Singularities and equicontinuity of certain families of set-valued mappings

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Abstract. In the present paper we establish an abstract principle of condensation of singularities for families consisting of set-valued mappings. By using it as a basic tool, the condensation of the singularities and the equicontinuity of certain families of generalized convex set-valued mappings are studied. In particular, a principle of condensation of the singularities of families of closed convex processes is derived. This principle immediately yields the uniform boundedness theorem stated in [1, Theorem 2.3.1].

Keywords: condensation of the singularities, equicontinuity, generalized convex set-valued mappings, closed convex processes

Classification: 46N10, 54C60, 26B25

1. Introduction

Many textbooks consider the following *principle of uniform boundedness* as one of the most important results in functional analysis.

Theorem 1.1. *Let X be a Banach space, let Y be a normed linear space, and let $\mathcal{F}$ be a family of continuous linear mappings from X into Y such that*

$$\sup\,\{\|f(x)\| \mid f \in \mathcal{F}\} < \infty \qquad \textit{for all}\ \ x \in X.$$

Then $\sup\{\|f\| \mid f \in \mathcal{F}\} < \infty$.

This theorem reveals that if $\sup\{\|f\| \mid f \in \mathcal{F}\} = \infty$, then there exists at least one singularity of $\mathcal{F}$, i.e. a point $x \in X$ such that $\sup\{\|f(x)\| \mid f \in \mathcal{F}\} = \infty$. More informations about the set of singularities of $\mathcal{F}$ can be obtained if the following *principle of condensation of the singularities* is applied instead of the uniform boundedness principle.

Theorem 1.2. *Let X be a Banach space, let Y be a normed linear space, and let $\mathcal{F}$ be a family of continuous linear mappings from X into Y such that*

$$\sup\{\|f\| \mid f \in \mathcal{F}\} = \infty.$$

Then the set $S_{\mathcal{F}}$ of all $x \in X$ for which

$$\sup\{\|f(x)\| \mid f \in \mathcal{F}\} = \infty$$

is residual.

Theorem 1.2 has been generalized by numerous authors and in several directions. Here we merely mention the investigations by W.W. Breckner [3], who established a principle of condensation of singularities for lower semicontinuous mappings defined on a topological space and taking values in the power set of a topological space. In the present paper we continue these investigations, but unlike W.W. Breckner we deal with families of set-valued mappings defined on topological spaces. In Section 2 we introduce the concept of a singularity for such families and after that we prove a general principle of condensation of the singularities of a family consisting of arbitrary upper semicontinuous set-valued mappings. In the following three sections we give several applications of this principle under the assumption that the involved set-valued mappings have some additional algebraic properties. They relate to families that consist either of (A, s)-convex set-valued mappings or of (A, s)-convex real-valued mappings that are not equicontinuous at the origin as well as to families of closed convex processes. An important corollary is the uniform boundedness theorem involving closed convex processes which has been stated by J.P. Aubin and H. Frankowska [1, Theorem 2.3.1].

Throughout the paper the set of all positive integers is denoted by $\mathbb{N}$. Given any subset M of a topological space, we denote by $\operatorname{cl} M$ its closure. Given any set Y, we denote by $\mathcal{P}_0(Y)$ the set consisting of all nonempty subsets of Y. Given a topological linear space X, we denote by $\mathcal{O}_{ac}(X)$ the class consisting of all nonempty, open and absolutely convex subsets of X.

2. An abstract principle of condensation of the singularities of families of set-valued mappings

Let X and Y be topological spaces, let F be a mapping from X to 2^Y, and let x_0 be any point of X. Recall that F is said to be *upper semicontinuous at* x_0 if for every open subset Y_0 of Y with $F(x_0) \subseteq Y_0$, there exists a neighbourhood V of x_0 such that $F(x) \subseteq Y_0$ for all $x \in V$. If F is upper semicontinuous at each point of X, then F is called *upper semicontinuous* (on X).

Let I be a nonempty set, let $B : I \times \mathbb{N} \to 2^Y$ be a mapping whose values are closed subsets of Y, and let $\mathcal{F}$ be a family of mappings from X to $\mathcal{P}_0(Y)$.

