \Upsilon^{\tau'}_{\mathcal{B}}\right),$$

is $\Upsilon^{\tau'}_{\mathcal{B}}$-complete. Let us represent by $\overline{H^{\Upsilon^{\tau'}_s}}$ (respectively $\overline{H^{\Upsilon^{\tau'}_{\mathcal{B}}}}$) the closure of H in $(\mathcal{F}(E;F),\Upsilon^{\tau'}_s)$ (respectively $(\mathcal{F}(E;F),\Upsilon^{\tau'}_{\mathcal{B}})$), where $\Upsilon^{\tau'}_{\mathcal{B}}$ also denotes the topology of $\mathcal{B}$-convergence on $\mathcal{F}(E;F)$. We claim that

$$\overline{H^{\Upsilon^{\tau'}_{\mathcal{B}}}} \subset \mathrm{Mor}_{\mathrm{Modtb}_A}((E,\tau,\mathcal{B}),(F,\tau',\mathcal{B}')).$$

Since $\mathcal{B}$ is a covering of E, $\Upsilon_s^{\tau'}$ is coarser than $\Upsilon_{\mathcal{B}}^{\tau'}$, and hence $\overline{H}^{\Upsilon_{\mathcal{B}}^{\tau'}} \subset \overline{H}^{\Upsilon_s^{\tau'}}$. By Proposition 6, p. 28 of [6], $\overline{H}^{\Upsilon_s^{\tau'}}$ is equicontinuous, and hence $\overline{H}^{\Upsilon_{\mathcal{B}}^{\tau'}}$ is equicontinuous. By the lemma,

$$\overline{H}^{\Upsilon_{\mathcal{B}}^{\tau'}} \subset \mathrm{Mor}_{\mathrm{Modt}_A}((E,\tau),(F,\tau')).$$

Now, let us see that $\overline{H}^{\Upsilon_{\mathcal{B}}^{\tau'}}$ is equibounded. In fact, let $B \in \mathcal{B}$. If $u \in \overline{H}^{\Upsilon_{\mathcal{B}}^{\tau'}}$ and $x \in B$, there exists a net $(u_\lambda)_{\lambda\in\Lambda}$ in H such that $(u_\lambda)_{\lambda\in\Lambda}$ converges to u for $\Upsilon_{\mathcal{B}}^{\tau'}$; thus $(u_\lambda(x))_{\lambda\in\Lambda}$ converges to $u(x)$ in (F,τ'), and so $u(x) \in \overline{H(B)}^{\tau'}$. We have just verified that $\overline{H}^{\Upsilon_{\mathcal{B}}^{\tau'}}(B) \subset \overline{H(B)}^{\tau'}$. On the other hand, $H(B) \in \mathcal{B}'$, and hence $\overline{H(B)}^{\tau'} \in \mathcal{B}'$ by hypothesis. Consequently, $\overline{H}^{\Upsilon_{\mathcal{B}}^{\tau'}}(B) \in \mathcal{B}'$, and $\overline{H}^{\Upsilon_{\mathcal{B}}^{\tau'}}$ is equibounded. In particular,

$$\overline{H}^{\Upsilon_{\mathcal{B}}^{\tau'}} \subset \mathrm{Mor}_{\mathrm{Modb}_A}((E,\mathcal{B}),(F,\mathcal{B}')).$$

Thus

$$\overline{H}^{\Upsilon_{\mathcal{B}}^{\tau'}} \subset \mathrm{Mor}_{\mathrm{Modtb}_A}((E,\tau,\mathcal{B}),(F,\tau',\mathcal{B}')),$$

which gives $\overline{H}^{\Upsilon_{\mathcal{B}}^{\tau'}} = H_1$. Finally, by Corollary 3, p. 16 of [6], $\overline{H}^{\Upsilon_{\mathcal{B}}^{\tau'}}$ is $\Upsilon_{\mathcal{B}}^{\tau'}$-complete in $\mathcal{F}(E;F)$ since $\overline{H(x)}^{\tau'}$ is τ'-complete for all $x \in E$. Therefore H_1 is $\Upsilon_{\mathcal{B}}^{\tau'}$-complete, which concludes the proof of the theorem. □

