The existence of initially ω_1-compact group topologies on free Abelian groups is independent of ZFC

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Abstract. It was known that free Abelian groups do not admit a Hausdorff compact group topology. Tkachenko showed in 1990 that, under CH, a free Abelian group of size $\mathfrak{c}$ admits a Hausdorff countably compact group topology.

We show that no Hausdorff group topology on a free Abelian group makes its ω-th power countably compact. In particular, a free Abelian group does not admit a Hausdorff p-compact nor a sequentially compact group topology. Under CH, we show that a free Abelian group does not admit a Hausdorff initially ω_1-compact group topology. We also show that the existence of such a group topology is independent of $\mathfrak{c} = \aleph_2$.

Keywords: free Abelian group, countable compactness, products, initially ω_1-compact, Martin's Axiom

Classification: 54H11, 22B99, 54D30

1. Introduction

Every topological space in this work is assumed to be infinite and Tychonoff. All the basic definitions will be given later in this section.

1.1 Motivation. It has been known long ago that it is impossible to introduce a compact group topology on any free Abelian group, but there are many different ways to endow such groups with pseudocompact group topologies (W.W. Comfort, D. Remus, D. Shakmatov, D. Dikranjan and M. Tkachenko made important contributions to the subject, see [3], [4], [11]). It is also known that under CH the free Abelian group of size $\mathfrak{c}$ admits a countably compact group topology ([11]). The later result gave rise to the problem whether there exists in ZFC a countably compact group topology on a free Abelian group ([2]). This problem still remains open.

In this paper we deal with two natural properties between countable compactness and compactness. First, we show that there is no group topology on a free Abelian group which makes the ω-th power of the group countably compact (correcting one of the statements made by M. Tkachenko in [11]). Then we prove the statement formulated in the title.

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We show that there exists a model of $\mathfrak{c} = \aleph_2$ in which every free Abelian group does not admit an initially ω_1-compact group topology. We show under $MA(\sigma\text{-}centered) + \mathfrak{c} = \aleph_2$ the existence of an initially ω_1-compact group topology on the free Abelian group of size $\mathfrak{c}$.

These results were obtained while I was a Ph.D student at York University ([15]). I thank my supervisor, Prof. Stephen Watson, for his guidance and encouragement during those years.

1.2 Basics. We will denote by $\mathbb{N}$ the set of all positive integers and $T \subseteq \mathbb{R}^2$ the unitary circle group with the subspace metric topology, but we will use the additive notation rather than the multiplicative.

Given a set I and a cardinal λ, define $[\kappa]^\lambda = \{A \subseteq I : |A| = \lambda\}$. Similarly, define $[\kappa]^{<\lambda} = \{A \subseteq I : |A| < \lambda\}$.

If α is an ordinal, $x \in T^\alpha$ and $a \in T$, then $x^\wedge a$ denotes the function $x \cup \{\langle \alpha, a\rangle\}$.

Definition 1. An infinite subset A of a topological space X has a complete accumulation point if there exists $x \in X$ such that $|A| = |A \cap U|$ for every neighbourhood U of x.

Definition 2. A space X is *initially ω_1-compact* if every open cover of X of size at most $\aleph_1$ has a finite subcover.

We will use the following equivalence: a space is initially ω_1-compact if and only if every infinite subset A of cardinality at most $\aleph_1$ has a complete accumulation point.

Definition 3. Let p be a free ultrafilter on ω. A point $x \in X$ is a *p-limit* of a sequence $\{x_n : n \in \omega\}$ in X if $\{n \in \omega : x_n \in U\} \in p$ for each neighbourhood U of x. We then write $x = p\text{-}\lim\{x_n : n \in \omega\}$.

Definition 4. Let p be a free ultrafilter on ω. A topological space X is *p-compact* if every sequence in X has a p-limit.

Every accumulation point of a sequence in X is a p-limit for some free ultrafilter p. Every p-compact space is countably compact and the product of p-compact spaces is p-compact. All powers of a space X are countably compact if and only if there exists a free ultrafilter p on ω such that X is p-compact (see [16]).

By abuse of terminology, "countably compact free Abelian group" will be a shortening for a "free Abelian group endowed with a countably compact group topology". Similarly, we will use "sequentially compact free Abelian group", "compact free Abelian group" and so on.

Definition 5. A partial order $\mathbb{P}$ is *σ-centered* if $\mathbb{P}$ can be represented as a union $\mathbb{P} = \bigcup_{n\in\omega} \mathbb{P}_n$ with each $\mathbb{P}_n$ centered (that is, every finite subset of $\mathbb{P}_n$ has a common extension in $\mathbb{P}_n$).