We say that $\mathcal{F}$ is *B-bounded at a point* $x_0 \in X$ if there exists a family $\{y_F \mid F \in \mathcal{F}\}$ with $y_F \in F(x_0)$ $(F \in \mathcal{F})$, satisfying the following condition: for every $i \in I$ one can select a positive integer n such that $\{y_F \mid F \in \mathcal{F}\} \subseteq B(i, n)$. If $\mathcal{F}$ is B-bounded at each point of X, then we say that $\mathcal{F}$ is *pointwise B-bounded* (on X). A point in X at which $\mathcal{F}$ is not B-bounded is called a *B-singularity* of $\mathcal{F}$. The set of all B-singularities of $\mathcal{F}$ will be denoted by $S_{\mathcal{F}}(B)$. Obviously, $\mathcal{F}$ is pointwise B-bounded if and only if the set $S_{\mathcal{F}}(B)$ is empty.

We say that $\mathcal{F}$ is *uniformly B-bounded* if for every $i \in I$ there exist a positive integer n and a nonempty open subset X_0 of X satisfying the following condition:

for each point $x \in X_0$ one can select a family $\{y_F \mid F \in \mathcal{F}\}$ with $y_F \in F(x)$ ($F \in \mathcal{F}$) such that $\{y_F \mid F \in \mathcal{F}\} \subseteq B(i,n)$.

The above definitions are inspired by [1] and [3]. After these preliminaries we are in position to state the main result of the paper.

Theorem 2.1. *Let X and Y be topological spaces, and let $\mathcal{F}$ be a family of upper semicontinuous mappings from X to $\mathcal{P}_0(Y)$ which is not uniformly B-bounded. Then the following assertions are true:*

1° $S_{\mathcal{F}}(B)$ is a residual set;

2° if in addition X is of second category, then $S_{\mathcal{F}}(B)$ is of second category and hence nonempty;

3° if in addition X is a Baire space satisfying the separation axiom T_1 and without isolated points, then $S_{\mathcal{F}}(B)$ is of second category and uncountable.

Proof: 1° Since $\mathcal{F}$ is not uniformly B-bounded, choose $i \in I$ such that for every positive integer n and every nonempty open subset X_0 of X there exist a point $x \in X_0$ and a mapping $F \in \mathcal{F}$ satisfying $F(x) \subseteq Y \setminus B(i,n)$. Put

$$G_n := \{x \in X \mid \exists\, F \in \mathcal{F}:\ F(x) \subseteq Y \setminus B(i,n)\} \tag{2.1}$$

for all $n \in \mathbb{N}$. We claim that all the sets G_n ($n \in \mathbb{N}$) are open and dense in X.

Indeed, let n be any positive integer, and let x_0 be any point in G_n. According to (2.1) there exists a mapping $F \in \mathcal{F}$ such that $F(x_0) \subseteq Y \setminus B(i,n)$. Taking into consideration that F is upper semicontinuous at x_0 and that $Y \setminus B(i,n)$ is open, it follows that there exists a neighbourhood V of x_0 such that $F(x) \subseteq Y \setminus B(i,n)$ for all $x \in V$. Hence $V \subseteq G_n$. Therefore x_0 is an interior point of G_n. Since x_0 was arbitrary in G_n, we can conclude that the set G_n is open.

Suppose now that there is a positive integer n for which G_n is not dense in X. Then $X \setminus \operatorname{cl} G_n$ is open and nonempty. The choice of i ensures that there exist a point $x \in X \setminus \operatorname{cl} G_n$ and a mapping $F \in \mathcal{F}$ such that $F(x) \subseteq Y \setminus B(i,n)$. But, in view of (2.1), we have $x \in G_n \subseteq \operatorname{cl} G_n$, which is a contradiction.

Consequently, all the sets G_n ($n \in \mathbb{N}$) are open and dense, as claimed. Since

$$\bigcap_{n\in\mathbb{N}} G_n \subseteq S_{\mathcal{F}}(B),$$

if follows that $S_{\mathcal{F}}(B)$ is a residual set.

The assertions 2° and 3° are immediate consequences of Proposition 2.4 and Proposition 2.6 in [3]. □

3. Singularities and equicontinuity of families of generalized convex set-valued mappings

Assume that A is a subset of the open interval $]0,1[$ having 0 as a cluster point, and that s is a positive real number. Let X and Y be topological linear spaces, and let M be a nonempty convex subset of X. According to W. W. Breckner [4] a mapping $F : M \to \mathcal{P}_0(Y)$ is said to be (A, s)-*convex* if

$$(1-a)^s F(x) + a^s F(y) \subseteq F((1-a)x + ay)$$

whenever $a \in A$ and $x, y \in M$.

Let X and Y be topological linear spaces, let o_X and o_Y denote the zero-elements of X and Y, respectively, let M be a nonempty subset of X, and let $\mathcal{F}$ be a family of mappings from M to $\mathcal{P}_0(Y)$.