Corollary 7. *Let $(E,\tau,\mathcal{B}) \in \mathrm{Ob}(\mathrm{Modtb}_A)$ such that the product of any neighborhood of zero in A by any neighborhood of zero in (E,τ) is a neighborhood of zero in (E,τ), and let $(F,\tau') \in \mathrm{Ob}(\mathrm{Modt}_A)$ such that $(F,\tau',\mathbf{B}(\tau'))$ is quasi-complete and (F,τ') is separated. Then*

$$\left(\mathrm{Mor}_{\mathrm{Modt}_A}((E,\tau),(F,\tau')),\Upsilon_{\mathcal{B}}^{\tau'},\mathcal{E}_{\tau,\tau'}\right)$$

is quasi-complete.

Proof: Immediate from Theorem 3 (recall Remark 5) since, for all $(E,\tau,\mathcal{B}) \in \mathrm{Ob}(\mathrm{Modtb}_A)$ and for all $(F,\tau') \in \mathrm{Ob}(\mathrm{Modt}_A)$,

$$\mathrm{Mor}_{\mathrm{Modt}_A}((E,\tau),(F,\tau')) = \mathrm{Mor}_{\mathrm{Modtb}_A}((E,\tau,\mathcal{B}),(F,\tau',\mathbf{B}(\tau')))$$

and

$$\mathcal{E}_{\tau,\tau'} = \mathcal{E}_{\tau,\tau'}^{\mathcal{B},\mathbf{B}(\tau')} \quad \text{by Theorem 25.5 of [17].}$$

□

Corollary 8. *Let $(E,\tau,\mathcal{B}) \in \mathrm{Ob}(\mathrm{Modtb}_A)$ such that the product of any neighborhood of zero in A by any neighborhood of zero in (E,τ) is a neighborhood of zero in (E,τ), and suppose that (E,τ) is barrelled* ([12])*. Let $(F,\tau') \in \mathrm{Ob}(\mathrm{Modt}_A)$ such that $(F,\tau',\mathbf{B}(\tau'))$ is quasi-complete and (F,τ') is separated. Then*

$$\left(\mathrm{Mor}_{\mathrm{Modt}_A}((E,\tau),(F,\tau')), \Upsilon^{\tau'}_{\mathcal{B}}, \mathbf{B}(\Upsilon^{\tau'}_{\mathcal{B}})\right)$$

is quasi-complete.

PROOF: We claim that $\mathbf{B}(\Upsilon^{\tau'}_{\mathcal{B}}) = \mathcal{E}_{\tau,\tau'}$. Indeed, since $\mathcal{E}_{\tau,\tau'}$ is finer than $\mathbf{B}(\Upsilon^{\tau'}_{\mathcal{B}})$, it remains to verify that $\mathbf{B}(\Upsilon^{\tau'}_{\mathcal{B}})$ is finer than $\mathcal{E}_{\tau,\tau'}$. But, if $H \in \mathbf{B}(\Upsilon^{\tau'}_{\mathcal{B}})$, then $H(x) \in \mathbf{B}(\tau')$ for all $x \in E$. By Theorem 3.1 of [12], $H \in \mathcal{E}_{\tau,\tau'}$. Therefore the result follows from Corollary 7. □

Corollary 9. *Let $(E,\tau) \in \mathrm{Ob}(\mathrm{Modt}_A)$ such that the product of any neighborhood of zero in A by any neighborhood of zero in (E,τ) is a neighborhood of zero in (E,τ), and suppose that (E,τ) is bornological* ([3])*. Let $(F,\tau') \in \mathrm{Ob}(\mathrm{Modt}_A)$ such that $(F,\tau',\mathbf{B}(\tau'))$ is quasi-complete and (F,τ') is separated. Then*

$$\left(\mathrm{Mor}_{\mathrm{Modt}_A}((E,\tau),(F,\tau')), \Upsilon^{\tau'}_{\mathbf{B}(\tau)}, \mathbf{B}(\Upsilon^{\tau'}_{\mathbf{B}(\tau)})\right)$$

is quasi-complete.