Definition 6. The Martin's Axiom for σ-centered partial orders, *MA*(σ-*centered*), is the following statement: If $\mathbb{P}$ is σ-centered, $\kappa < \mathfrak{c}$ and $\{\mathcal{D}_\xi : \xi < \kappa\}$ is a family of dense subsets of $\mathbb{P}$, then there exists a filter $\mathbb{G}$ in $\mathbb{P}$ such that $\mathbb{G} \cap \mathcal{D}_\xi \neq \emptyset$ for each $\xi < \kappa$.

We recall that *MA*(σ-*centered*) is a weaker version of Martin's Axiom [18].

2. Free abelian groups and sequential compactness

We will first show that every sequentially compact free Abelian group does not contain a proper open subgroup. We will then show that a free Abelian group does not admit a sequentially compact group topology.

Theorem 7. *Suppose that a free Abelian group G endowed with a group topology contains a proper open subgroup. Then G is not sequentially compact.*

Proof: Let H be a proper open subgroup of G and let $x \in G \setminus H$.

Claim 1. *There exists a prime p such that $p^k x \notin H$ for each $k \in \mathbb{N}$.*

Proof of Claim 1: If $\{mx : m \in \mathbb{N}\} \cap H \neq \{0\}$, let n be the smallest positive integer such that $nx \in H$. Fix a prime p such that p does not divide n.

Assume that there exists $k \in \mathbb{N}$ such that $p^k x \in H$. Let $a, b \in \mathbb{Z}$ be such that $ap^k + bn = 1$. Then $x = ap^k x + bnx \in H$, which is a contradiction.

If $\{nx : n \in \mathbb{N}\} \cap H = \{0\}$, then the claim is valid for any prime p. □

Let p be as in Claim 1.

Claim 2. *Suppose that $\{k_n : n \in \omega\}$ is a strictly increasing sequence in $\mathbb{N}$ such that the sequence $\{p^{k_n} x : n \in \omega\}$ converges. Then this sequence must converge to* 0.

Proof of Claim 2: Suppose that $\{p^{k_n} x : n \in \omega\}$ converges to $y \in G$. Let $\{y_\alpha : \alpha \in I\}$ be a set of free generators for G and let $y = \sum_{i=1}^{l} a_i y_{\alpha_i}$, where each each a_i is an integer. Let $n_0 \in \mathbb{N}$ be such that $p^{k_{n_0}} > |a_i|$ for every $i \in \{1, \dots, l\}$. By sequential compactness of G, the sequence $\{p^{k_n - k_{n_0}} x : n > n_0\}$ has a subsequence $\{p^{k_{n_j} - k_{n_0}} x : j \geq 1\}$ converging to a point $z \in G$.

Clearly $\{p^{k_{n_j}} x : j \geq 1\} = \{p^{k_{n_0}} (p^{k_{n_j} - k_{n_0}}) x : j \geq 1\}$ converges to $p^{k_{n_0}} z$. Hence, by the uniqueness of the limit, $p^{k_{n_0}} z = y$. Thus, for each $i \in \{1, .., l\}$, $p^{k_{n_0}}$ divides a_i. Hence $a_i = 0$ for each $i \in \{1, .., l\}$. Therefore, $y = 0$. □

It follows from Claim 1 and Claim 2 that $\{p^n x : x \in \omega\}$ does not have a convergent subsequence. Thus G is not sequentially compact. □

We will show now that for each cardinal κ, every infinite sequentially compact subgroup of T^κ is not free Abelian. I thank Prof. Comfort for pointing out to me that this implies that there are no infinite sequentially compact free Abelian groups.

Theorem 8. *Let G be an infinite sequentially compact group. Then G is not free Abelian.*

Proof: Every countably compact Abelian group can be embedded into T^κ as a topological subgroup for a sufficiently large cardinal κ (see [1]). Therefore, we can assume that G is a subgroup of T^κ.

Suppose by contradiction that G is free Abelian. We will denote by π_α the projection of G on the α-th coordinate. Since G is infinite, we can fix $\beta < \kappa$ such that $\pi_\beta(G)$ is not trivial, i.e. contains at least two points.

Claim 3. *There exists $x \in G$ such that $\{\pi_\beta(nx) : n \in \mathbb{Z}\} = \{nx(\beta) : n \in \mathbb{Z}\}$ is a non-trivial finite group.*

Proof of Claim 3: Let $y \in G$ be such that $y(\beta) \neq 0$. If the subgroup of T generated by $y(\beta)$ is finite, one can simply take $x = y$. Otherwise, the subgroup generated by $y(\beta)$ is dense in T. By countable compactness of G, $\pi_\beta(G) = T$. Fix a non-zero element $a \in T$ of finite order and choose $x \in G$ such that $\pi_\beta(x) = a$. □

Let $x \in G$ be as in Claim 3, and let K be the subgroup of T generated by $x(\beta)$. Since K is a closed subgroup of T and π_β is a continuous homomorphism, $H = \pi_\beta^{-1}(K)$ is a closed subgroup of G, hence a sequentially compact free Abelian group. Furthermore, $H_0 = \pi_\beta^{-1}\{0\}$ is an open subgroup of H. The existence of such H and H_0 contradicts Theorem 7, which completes the proof. □

The above Claim 3 implies the following result.

