# On the selector of twin functions

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*Abstract.* A theorem is proved which could be considered as a bridge between the combinatorics which have a beginning in the dyadic spaces theory and the partition calculus.

*Keywords:* twin functions, selector of twin functions, transfixed selector

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The aim of this paper is to prove a pure set-theoretical theorem which could be considered as a bridge between the combinatorics which have a beginning in the dyadic spaces theory (see [1]) and partition calculus (see [3]). As an application of this theorem, the proofs of Erdös-Rado Theorem [2], Strong Sequences Theorem [5] and the Bolzano-Weierstrass Method [4] will be given.

The Erdös-Rado Theorem has been used several times for proving important theorems (for more information see [3]). The same we can say about the role of the strong sequences theorem in dyadic spaces theory (see [1], [6]).

## Main theorem

Let $X$ be a set and $\phi$ be an ordinal.

A pair $(F, G)$ of two multifunctions

$$G : \phi \longrightarrow 2^X$$
$$F : X \longrightarrow 2^\phi$$

such that

(∗) for any conditions $\beta < \alpha < \phi$ there exists $b \in G(\beta)$ such that $\alpha \in F(b)$

is said to be **twin functions**.

A map $g : K \longrightarrow X,\ K \subset \phi$, is said to be **a selector of twin functions** if

(1) for any $\beta \in K$ there is $g(\beta) \in G(\beta)$,
(2) for any $\alpha, \beta \in K$; $\beta < \alpha < \phi$ implies $\alpha \in F(g(\beta))$.

Fix twin functions $(F, G)$. The main result of this note is the following:

**Theorem.** *If $\phi = (\kappa^\lambda)^+$ and $card(G(\alpha)) \leq \kappa$ for each $\alpha < \phi$ then there is a selector $g : K \longrightarrow X$ of the twin functions such that $card(K) \geq \lambda^+$.*

A selector $g : K \longrightarrow X$ is said to be **transfixed** if there is an $\alpha > \sup K$ with
$$\alpha \in \bigcap\{F(g(\beta)) : \beta \in K\}.$$
Denote by $\alpha(g)$ the least ordinal of this property.

For a given $\lambda$ let us denote by $\lambda^*$, $\lambda < \lambda^* \leq \phi$, the ordinal having the following property:

(I) if $g : K \longrightarrow X$ is a transfixed selector such that $K \subset \lambda^*$, $card(K) \leq \lambda$, then $\alpha(g) < \lambda^*$.

**Lemma 1.** *If $\lambda^* < \phi$, then there is a selector $g : K \longrightarrow X$ of the twin functions such that $card(K) \geq \lambda^+$.*

PROOF: Consider the family $\mathcal{K}$ of all transfixed selectors $g : K \longrightarrow X$ such that
$$\lambda^* \in \bigcap\{F(g(\beta)) : \beta \in K,\ K \subset \lambda^*,\ card(K) \leq \lambda\}$$
with the partial ordering
$$g_1 \leq g_2 \text{ if } \operatorname{dom} g_1 \subset \operatorname{dom} g_2 \text{ and } g_2 \mid \operatorname{dom} g_1 = g_1.$$
The set $\mathcal{K}$ is non-empty because for each $\alpha < \lambda^*$ in view of $(*)$ we have $\lambda^* \in F(G(\alpha))$. Hence there exists $a \in G(\alpha)$ such that $\lambda^* \in F(a)$. It is clear that the map $g : \{\alpha\} \longrightarrow a$, $g(\alpha) = a$, belongs to $\mathcal{K}$.

Let us observe that there are no maximal elements in $\mathcal{K}$. To see this, fix $g \in \mathcal{K}$, $g : \mathcal{K} \longrightarrow X$ and define $g_1 : K_1 \longrightarrow X$ with $K_1 = K \cup \{\alpha(g)\}$, $g_1(\beta) = g(\beta)$ for $\beta \in K$ and $g_1(\alpha(g)) = x$, where according to the condition $(*)$ one can find $x \in G(\alpha(g))$ such that $\lambda^* \in F(x)$. Since $\alpha(g) < \lambda^*$, the map $g_1 : K \longrightarrow X$ is well defined, $g_1 \in \mathcal{K}$, $g_1 \neq g$ and $g \leq g_1$.