We say that $\mathcal{F}$ is *bounded at a point* $x_0 \in M$ if there exists a family $\{y_F \mid F \in \mathcal{F}\}$ with $y_F \in F(x_0)$ $(F \in \mathcal{F})$ which is bounded, i.e. for every neighbourhood V of o_Y one can find a positive integer n such that $\{y_F \mid F \in \mathcal{F}\} \subseteq nV$. If $\mathcal{F}$ is bounded at each point of M, then we say that $\mathcal{F}$ is *pointwise bounded* (on M). Any point in M at which $\mathcal{F}$ is not bounded is called a *singularity* of $\mathcal{F}$. The set of all singularities of $\mathcal{F}$ will be denoted by $S_{\mathcal{F}}$. If the set M is balanced, then we call a point $x_0 \in M$ a *weak singularity of* $\mathcal{F}$ if $\mathcal{F}$ is not bounded either at x_0 or at $-x_0$. The set of all weak singularities of $\mathcal{F}$ will be denoted by $WS_{\mathcal{F}}$. Obviously $WS_{\mathcal{F}} = S_{\mathcal{F}} \cup (-S_{\mathcal{F}})$ holds. The concept of weak singularity has already been introduced in [5].

We say that $\mathcal{F}$ is *equi-lower semicontinuous* (respectively *equi-upper semicontinuous*) *at a point* $x_0 \in M$ if for every neighbourhood V of o_Y there exists a neighbourhood U of x_0 such that

$$F(x_0) \subseteq F(x) + V \quad (\text{respectively} \quad F(x) \subseteq F(x_0) + V)$$

for all $F \in \mathcal{F}$ and all $x \in U \cap M$. We say that $\mathcal{F}$ is *equicontinuous at* x_0 if it is both equi-lower semicontinuous and equi-upper semicontinuous at this point.

Proposition 3.1. *Let X and Y be topological linear spaces, let $\mathcal{B}$ be a neighbourhood base at o_Y composed of closed sets, and let $B : \mathcal{B} \times \mathbb{N} \to 2^Y$ be the mapping defined by*

$$B(V, n) := nV \quad \textit{for all} \quad (V, n) \in \mathcal{B} \times \mathbb{N}.$$

Further let $\mathcal{F}$ be a family of (A, s)-convex mappings from a set $M \in \mathcal{O}_{ac}(X)$ to $\mathcal{P}_0(Y)$, and let $\mathcal{G} := \mathcal{F} \cup \{G_F \mid F \in \mathcal{F}\}$, where $G_F : M \to \mathcal{P}_0(Y)$ is the mapping defined by $G_F(x) := F(-x)$ $(F \in \mathcal{F})$. Then the following assertions are true:

1° *if all mappings in $\mathcal{F}$ are upper semicontinuous on M, then all mappings in $\mathcal{G}$ are upper semicontinuous on M, too;*

2° *$\mathcal{F}$ is bounded at a point $x_0 \in M$ if and only if $\mathcal{F}$ is B-bounded at x_0;*

3° *$WS_{\mathcal{F}} = S_{\mathcal{G}}(B)$;*

4° *if $\mathcal{F}$ is bounded and equi-lower semicontinuous at o_X, then $\mathcal{G}$ is uniformly B-bounded on M;*

5° *if $\mathcal{G}$ is uniformly B-bounded on M, and*

$$\frac{1}{\lambda}F(o_X) \subseteq F(o_X) \subseteq \lambda F(o_X) \quad \textit{for all} \quad F \in \mathcal{F} \quad \textit{and all} \quad \lambda \in \,]0,1],$$

then $\mathcal{F}$ is equicontinuous at o_X.

Proof: The assertions 1°, 2° and 3° are obvious.

4° Let V be any closed neighbourhood of o_Y. Choose a balanced neighbourhood V_0 of o_Y such that $V_0+V_0 \subseteq V$. Since $\mathcal{F}$ is bounded at o_X, there exists a bounded family $\{x_F \mid F \in \mathcal{F}\}$ with $x_F \in F(o_X)$ $(F \in \mathcal{F})$. Therefore one can find a positive integer n so that

$$\{x_F \mid F \in \mathcal{F}\} \subseteq nV_0.$$

On the other hand, taking into consideration that $\mathcal{F}$ is equi-lower semicontinuous at o_X, we can select a balanced neighbourhood U of o_X such that

$$F(o_X) \subseteq F(x) + nV_0 \quad \text{for all} \quad F \in \mathcal{F} \quad \text{and all} \quad x \in U \cap M.$$