Corollary 9. *Let G be an infinite countably compact Abelian group. Then there exists a non-trivial closed subgroup H of G such that H contains a proper open subgroup.*

This corollary will be used to generalize Theorem 8.

3. The ω-th power of free Abelian groups and countable compactness

S. Watson pointed out that the proof of Theorem 8 does not rely strongly on sequential compactness. He suggested that sequential compactness of G could be replaced by a weaker condition, such as the square of G being countably compact.

In the case of sequential compactness, it was essential to use the fact that the limit of a convergent sequence is unique. It turns out that we can use a similar argument if the ω-th power of a topological group is countably compact. We will prove a slightly more general result which will include certain semigroups.

Theorem 10. *Let G be a non-trivial Abelian semigroup. If G has the neutral element, denote it by 0. Suppose that G is endowed with a semigroup topology which satisfies the following conditions:*

(1) *G is torsion free, that is, for each $x \in G \setminus \{0\}$, $nx \neq mx$ whenever m and n are distinct non-negative integers;*

(2) *for every* $x \neq 0$, *there exist infinitely many primes* p *such that* $p^n \nmid x$ *for some* $n \in \mathbb{N}$ *(that is,* $p^n y \neq x$ *for each* $y \in G$*);*

(3) *there are* $x \in G \setminus \{0\}$, $A \in [\omega]^\omega$ *and an enumeration* $p_1, p_2, \dots$ *of all but finitely many prime numbers such that* $0 \notin \overline{\{(p_1 \dots p_n)^n x : n \in A\}}$.

Then G^ω *is not countably compact.*

PROOF: Suppose by contradiction that G^ω is countably compact. Fix $x \in G$, $A \subseteq \omega$ and $p_1, p_2, \dots$ satisfying (3). Define $\{\vec{x}_n : n \in A\} \subseteq G^{\omega \setminus \{0\}}$ as follows:

$$\vec{x}_n(m) = \begin{cases} \frac{(p_1 \dots p_n)^n}{m} x & \text{if } m \text{ divides } (p_1 \dots p_n)^n, \\ x & \text{otherwise.} \end{cases}$$

Since G^ω is countably compact, the sequence $\{\vec{x}_n : n \in A\}$ has an accumulation point $\vec{x} \in G^\omega$.

By (3), $\vec{x}(1) \neq 0$. We will show that this leads to a contradiction. By (2), there exist a prime $p \in \{p_1, p_2, \dots\}$ and $k \in \mathbb{N}$ such that $p^k \nmid \vec{x}(1)$. Put $m^* = p^k$ and fix a free ultrafilter $\mathcal{U}$ on A such that $\vec{x}$ is a $\mathcal{U}$-limit of $\{\vec{x}_n : n \in A\}$. Then

$(*)$ for each $i \in \omega \setminus \{0\}$, $\vec{x}(i)$ is a $\mathcal{U}$-limit of $\{\vec{x}_n(i) : n \in A\}$.

Note that by definition, $\{\vec{x}_n(1) : n \in A\} = \{(p_1 \dots p_n)^n x : n \in A\}$. Let $n^* \in \omega$ be such that $p \in \{p_1, \dots, p_{n^*}\}$ and $m^* < n^*$. By definition,

$$\{\vec{x}_n(m^*) : n \in A,\ n > n^*\} = \Big\{\frac{(p_1 \dots p_n)^n}{m^*} : n \in A,\ n > n^*\Big\}.$$

Therefore, there are only finitely many $n \in A$ with $m^* \vec{x}_n(m^*) \neq (p_1 \dots p_n)^n x$. Thus, by $(*)$, the points $\vec{x}(1)$ and $m^* \vec{x}(m^*)$ are $\mathcal{U}$-limits of $\{(p_1 \dots p_n)^n x : n \in A\}$. Since the space X is Hausdorff, we conclude that $\vec{x}(1) = m^* \vec{x}(m^*)$. This contradicts the equality $m^* = p^k$. □

Corollary 11. *Let* S *be a topological semigroup without a neutral element. Suppose that the conditions* (1) *and* (2) *of Theorem* 10 *are satisfied. Then* S^ω *is not countably compact.*

PROOF: Note that the condition (3) of Theorem 10 is trivially satisfied for a semigroup without the neutral element. □

Corollary 12. *Let* S *be a subsemigroup of a free Abelian group and suppose that* S *does not have the neutral element. If* S *is endowed with a semigroup topology, then* S^ω *is not countably compact.*