Now let $\mathcal{L} \subset \mathcal{K}$ be a chain. Denote by $g_{\mathcal{L}} : K_{\mathcal{L}} \longrightarrow X$ a selector such that $K_{\mathcal{L}} = \bigcup\{\operatorname{dom} g : g \in \mathcal{L}\}$ and $g_{\mathcal{L}} \mid \operatorname{dom} g = g$. Observe that if $card(\mathcal{L}) \leq \lambda$, then $g_{\mathcal{L}} \in \mathcal{K}$. Since there are no maximal elements in $\mathcal{K}$, by the Zorn Lemma there is a chain $\mathcal{L} \subset \mathcal{K}$ such that $\lambda^+ \leq card(\mathcal{L})$. It is clear that $g_{\mathcal{L}}$ is a selector with $\lambda^+ \leq card(\operatorname{dom} g_{\mathcal{L}})$. □

**Lemma 2.** *If $card(G(\alpha)) \leq \kappa$, then for each $\lambda$ such that $(\kappa^\lambda)^+ \leq \phi$ we have $\lambda^* < \phi$.*

PROOF: Let us observe that if $(G, F)$ are twin functions, then for each $\kappa$ and $\lambda$ such that $(\kappa^\lambda)^+ \leq \phi$, the system $(G, F)$, for which we take $\phi = (\kappa^\lambda)^+$, is a system of twin functions.

Consider the set $\mathcal{M}$ of all transfixed selectors. By induction we shall define an increasing sequence of ordinals $\{\lambda_\alpha : \alpha < \lambda^+\}$ satisfying the following conditions:

$1^o$ $\lambda_0 = \lambda$,
$2^o$ if $\alpha$ is a limit ordinal then $\lambda_\alpha = \sup\{\lambda_\beta : \beta < \alpha\}$,
$3^o$ if $\alpha = \beta + 1$ then $\lambda_\alpha = \sup\{\alpha(g) : g \in \mathcal{M},\ \operatorname{dom} g \subset \lambda_\beta,\ card(\operatorname{dom} g) \leq \lambda\} + 1$.

Let us put $\lambda^* = \sup\{\lambda_\alpha : \alpha < \lambda^+\}$. To see that $\lambda^* < (\kappa^\lambda)^+$, let us observe that if $\lambda_\beta < (\kappa^\lambda)^+$ then the set

$$\mathcal{M}_\beta = \{g \in \mathcal{M} : \operatorname{dom} g \subset \lambda_\beta,\ card(g) \leq \lambda\}$$

has cardinality less or equal to $\kappa^\lambda$. Therefore $\lambda_{\beta+1} < (\kappa^\lambda)^+$.

Now let us verify that if $g \in \mathcal{M}$, $card(\operatorname{dom} g) \leq \lambda$ and $\operatorname{dom} g \subset \lambda^*$, then $\alpha(g) < \lambda^*$. Indeed, if $card(\operatorname{dom} g) \leq \lambda$, $\operatorname{dom} g \subset \lambda^*$, then there is $\beta < \lambda^+$ such that $\operatorname{dom} g \subset \lambda_\beta$. By our construction we have $\alpha < \lambda_{\beta+1} < \lambda^*$. □

The Theorem is an easy corollary of Lemmas 1 and 2.

Proof of the Theorem: From Lemma 2 it follows that $\lambda^* < (\kappa^\lambda)^+$. Hence, by Lemma 1, there exists a selector $g : K \longrightarrow X$ of the twin functions such that $card(K) \geq \lambda^+$. □

**Applications**

We shall prove the following theorem of P. Erdös and R. Rado [2]. By $[X]^2$ denote the family of all exactly two points subsets of $X$.