Put $M_0 := \operatorname{int}(U \cap M)$. Then M_0 is a nonempty open subset of M. Let x be any point in M_0. For every $F \in \mathcal{F}$ we have

$$x_F \in F(o_X) \subseteq F(x) + nV_0 \quad \text{and} \quad x_F \in F(o_X) \subseteq F(-x) + nV_0.$$

Therefore, for every $F \in \mathcal{F}$ we can select elements $y_F \in F(x)$ and $z_F \in F(-x) = G_F(x)$ such that $x_F \in y_F + nV_0$ and $x_F \in z_F + nV_0$. Consequently

$$y_F \in x_F - nV_0 \subseteq nV_0 + nV_0 \subseteq nV$$

and

$$z_F \in x_F - nV_0 \subseteq nV_0 + nV_0 \subseteq nV.$$

Hence

$$\{y_F \mid F \in \mathcal{F}\} \cup \{z_F \mid F \in \mathcal{F}\} \subseteq nV = B(V,n).$$

Since x was arbitrary in M_0, we can conclude that $\mathcal{G}$ is uniformly B-bounded on M.

5° Let V be any neighbourhood of o_Y. Select a balanced neighbourhood V_0 of o_Y such that $V_0+V_0 \subseteq V$ and a closed neighbourhood W of o_Y such that $W \subseteq V_0$. Since the family $\mathcal{G}$ is uniformly B-bounded on M, there exist a positive integer n and a nonempty open subset M_0 of M satisfying the following condition: for each point $x \in M_0$ one can select the families $\{y_F \mid F \in \mathcal{F}\}$ and $\{z_F \mid F \in \mathcal{F}\}$ with $y_F \in F(x)$, $z_F \in F(-x)$ $(F \in \mathcal{F})$ such that

$$\{y_F \mid F \in \mathcal{F}\} \cup \{z_F \mid F \in \mathcal{F}\} \subseteq nW. \tag{3.1}$$

Let x_0 be any point in M_0. The choice of M_0 ensures that $-x_0 \in M_0$. Since M_0 is open, we can find a balanced neighbourhood U of o_X such that $x_0 + U \subseteq M_0$ and $-x_0+U \subseteq M_0$. Finally, choose $a \in A$ so that $a^s n/2^s < 1$, and put $U_0 := aU$.

Let x be an arbitrary point in U. From the equality

$$ax = \frac{a}{2}(x_0 + x) + \frac{a}{2}(-x_0 + x) + (1-a)o_X,$$

we get

$$\left(\frac{a}{2}\right)^s F(x_0 + x) + \left(\frac{a}{2}\right)^s F(-x_0 + x) + (1-a)^s F(o_X) \subseteq F(ax)$$

for all $F \in \mathcal{F}$. Taking into account that

$$F(o_X) \subseteq (1-a)^s F(o_X) \quad (F \in \mathcal{F}),$$

we obtain

$$\left(\frac{a}{2}\right)^s F(x_0 + x) + \left(\frac{a}{2}\right)^s F(-x_0 + x) + F(o_X) \subseteq F(ax) \tag{3.2}$$

for all $F \in \mathcal{F}$. But $x_0 + x$ and $-x_0 + x$ are points in M_0. Therefore there exist families $\{y_F \mid F \in \mathcal{F}\}$ and $\{z_F \mid F \in \mathcal{F}\}$ such that $y_F \in F(x_0 + x)$, $z_F \in F(-x_0 + x)$ $(F \in \mathcal{F})$ and which satisfy (3.1).

Let F be any mapping in $\mathcal{F}$. Taking into account (3.2) we get

$$\left(\frac{a}{2}\right)^s y_F + \left(\frac{a}{2}\right)^s z_F + F(o_X) \subseteq F(ax).$$

Consequently

$$\begin{aligned} F(o_X) &\subseteq F(ax) - \left(\frac{a}{2}\right)^s y_F - \left(\frac{a}{2}\right)^s z_F \subseteq \\ &\subseteq F(ax) - \left(\frac{a}{2}\right)^s nW - \left(\frac{a}{2}\right)^s nW \subseteq \\ &\subseteq F(ax) - \left(\frac{a}{2}\right)^s nV_0 - \left(\frac{a}{2}\right)^s nV_0 \subseteq F(ax) + V_0 + V_0 \subseteq \\ &\subseteq F(ax) + V. \end{aligned}$$

Hence we have proved that

$$F(o_X) \subseteq F(x) + V \quad \text{for all} \quad F \in \mathcal{F} \quad \text{and all} \quad x \in U_0.$$

Since V was an arbitrary neighbourhood of o_Y, we can conclude that $\mathcal{F}$ is equi-lower semicontinuous at o_X.