PROOF: It suffices to note that a subsemigroup of a free Abelian group satisfies conditions (1) and (2) of Theorem 10. □

Theorem 13. *Let S be a topological semigroup satisfying the conditions* (1) *and* (2) *of Theorem* 10. *Suppose also that S has no non-trivial convergent sequences. Then S^ω is not countably compact.*

PROOF: Let x be any element of G which is not the neutral element. Let $p_1, p_2, \dots$ be an enumeration of all prime numbers. By (1), $X = \{(p_1 \dots p_n)^n x : n \in \omega \setminus \{0\}\}$ is a sequence whose elements are pairwise distinct. By hypothesis, X does not converge. Therefore, there exists $A \in [\omega]^\omega$ such that $0 \notin \overline{\{(p_1 \dots p_n)^n x : n \in A\}}$. Thus, the condition (3) of Theorem 10 holds. □

Corollary 14. *Let S be a subsemigroup of a free Abelian group. Suppose that S is endowed with a semigroup topology in which S has no non-trivial convergent sequences. Then S^ω is not countably compact.*

Corollary 15. *Suppose that a topological Abelian group G has no non-trivial convergent sequences. If G is algebraically free and $S \subseteq G$ is a subsemigroup of G endowed with the subspace topology, then S^ω is not countably compact.*

Proposition 16. *Let G be a non-trivial Abelian semigroup with the neutral element* 0. *Suppose that G is endowed with a semigroup topology which contains a proper open subgroup. If conditions* (1) *and* (2) *of Theorem* 10 *are satisfied, then G^ω is not countably compact.*

PROOF: It suffices to show that condition (3) of Theorem 10 is satisfied. Let H be a proper open subgroup of G and let $x \in G \setminus H$.

Claim 4. *There exists $k \in \omega \setminus \{0\}$ such that $\{mx : (m, k) = 1 \wedge m \geq 2\} \cap H = \emptyset$, where $(m, k) = 1$ means that m and k do not have common prime divisors.*

PROOF OF CLAIM 4: If $\{nx : n \in \omega\} \cap H \neq \{0\}$, let k be the smallest positive integer such that $kx \in H$. Suppose that there exists $m \in \omega \setminus 2$ such that $(m, k) = 1$ and $mx \in H$. Then one can find $a, b \in \mathbb{Z}$ such that $x = amx + bkx \in H$, a contradiction.

If $\{nx : n \in \omega\} \cap H = \{0\}$, then the Claim is valid for any $k \geq 1$. □

Since H is an open neighbourhood of 0, by Claim 4, the neutral element 0 cannot be an accumulation point of $A = \{mx : m \in \omega \wedge (m, k) = 1\}$. Let $p_1, p_2, \dots$ be an enumeration of all prime numbers that do not divide k. Clearly, x, A and $p_1, p_2 \dots$ satisfy the condition (3). It follows from Theorem 10 that G^ω is not countably compact. □

Theorem 17. *Let G be an infinite free Abelian group endowed with a group topology. Then G^ω is not countably compact.*

PROOF: By Corollary 9, we can assume that G contains a proper open subgroup. Clearly, the conditions (1) and (2) of Theorem 10 are satisfied, thus the result follows from Proposition 16. □

Corollary 18. *There exists no p-compact group topology on a free Abelian group.*

4. Countably compact group topologies and $MA(\sigma\text{-}centered)$

We will construct under $MA(\sigma\text{-}centered)$ a subgroup $G \subseteq T^{\mathfrak{c}}$ of size $\mathfrak{c}$ which is algebraically free Abelian and countably compact when endowed with the subspace topology. This yields the following.

Example 19 ($MA(\sigma\text{-}centered)$). *There exists a countably compact group topology on the free Abelian group generated by $\mathfrak{c}$ elements.*

The construction is a modification of van Douwen's group ([5]). It is related to Tkachenko's modification ([11]) of Hajnal and Juhász' group ([8]).

4.1 Main ideas. We will construct a subset $X = \{x_\alpha : \alpha < \mathfrak{c}\} \subseteq T^{\mathfrak{c}}$ and the group G will be the subgroup of $T^{\mathfrak{c}}$ generated by X.

Every element of G can be written as a finite sum of elements of X and each element of X is indexed by an ordinal. Thus, for each $x \in G$ there exists a function f from a finite subset of $\mathfrak{c}$ to $Z \setminus \{0\}$ such that $x = \sum_{\xi \in \mathrm{dom}\, f} f(\xi) x_\xi$. We will call such an f the coding of x (the coding of the neutral element is the empty function).

Definition 20. Define $\mathcal{F}$ as the set of all functions f from a finite subset of $\mathfrak{c}$ into $\mathbb{Z} \setminus \{0\}$.