PROOF: We claim that $\mathbf{B}(\Upsilon^{\tau'}_{\mathbf{B}(\tau)}) = \mathcal{E}_{\tau,\tau'}$. Indeed, since $\mathcal{E}_{\tau,\tau'}$ is finer than $\mathbf{B}(\Upsilon^{\tau'}_{\mathbf{B}(\tau)})$, it remains to verify that $\mathbf{B}(\Upsilon^{\tau'}_{\mathbf{B}(\tau)})$ is finer than $\mathcal{E}_{\tau,\tau'}$. But, if $H \in \mathbf{B}(\Upsilon^{\tau'}_{\mathbf{B}(\tau)})$, then $H(B) \in \mathbf{B}(\tau')$ for all $B \in \mathbf{B}(\tau)$. By the theorem proved in [3], $H \in \mathcal{E}_{\tau,\tau'}$. Therefore the result follows from Corollary 7. □

Remark 7. Corollary 8 (respectively Corollary 9) was suggested by Corollary 2, p. 31 of [7] (respectively (4), p. 143 of [10]).

REFERENCES

[1] Barboza de Oliveira L.R., *Compatibilité entre bornologies linéaires et topologies linéaires*, C.R. Acad. Sci. Paris **278** (1974), 1585–1586.

[2] Barboza de Oliveira L.R., *Les espaces vectoriels linéairement bornologisés et topologisés*, C.R. Acad. Sci. Paris **278** (1974), 1627–1628.

[3] Bernardes N.C., Jr., Pombo D.P., Jr., *Bornological topological modules*, Math. Japonica **40** (1994), 455–459.

[4] Bourbaki N.,, *Algèbre*, Chapter II, 3rd edition, Actualités Scientifiques et Industrielles 1236, Hermann, 1967.

[5] Bourbaki N., *Topologie Générale*, Chapter II, 4th edition, Actualités Scientifiques et Industrielles 1142, Hermann, 1965.

[6] Bourbaki N., *Topologie Générale*, Chapter X, 2nd edition, Actualités Scientifiques et Industrielles 1084, Hermann, 1967.

[7] Bourbaki N., *Espaces Vectoriels Topologiques*, Chapter III, Actualités Scientifiques et Industrielles 1229, Hermann, 1967.

[8] Grothendieck A., Dieudonné J., *Eléments de Géometrie Algébrique I*, Grundlehren der mathematischen Wissenschaften 166, Springer-Verlag, 1971.
[9] Hogbe-Nlend H., *Bornologies and Functional Analysis*, Notas de Matemática 62, North-Holland Publishing Company, 1977.
[10] Köthe G., *Topological Vector Spaces II*, Grundlehren der mathematischen Wissenschaften 237, Springer-Verlag, 1979.
[11] Pombo D.P., Jr., *On the completion of certain topological modules*, Math. Japonica **37** (1992), 333-336.
[12] Pombo D.P., Jr., *On barrelled topological modules*, Internat. J. Math. & Math. Sci. **19** (1996), 45–52.
[13] Pombo D.P., Jr., *Module bornologies*, Comment. Math. **36** (1996), 179–188.
[14] Schaefer H.H., *Topological Vector Spaces*, Graduate Texts in Mathematics 3, Springer-Verlag, 1971.
[15] Séminaire Banach, Lecture Notes in Mathematics 277, Springer-Verlag, 1972.
[16] Waelbroeck L., *Topological Vector Spaces and Algebras*, Lecture Notes in Mathematics 230, Springer-Verlag, 1971.
[17] Warner S., *Topological Fields*, Notas de Matemática 126, North-Holland Publishing Company, 1989.

Instituto de Matemática, Universidade Federal Fluminense, Rua São Paulo, s/n°, 24020-005 Niterói, RJ Brasil

(*Received* November 13, 1996)