Let again x be any point in U. From the equality

$$o_X = \frac{a}{2(1+a)}(x_0 - x) + \frac{a}{2(1+a)}(-x_0 - x) + \frac{1}{1+a}(ax),$$

we get

$$\left(\frac{a}{2}\right)^s F(x_0 - x) + \left(\frac{a}{2}\right)^s F(-x_0 - x) + F(ax) \subseteq (1+a)^s F(o_X)$$

for all $F \in \mathcal{F}$. Taking into account that

$$(1+a)^s F(o_X) \subseteq F(o_X) \quad (F \in \mathcal{F}),$$

we obtain

$$\left(\frac{a}{2}\right)^s F(x_0 - x) + \left(\frac{a}{2}\right)^s F(-x_0 - x) + F(ax) \subseteq F(o_X) \tag{3.3}$$

for all $F \in \mathcal{F}$. But $x_0 - x$ and $-x_0 - x$ are points in M_0. Therefore there exist families $\{y'_F \mid F \in \mathcal{F}\}$ and $\{z'_F \mid F \in \mathcal{F}\}$ such that $y'_F \in F(x_0 - x)$, $z'_F \in F(-x_0 - x)$ $(F \in \mathcal{F})$ and

$$\{y'_F \mid F \in \mathcal{F}\} \cup \{z'_F \mid F \in \mathcal{F}\} \subseteq nW. \tag{3.4}$$

Let F be any mapping in $\mathcal{F}$. From (3.3) we obtain

$$\left(\frac{a}{2}\right)^s y'_F + \left(\frac{a}{2}\right)^s z'_F + F(ax) \subseteq F(o_X).$$

Taking now into account relation (3.4) we get

$$\begin{aligned} F(ax) &\subseteq F(o_X) - \left(\frac{a}{2}\right)^s y'_F - \left(\frac{a}{2}\right)^s z'_F \subseteq \\ &\subseteq F(o_X) - \left(\frac{a}{2}\right)^s nW - \left(\frac{a}{2}\right)^s nW \subseteq \\ &\subseteq F(o_X) - \left(\frac{a}{2}\right)^s nV_0 - \left(\frac{a}{2}\right)^s nV_0 \subseteq F(o_X) + V_0 + V_0 \subseteq \\ &\subseteq F(o_X) + V. \end{aligned}$$

Hence we have proved that

$$F(x) \subseteq F(o_X) + V \quad \text{for all} \quad F \in \mathcal{F} \quad \text{and all} \quad x \in U_0.$$

Since V was an arbitrary neighbourhood of o_Y, we can conclude that $\mathcal{F}$ is equi-upper semicontinuous at o_X. Consequently, $\mathcal{F}$ is equicontinuous at o_X.

□

Theorem 3.2. *Let X and Y be topological linear spaces, and let $\mathcal{F}$ be a family of upper semicontinuous (A,s)-convex mappings from a set $M \in \mathcal{O}_{ac}(X)$ to $\mathcal{P}_0(Y)$ which is not equicontinuous at o_X and satisfies the condition*

$$\frac{1}{\lambda} F(o_X) \subseteq F(o_X) \subseteq \lambda F(o_X) \quad \textit{for all} \quad F \in \mathcal{F} \quad \textit{and all} \quad \lambda \in \,]0,1].$$

Then the following assertions are true:

1° *$WS_{\mathcal{F}}$ is residual in M;*

2° *if in addition X is of second category, then $WS_{\mathcal{F}}$ is of second category in M and hence nonempty;*

3° *if in addition X is of second category and satisfies the separation axiom T_1, then $WS_{\mathcal{F}}$ is of second category in M and uncountable.*

Proof: Follows immediately from Theorem 2.1 and Proposition 3.1. □

Remark. The assertion 1° of Theorem 3.2 does not remain true if we replace the set $WS_{\mathcal{F}}$ by $S_{\mathcal{F}}$. This is shown by the following example. Let $A :=]0,1[$, and let $s := 1$. For each positive integer n, define the function $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) := \begin{cases} 0 & \text{if } x < 0 \\ nx & \text{if } x \geq 0, \end{cases}$$

and then define the mapping $F_n : \mathbb{R} \to \mathcal{P}_0(\mathbb{R})$ by $F_n(x) := [f_n(x), \infty[$. Obviously, the family $\mathcal{F} := \{F_n \mid n \in \mathbb{N}\}$ satisfies the conditions of Theorem 3.2. Nevertheless $S_{\mathcal{F}}$ is not residual because $S_{\mathcal{F}} =]0, \infty[$.