At stage $\alpha \leq \mathfrak{c}$, we will have defined $\{x_\beta \restriction_\alpha \colon \beta < \alpha\}$. Thus, at each intermediate stage we only know a fragment of X. We will use $\mathcal{F}$ to code the fragments of G.

Let us see now which are the inductive assumptions we have to impose on X.

Making G free Abelian

We will construct X to witness that G is free Abelian. That is, if $f \in \mathcal{F} \setminus \{\emptyset\}$, then $\sum_{\xi \in \mathrm{dom}\, f} f(\xi) x_\xi \neq 0$. This condition will be satisfied if for each $f \in \mathcal{F} \setminus \{\emptyset\}$ there exists $\beta < \mathfrak{c}$ such that $\sum_{\xi \in \mathrm{dom}\, f} f(\xi) x_\xi(\beta) \neq 0$.

Since $|\mathcal{F} \setminus \{\emptyset\}| = \mathfrak{c}$, we can take care of one $f \in \mathcal{F}$ at each step.

Definition 21. Let $\{f_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of $\mathcal{F}$ such that $\mathrm{dom}\, f_\alpha \subseteq \alpha + 1$ for each $\alpha < \mathfrak{c}$.

Then the group G is free Abelian if

(1) for each $\alpha < \mathfrak{c}$, $\sum_{\xi \in \mathrm{dom}\, f_\alpha} f_\alpha(\xi) x_\xi(\alpha) \neq 0$.

Note that at stage $\alpha + 1$, we will have defined $x_\xi \restriction_{\alpha+1}$ for each $\xi \in \mathrm{dom}\, f_\alpha$. Thus (1) makes sense at stage $\alpha + 1$.

Definition 22. If $f \in \mathcal{F}$ and $\mathrm{dom}\, f \subseteq \gamma$, then denote by $\sum_{\xi \in \mathrm{dom}\, f} f(\xi) x_\xi \restriction_\gamma$ the element of T^γ coded by f.

Making G countably compact

We have to guarantee that every infinite subset of G has an accumulation point. Each countably infinite subset of G can be coded by a countable subset of $\mathcal{F}$. There are $\mathfrak{c}$ many codings for a sequences in G. For technical reasons, we can only take care of fewer than $\mathfrak{c}$ sequences at each stage.

Definition 23. Let $\{\mathcal{F}_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of $[\mathcal{F}]^\omega$ such that $\operatorname{dom} f \subseteq \alpha + 1$ for each $f \in \mathcal{F}_\alpha$.

At stage $\alpha + 1$, we promise that x_α will be the accumulation point of the sequence $\{\sum_{\xi \in \operatorname{dom} f} f(\xi) x_\xi : f \in \mathcal{F}_\alpha\}$. Thus, at each stage $\beta < \mathfrak{c}$, we will only worry about the sequences coded by some $\mathcal{F}_\alpha$ with $\alpha < \beta$.

To keep the promise we will need the following.

Lemma 24. *Let y be an element of T^β and $\{y_n : n \in \omega\} \subseteq T^\beta$, where $\beta < \mathfrak{c}$ is a limit ordinal. Then y is an accumulation point of $\{y_n : n \in \omega\}$ if and only if $y{\restriction}_\alpha$ is an accumulation point of $\{y_n{\restriction}_\alpha : n \in \omega\}$ for each $\alpha < \beta$.*

Definition 25. If $\alpha \leq \gamma$, then denote by $\{\sum_{\xi \in \operatorname{dom} f} f(\xi) x_\xi{\restriction}_\gamma : f \in \mathcal{F}_\alpha\}$ the sequence in T^γ coded by $\mathcal{F}_\alpha$.

Using Lemma 24, it suffices to choose $x_\alpha{\restriction}_\alpha$ to be an accumulation point of the sequence in T^α coded by $\mathcal{F}_\alpha$ and by induction show that if $x_\alpha{\restriction}_\gamma$ is an accumulation point for the sequence in T^γ coded by $\mathcal{F}_\alpha$, then we can define the γ-th coordinates of the points in question so that $x_\alpha{\restriction}_{\gamma+1}$ will be an accumulation point for the sequence in $T^{\gamma+1}$ coded by $\mathcal{F}_\alpha$.

We will rewrite this last condition in terms of subsets of $\mathcal{F}_\alpha$. In what follows $\|t\|$ stands for the length of a vector $t \in \mathbb{R}^2$.

Definition 26. For each $\alpha \leq \gamma$, $F \in [\gamma]^{<\omega}$ and $k \in \omega$, define

$$E(\alpha, F, k) = \Big\{ f \in \mathcal{F}_\alpha : \forall\, \mu \in F \Big(\Big\| \sum_{\xi \in \operatorname{dom} f} f(\xi) x_\xi(\mu) - x_\alpha(\mu) \Big\| < \frac{1}{k+1} \Big) \Big\}.$$

The proof of the following lemma is left to the reader.

Lemma 27. *Let $\alpha \leq \gamma$. Then $x_\alpha{\restriction}_\gamma$ is an accumulation point of the sequence in T^γ coded by $\mathcal{F}_\alpha$ if and only if $|E(\alpha, F, k)| = \omega$ for every $F \in [\gamma]^{<\omega}$ and $k \in \omega$.*

Thus, G will be countably compact if the following two conditions hold:

(2) the set $E(\alpha, F, k)$ is infinite for all $\alpha < \mathfrak{c}$, $F \in [\gamma]^{<\omega}$ and $k \in \omega$;

(3) if $\alpha < \gamma$ and (2) holds for α, then

$$\{f \in E(\alpha, F, k) : \| \sum_{\xi \in \operatorname{dom} f} f(\xi) x_\xi(\gamma) - x_\alpha(\gamma) \| < \frac{1}{k+1}\}$$

is infinite for all $F \in [\gamma]^{<\omega}$ and $k \in \omega$.

A construction of the group as in Example 19 is given below.

Theorem 28 (MA(*σ-centered*))**.** *There exists a family $\{x_\alpha : \alpha < \mathfrak{c}\} \subseteq T^{\mathfrak{c}}$ which satisfies the conditions* (1)–(3) *stated above.*

Proof: At stage 0 there is nothing to do. Suppose we have defined $\{x_\alpha{\restriction}_\beta : \alpha < \beta\}$ for each $\beta < \gamma$ such that the conditions (1)–(3) hold.

If γ is limit, define $x_\alpha\restriction_\gamma= \bigcup_{\alpha<\beta<\gamma} x_\alpha\restriction_\beta$. Clearly $\{x_\alpha\restriction_\gamma\colon\ \alpha<\gamma\}$ satisfies the conditions (1)–(3).

Let us suppose that $\gamma=\beta+1$. Then we have defined $\{x_\alpha\restriction_\beta\colon\ \alpha<\beta\}$ satisfying the conditions (1)–(3). Then the sequence in T^β coded by $\mathcal{F}_\beta$ is already defined. Fix an accumulation point $y\in T^\beta$ and define $x_\beta\restriction_\beta= y$. Thus, (2) is valid for β.

Let ϕ be as in Lemma 29. Then the family $\{x_\alpha\restriction_\gamma\colon\ \alpha<\gamma\}$ satisfies the conditions (1)–(3) if we define $x_\alpha\restriction_\gamma=(x_\alpha\restriction_\beta)^\wedge\phi(\alpha)$ for each $\alpha<\gamma$.

Lemma 29 (*MA*(σ-*centered*))**.** *There exists a function $\phi:\gamma\longrightarrow T$ satisfying the following conditions:*

(A) $\sum_{\xi\in\operatorname{dom} f_\beta} f_\beta(\xi)\phi(\xi)\neq 0$;

(B) *the set* $\{f\in E(\alpha,F,k):\ \|\sum_{\xi\in\operatorname{dom} f} f(\xi)\phi(\xi)-\phi(\alpha)\|<\frac{1}{k+1}\}$ *is infinite for all* $\alpha<\gamma$, $F\in[\beta]^{<\omega}$ *and* $k\in\omega$.

The function ϕ will be constructed in the next subsection. □

4.2 Proof of Lemma 29. We will define a partial order $\mathbb{P}$ and dense sets which will be used to define a function ϕ satisfying the conditions of Lemma 29. Recall that β is a fixed ordinal with $\beta<\mathfrak{c}$, and $\gamma=\beta+1$.

Fix a countable basis $\mathcal{B}$ for T consisting of non-empty open subsets. We assume that $T\in\mathcal{B}$.

Definition 30. Let $\mathbb{P}$ be the family of all functions p from a finite subset of γ into $\mathcal{B}$.

Given $p,q\in\mathbb{P}$, define $p\leq q$ if $\operatorname{dom} p\supseteq\operatorname{dom} q$ and either $p(\xi)=q(\xi)$ or $\overline{p(\xi)}\subseteq q(\xi)$ for each $\xi\in\operatorname{dom} q$.

Lemma 31. *The partial order $\mathbb{P}$ is σ-centered.*

Proof: Let $\{s_n:\ n\in\omega\}$ be a dense subset of $\mathcal{B}^\gamma$, where $\mathcal{B}$ is endowed with the discrete topology. Let $\mathbb{P}_n=\{p\in\mathbb{P}:\ p\leq s_n\}$ for every $n\in\omega$. Then clearly $\mathbb{P}_n$ is centered and $\mathbb{P}=\bigcup_{n\in\omega}\mathbb{P}_n$. □

Suppose that $\mathbb{G}$ is a filter on $\mathbb{P}$. For every $\xi<\lambda$, define $\Phi(\xi)=\{p(\xi):\ \xi\in\operatorname{dom} p\}$. If $\Phi(\xi)$ is not empty, then $\bigcap\Phi(\xi)\neq\emptyset$ and we can choose $\phi_{\mathbb{G}}(\xi)\in\bigcap\Phi(\xi)$.