Theorem 3.3. *Let X be a topological linear space of second category, let Y be a topological linear space, and let $\mathcal{F}$ be a pointwise bounded family of upper semi-continuous (A,s)-convex mappings from a set $M \in \mathcal{O}_{ac}(X)$ to $\mathcal{P}_0(Y)$ satisfying the condition*

$$\frac{1}{\lambda} F(o_X) \subseteq F(o_X) \subseteq \lambda F(o_X) \quad \text{for all} \quad F \in \mathcal{F} \quad \text{and all} \quad \lambda \in]0,1].$$

Then $\mathcal{F}$ is equicontinuous at o_X.

Proof: Follows immediately from Theorem 3.2. □

4. Singularities and equicontinuity of families of generalized convex real-valued functions

Let X be a linear space, and let M be a nonempty convex subset of X. Assume that A is a subset of $]0,1[$ having 0 as a cluster point, and that s is a positive real number. Having in mind the definitions of the s-convex and rationally s-convex functions given by W.W. Breckner [2], we say that a function $f : M \to \mathbb{R}$ is (A,s)-*convex* if

$$f((1-a)x + ay) \leq (1-a)^s f(x) + a^s f(y)$$

whenever $a \in A$ and $x, y \in M$.

It is easily seen that if $f : M \to \mathbb{R}$ is an (A,s)-convex function, then the mapping $F : M \to \mathcal{P}_0(\mathbb{R})$ defined by $F(x) := [f(x), \infty[$ is (A,s)-convex, too.

Let X and Y be topological linear spaces, let o_X and o_Y be the zero-elements of X and Y, respectively, let M be a nonempty subset of X, and let $\mathcal{F}$ be a family of mappings from M into Y.

If x_0 is a point in M, then $\mathcal{F}$ is said to be:

(i) *equicontinuous at* x_0 if for every neighbourhood V of o_Y there exists a neighbourhood U of x_0 such that

$$\{f(x) - f(x_0) \mid f \in \mathcal{F}\} \subseteq V \quad \text{for every} \quad x \in U \cap M;$$

(ii) *bounded at* x_0 if the set $\{f(x_0) \mid f \in \mathcal{F}\}$ is bounded, i.e. for every neighbourhood V of o_Y there exists a positive integer n such that

$$\{f(x_0) \mid f \in \mathcal{F}\} \subseteq nV.$$

If $\mathcal{F}$ is equicontinuous (respectively bounded) at each point of M, then $\mathcal{F}$ is called *equicontinuous* (respectively *pointwise bounded*) *on* M.

Any point in M at which $\mathcal{F}$ is not bounded is said to be a *singularity* of $\mathcal{F}$. The set of all singularities of $\mathcal{F}$ is denoted by $S_{\mathcal{F}}$.

If M is balanced, then we say that a point $x_0 \in M$ is a *weak singularity* of $\mathcal{F}$ if the set

$$\{f(x_0) \mid f \in \mathcal{F}\} \cup \{f(-x_0) \mid f \in \mathcal{F}\}$$

is unbounded.

The set of all weak singularities of $\mathcal{F}$ is denoted by $WS_{\mathcal{F}}$. Obviously, $WS_{\mathcal{F}} = S_{\mathcal{F}} \cup (-S_{\mathcal{F}})$ holds.

The next theorem is an improvement of Theorem 3.1 in [5].

Theorem 4.1. *Let X be a topological linear space, and let $\mathcal{F}$ be a family of lower semicontinuous (A, s)-convex functions from a set $M \in \mathcal{O}_{ac}(X)$ to $\mathbb{R}$ which is not equicontinuous at o_X. If either $s \in]0, 1[$, or $s = 1$ and $\mathcal{F}$ is bounded at o_X, then the following assertions are true:*

1° *$WS_{\mathcal{F}}$ is residual in M;*

2° *if in addition X is of second category, then $WS_{\mathcal{F}}$ is of second category in M and hence nonempty;*

3° *if in addition X is of second category and satisfies the separation axiom T_1, then $WS_{\mathcal{F}}$ is of second category in M and uncountable.*

Proof: Construct a new family $\mathcal{G} := \{G_f \mid f \in \mathcal{F}\}$ of set-valued mappings, where $G_f : M \to \mathcal{P}_0(\mathbb{R})$ is defined by

$$G_f(x) := \begin{cases} [f(x), \infty[& \text{if } \ s \in]0, 1[\\ [f(x) - f(o_X), \infty[& \text{if } \ s = 1. \end{cases}$$