Lemma 32. *For each $\xi<\gamma$, the set $\mathcal{D}_\xi=\{p\in\mathbb{P}:\ \xi\in\operatorname{dom} p\}$ is dense in $\mathbb{P}$.*

Proof: Fix $q\in\mathbb{P}$. If $\xi\notin\operatorname{dom} q$, then define $p=q\cup\{\langle\xi,T\rangle\}$. □

Applying Lemma 32 and *MA*(σ-*centered*) to $\mathbb{P}$, we choose a filter $\mathbb{G}$ on $\mathbb{P}$ such that $\operatorname{dom}\phi_{\mathbb{G}}=\gamma$.

If there exists $p\in\mathbb{G}$ such that $0\notin\sum_{\xi\in\operatorname{dom} f_\beta} f_\beta(\xi)p(\xi)$, then the condition (A) of Lemma 29 holds.

The proof of the following simple fact is omitted.

Lemma 33. *The set $\{p\in\mathbb{P}:\ 0\notin\sum_{\xi\in\operatorname{dom} f_\beta} f_\beta(\xi)p(\xi)\}$ is dense and open in $\mathbb{P}$.*

Condition (3) presents more difficulties.

Definition 34. For $\alpha < \gamma$, $F \in [\beta]^{<\omega}$ and $k \in \omega$, let $\{E(\alpha, F, k, n) : n \in \omega\}$ be a partition of $E(\alpha, F, k)$ into disjoint infinite subsets.

Let $\mathfrak{E} = \{E(\alpha, F, k, n) : \alpha < \gamma, F \in [\beta]^{<\omega},\ k, n \in \omega\}$.

To guarantee that the condition (3) holds, it suffices to show that for all $U \in \mathcal{B}$ and $f \in E(\alpha, F, k, n)$ there exists $p \in \mathbb{G}$ such that $\operatorname{dom} f \subseteq \operatorname{dom} p$ and $\sum_{\xi \in \operatorname{dom} f} f(\xi)p(\xi) \subseteq U$.

Lemma 35. *For all $E \in \mathfrak{E}$ and $U \in \mathcal{B}$, the set $\{p \in \mathbb{P} : (\exists f \in E)(\sum_{\xi \in \operatorname{dom} f} f(\xi)p(\xi) \subseteq U)\}$ is a dense subset of $\mathbb{P}$.*

Proof: The argument which follows is similar to Tkachenko's ([11]).

Fix $E \in \mathfrak{E}$, $U \in \mathcal{B}$ and $q \in \mathbb{P}$. We have two cases to consider.

Case 1. There exists an infinite subset $E_1 \subseteq E$ such that $\operatorname{dom} f \neq \operatorname{dom} g$ whenever $f, g \in E_1$ are distinct.

Then there are $f \in E_1$ and $\mu < \gamma$ such that $\mu \in \operatorname{dom} f \setminus \operatorname{dom} q \neq \emptyset$. We will define p such that its domain will be $D = \operatorname{dom} q \cup \operatorname{dom} f$. For every $\xi \in \operatorname{dom} q$, fix $a_\xi \in q(\xi)$ and for every $\xi \in D \setminus (\operatorname{dom} q \cup \{\mu\})$, put $a_\xi = 0$. Choose $a_\mu \in T$ such that $\sum_{\xi \in \operatorname{dom} f} f(\xi)a_\xi \in U$. For every $\xi \in D$, fix $U_\xi \in \mathcal{B}$ such that $\sum_{\xi \in \operatorname{dom} f} f(\xi)U_\xi \subseteq U$, $a_\xi \in U_\xi$ and $\overline{U}_\xi \subseteq q(\xi)$ if $\xi \in \operatorname{dom} q$. Define $p = \{\langle \xi, U_\xi \rangle : \xi \in D\}$.

Case 2. There exists an infinite subset $E_2 \subseteq E$ such that $|\{\operatorname{dom} f : f \in E_2\}| = 1$.

Let D be the domain of every $f \in E_2$. We can suppose without loss of generality that $\operatorname{dom} q \supseteq D$. There exists $\mu \in D$ such that $\{f(\mu) : f \in E_2\}$ is unbounded in $\mathbb{Z}$. Let $g \in E_2$ be such that $g(\mu)q(\mu) = T$. Fix $a_\xi \in q(\xi)$ for every $\xi \in D \setminus \{\mu\}$. Choose $a_\mu \in q(\mu)$ such that $\sum_{\xi \in D \setminus \{\mu\}} g(\xi)a_\xi + g(\mu)a_\mu \in U$. Then we can proceed as in Case 1. $\square$

Observation. The set $\{nx_0 : n \in \omega\}$ can be made dense in $T^{\mathfrak{c}}$ by adding some new dense subsets to $\mathfrak{E}$.

5. Initially ω_1-compact group topologies on the free Abelian group

Finally, we prove the result formulated in the title.

Theorem 36. *The existence of an initially ω_1-compact group topology on some free Abelian group is independent of $\mathfrak{c} = \aleph_2$.*

This follows from Example 37 and Theorem 41 below.

The existence of an initially ω_1-compact group topology.

Example 37 ($MA(\sigma$-*centered*$) + \mathfrak{c} = \aleph_2$)**.** *There exists an initially $\aleph_1$-compact group topology on the free Abelian group of size $\mathfrak{c}$.*

We will point out small modifications which should be made in the construction presented in Section 4.

It suffices to construct a subset $X \subseteq T^{\mathfrak{c}}$ in such a way that every infinite subset G of size at most $\aleph_1$ has a complete accumulation point in G.

Since $2^{\aleph_1} = \mathfrak{c}$, we can enumerate all infinite subsets of $\mathcal{F}$ of size at most $\aleph_1$ in length $\mathfrak{c}$. In the inductive condition (2) we require that $|E(\alpha, F, k)| = |\mathcal{F}_\alpha|$. During the successor stage, if $E(\alpha, F, k)$ has size $\aleph_1$, then we split it into $\aleph_1$ pieces of size $\aleph_1$.

5.2 Non-existence of such a group topology.

Let p be a free ultrafilter on ω having a base of cardinality $\aleph_1$. It is not difficult to verify that every infinite ω_1-compact space is p-compact. Since the class of p-compact spaces is productive, we have the following.

Lemma 38. *Suppose that there exists a free ultrafilter p on ω generated by a basis of size $\aleph_1$. Then the product of arbitrarily many initially ω_1-compact spaces is p-compact. In particular, the product of initially ω_1-compact spaces is countably compact.*

Theorem 39 (CH)**.** *No free Abelian group can be endowed with an initially ω_1-compact group topology.*

PROOF: Let p be any free ultrafilter on ω. By our assumption, $\mathfrak{c} = \aleph_1$, so that p has a base of cardinality $\aleph_1$. However, the ω-th power of a "topological" free Abelian group G is not countably compact (Theorem 17). Thus, by Lemma 38, there exists no initially ω_1-compact group topology on G. $\square$

Thus, the absence of such group topologies is related with the existence of free ultrafilters p as in Lemma 38.

Definition 40. Kunen's Axiom (KA) is the following statement: *There exists a free ultrafilter on ω generated by a basis of size $\aleph_1$.*

KA is consistent with $\mathfrak{c} = \aleph_2$ (see Chapter VIII of [9] or [5]). Therefore, we have the following result.

Theorem 41 ($KA+\mathfrak{c} = \aleph_2$)**.** *There exists no initially ω_1-compact group topology on any free Abelian group.*

6. Final remarks

Some comments on Wallace semigroups

Definition 42. A cancellative semigroup S endowed with a countably compact semigroup topology which does not make S a topological group is called a *Wallace semigroup.*

Using the terminology above, Wallace [17] asked whether there is no Wallace semigroup (see [1], [2]).

For any Wallace semigroup, its $2^{\mathfrak{c}}$-th power is not countably compact ([12]). The Wallace semigroups constructed in [10] and [12] do not have their ω-th power countably compact. Indeed, the Robbie and Svetlichny's counterexample for the Wallace's Problem constructed under CH (see [10]) satisfies the conditions of Corollary 12, so its ω-th power is not countably compact.

The countably compact group topology on the free Abelian group G that we constructed in Section 4 does not have non-trivial convergent sequences. We note that every subsemigroup of G containing the subset $X \subseteq G$ and which is not a group will be a Wallace semigroup. By Corollary 15, its ω-th power is not countably compact.

The Wallace semigroups we have constructed in [12] cannot have its ω-th power countably compact either.

Thus, it is natural to ask which is the least κ such that the κ-th power of any Wallace semigroup is not countably compact.

Finite powers of free Abelian groups

In [14] we have constructed a topological group whose square is countably compact but whose cube is not. In [13] we showed that for every integer k there exists an integer $n \geq k$ and a topological group G such that G^n is countably compact but G^{n+1} is not.

It is not clear for us whether these constructions could be used to obtain a topological group topology on an infinite free Abelian group G which makes G^2 countably compact.

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