It is immediately seen that $\mathcal{G}$ is a family of upper semicontinuous (A, s)-convex mappings from M to $\mathcal{P}_0(\mathbb{R})$ which is not equicontinuous at o_X and satisfies the condition

$$\frac{1}{\lambda} G(o_X) \subseteq G(o_X) \subseteq \lambda G(o_X) \ \text{ for all } \ G \in \mathcal{G} \ \text{ and all } \ \lambda \in]0, 1].$$

On the other hand, it is clear that $\mathcal{F}$ is bounded at a point $x_0 \in M$ if and only if $\mathcal{G}$ is bounded at x_0. Hence $WS_{\mathcal{F}} = WS_{\mathcal{G}}$. Therefore, the assertions of the theorem are consequences of Theorem 3.2. $\square$

Theorem 4.2. *Let X be a topological linear space of second category, let $s \in]0,1]$, and let $\mathcal{F}$ be a pointwise bounded family of lower semicontinuous (A,s)-convex functions from a set $M \in \mathcal{O}_{ac}(X)$ to $\mathbb{R}$. Then $\mathcal{F}$ is equicontinuous on M.*

Proof: Let x_0 be any point in M. We choose a balanced neighbourhood U of x_0 such that $x_0 + U \subseteq M$. Then $M_0 := \operatorname{int}(\operatorname{conv} U)$ lies in $\mathcal{O}_{ac}(X)$ and satisfies $x_0 + M_0 \subseteq M$. To each $f \in \mathcal{F}$ we assign the function $g_f : M_0 \to \mathbb{R}$ defined by $g_f(x) := f(x_0 + x)$. Obviously, all the functions g_f $(f \in \mathcal{F})$ are lower semicontinuous and (A,s)-convex. Moreover, the family $\mathcal{G} := \{g_f \mid f \in \mathcal{F}\}$ is bounded on M_0. Therefore, in view of Theorem 4.1, the family $\mathcal{G}$ must be equicontinuous at o_X. In other words, $\mathcal{F}$ must be equicontinuous at x_0. □

5. Singularities and uniform boundedness of families of closed convex processes

Let X and Y be normed linear spaces. In the sequel we shall denote by $B_X(r)$ (respectively by $B_Y(r)$) the closed ball centered at o_X (respectively at o_Y) and of radius r. The balls $B_X(1)$ and $B_Y(1)$ will be simply denoted by B_X and B_Y, respectively.

A mapping $F : X \to 2^Y$ is called a *convex process* if the following conditions are satisfied:

(i) $o_Y \in F(o_X)$;
(ii) $F(\lambda x) = \lambda F(x)$ for all $x \in X$ and all $\lambda \in]0, \infty[$;
(iii) $F(x_1) + F(x_2) \subseteq F(x_1 + x_2)$ for all $x_1, x_2 \in X$.

The norm $\|F\|$ of a convex process F is defined by

$$\|F\| := \sup_{x \in \operatorname{Dom}(F) \cap B_X} \inf_{y \in F(x)} \|y\|.$$

The mapping F is called a *closed convex process* if it is a convex process whose graph is closed.

Proposition 5.1. *Let X and Y be normed linear spaces, and let $\mathcal{F}$ be a family of convex processes from X to $\mathcal{P}_0(Y)$. Then the following assertions are equivalent:*

1° *$\mathcal{F}$ is equicontinuous at o_X;*

2° *there exists a real number k such that $\|F\| \le k$ for all $F \in \mathcal{F}$.*

Proof: $1^\circ \Rightarrow 2^\circ$ Since $\mathcal{F}$ is equicontinuous at o_X, there exists a positive real number r such that

$$F(o_X) \subseteq F(x) + B_Y \quad \text{for all} \quad F \in \mathcal{F} \quad \text{and all} \quad x \in B_X(r).$$

We shall prove that

$$\|F\| \le \frac{1}{r} \quad \text{for all} \quad F \in \mathcal{F}. \tag{5.1}$$

Let F be any mapping in $\mathcal{F}$, and let x be any point in B_X. Because of $\|rx\| \leq r$, we have

$$F(o_X) \subseteq F(rx) + B_Y = rF(x) + B_Y.$$

Since $o_Y \in F(o_X)$, we can choose the points $y_0 \in F(x)$ and $z_0 \in B_Y$ such that $ry_0 + z_0 = o_Y$. Hence $\|y_0\| = 1/r\|z_0\| \leq 1/r$. Therefore we have

$$\inf_{y \in F(x)} \|y\| \leq \frac{1}{r}.$$

Since x was arbitrarily chosen in B_X, we can conclude that $\|F\| \leq 1/r$. Consequently (5.1) holds.

$2^\circ \Rightarrow 1^\circ$ Let V be any neighbourhood of o_Y. We choose a positive real number r such that $B_Y(r) \subseteq V$. Put $\alpha := \frac{k+1}{r}$ and $U := \frac{1}{\alpha} B_X$. We shall prove that

$$F(o_X) \subseteq F(x) + V \quad \text{and} \quad F(x) \subseteq F(o_X) + V \tag{5.2}$$

whenever $F \in \mathcal{F}$ and $x \in U$.

Let F be any mapping in $\mathcal{F}$, and let x be any point in U. Then $\alpha x \in B_X$. Since $\|F\| \leq k$, we can find a point $y_0 \in F(\alpha x)$ such that $\|y_0\| \leq k+1$. Hence we can find a point $z_0 \in F(x) \cap B_Y(r)$ such that $y_0 = \alpha z_0$. Then

$$z_0 + F(o_X) \subseteq F(x) + F(o_X) \subseteq F(x).$$

Consequently

$$F(o_X) \subseteq F(x) - z_0 \subseteq F(x) + B_Y(r) \subseteq F(x) + V.$$

On the other hand, we have $-\alpha x \in B_X$. Hence we can find a point $y_0' \in F(-\alpha x)$ such that $\|y_0'\| \leq k+1$. Therefore we can find a point $z_0' \in F(-x) \cap B_Y(r)$ such that $y_0' = \alpha z_0'$. We have

$$z_0' + F(x) \subseteq F(-x) + F(x) \subseteq F(o_X).$$

Hence

$$F(x) \subseteq F(o_X) - z_0' \subseteq F(o_X) + B_Y(r) \subseteq F(o_X) + V.$$

Consequently (5.2) holds as claimed. Therefore $\mathcal{F}$ is equicontinuous at o_X. $\square$

Theorem 5.2. *Let X be a Banach space, let Y be a normed linear space, and let $\mathcal{F}$ be a family of closed convex processes from X to $\mathcal{P}_0(Y)$ such that*

$$\sup\{\|F\| \mid F \in \mathcal{F}\} = \infty.$$

Then $WS_{\mathcal{F}}$ is an uncountable residual subset of X.

Proof: For every $F \in \mathcal{F}$ define the function $f_F : X \to \mathbb{R}$ by

$$f_F(x) := \inf_{y \in F(x)} \|y\|.$$

Since F is a closed convex process, it is easily seen that f_F is a sublinear lower semicontinuous function. Put $\mathcal{G} := \{f_F \mid F \in \mathcal{F}\}$. Obviously $\mathcal{G}$ is bounded at o_X because $f_F(o_X) = 0$ for all $F \in \mathcal{F}$.

Suppose that $\mathcal{G}$ is equicontinuous at o_X. Then we can find a positive real number δ such that $f_F(x) < 1$ for all $x \in B_X(\delta)$ and all $F \in \mathcal{F}$. Choose $F_0 \in \mathcal{F}$ such that $\|F_0\| > 2/\delta$. Then we can find a point $x_0^* \in B_X$ such that

$$\inf_{y \in F_0(x_0^*)} \|y\| > \frac{2}{\delta}.$$

Therefore we have $\|y\| > 2/\delta$ for all $y \in F_0(x_0^*)$. Put $x_0 := \delta x_0^*$. Then $x_0 \in B_X(\delta)$ and $\|y\| > 2$ for all $y \in F_0(x_0)$. Hence $f_{F_0}(x_0) \geq 2$, which is a contradiction. Summing up, we conclude that the family $\mathcal{G}$ is not equicontinuous at o_X.

Since $WS_{\mathcal{F}} = WS_{\mathcal{G}}$, the conclusion of the theorem follows immediately from Theorem 4.1. □

The next result is Theorem 2.3.1 in [1].

Theorem 5.3. *Let X be a Banach space, let Y be a normed linear space, and let $\mathcal{F}$ be a pointwise bounded family of closed convex processes from X to $\mathcal{P}_0(Y)$. Then*

$$\sup\{\|F\| \mid F \in \mathcal{F}\} < \infty.$$

PROOF: Follows immediately from Theorem 5.2. □

The following theorem is similar to Theorem 5.3, but upper semicontinuous convex processes are considered instead of closed convex ones.

Theorem 5.4. *Let X be a Banach space, let Y be a normed linear space, and let $\mathcal{F}$ be a pointwise bounded family of upper semicontinuous convex processes from X to $\mathcal{P}_0(Y)$. Then*

$$\sup\{\|F\| \mid F \in \mathcal{F}\} < \infty.$$

PROOF: Follows immediately from Theorem 3.3 and Proposition 5.1. □

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