**Theorem** (Erdös-Rado [2])**.** *Suppose $\lambda$ is an infinite cardinal number and $F$ is a partition of $[X]^2$ of cardinality not greater than $\lambda$. If the cardinality of the set $X$ is greater than $2^\lambda$, then there exists a subset $\Gamma \subset X$ of cardinality greater than $\lambda$ such that the family $[\Gamma]^2$ is contained in some element of $F$.*

Proof: Order well the elements of $F$ into the size $\lambda$, i.e. $F = \{F_\beta : \beta < \lambda\}$. Order well the set $X$ into the size $(2^\lambda)^+$, i.e. $X = \{\alpha : \alpha < (2^\lambda)^+\}$. For each $\alpha < (2^\lambda)^+$ let $F_\gamma(\alpha) = \{\beta : \{\alpha, \beta\} \in F_\gamma\}$. Let $Z = \{\{F_\gamma(\alpha)\} : \alpha < (2^\lambda)^+$ and $\gamma < \lambda\}$.

Let us define the functions

$$G : (2^\lambda)^+ \longrightarrow 2^Z;\ \alpha \longmapsto \{\{F_\gamma(\alpha)\} : \gamma < \lambda\}$$

and

$$F : Z \longrightarrow 2^{(2^\lambda)^+} : \{F_\gamma(\alpha)\} \longmapsto F_\gamma(\alpha).$$

We shall show that $(F, G)$ are twin functions. For this purpose, take $\beta < \alpha$,

$$G(\beta) = \{\{F_\gamma\} : \gamma < \lambda\} \text{ and } \bigcup\{F_\gamma(\beta) : \{F_\gamma(\beta)\} \in G(\beta)\} = (2^\lambda)^+ \setminus \{\beta\}.$$

Hence we have $\alpha \in F_\gamma(\beta) = F(\{F_\gamma(\beta)\})$ for some $\gamma < \lambda$. Hence, by the Theorem there exists a selector $g : K \longrightarrow Z$ such that $\lambda^+ \leq card(K)$. From this it follows that there exist $\gamma < \lambda$ and $\Gamma \subset K$, $card(\Gamma) = \lambda^+$ such that $g(\beta) = \{F_\gamma(\beta)\}$ for each $\beta \in \Gamma$. Hence for each $\alpha$ and $\beta$ from $\Gamma$, the condition $\beta < \alpha$ implies that $\alpha \in F_\gamma(\beta)$. This means that for each $\alpha, \beta$ from $\Gamma$ we have $\{\alpha, \beta\} \in F_\gamma$. Hence $[\Gamma]^2 \subset F_\gamma$. □

Let $X$ be a set. Let $r \subset [X]^{<\omega} \times [X]^{<\omega}$. Let $S_\phi$ be a finite subset of $X$ and $H_\phi \subset X$ for $\phi < \alpha$.

**Definition.** *A sequence* $(S_\phi, H_\phi); \phi < \alpha$ *is called a* **strong sequence** *if*

$1^o$ *for each* $T, S \in [S_\phi \cup H_\phi]^{<\omega}$ *there is* $TrS$*,*

$2^o$ *for each* $\beta > \phi$ *there exist* $T, S \in [S_\beta \cup H_\phi]^{<\omega}$ *such that* $\sim (TrS)$*.*

**Theorem** (On strong sequences [1], [5], [6])**.** *Let* $X$ *be a set and* $r$ *be a relation on* $[X]^{<\omega}$*. Let* $(S_\phi, H_\phi); \phi < (\kappa^\lambda)^+$ *be a strong sequence such that* $card(H_\phi) \leq \kappa$ *for each* $\phi < (\kappa^\lambda)^+$*. Then there exists a strong sequence* $(S_\phi, T_\phi)$*;* $\phi < \lambda^+$*, where* $card(T_\phi) < \omega$ *for each* $\phi < \lambda^+$*.*

Proof: For each $H_\phi$ let

$$G(\phi) = \{T : T \subset H_\phi,\ card(T) < \omega \text{ and there exists } \beta > \phi \text{ such that } \sim (TrS_\beta)\}.$$

Let $\mathcal{X} = \{T : T \in G(\phi)$ for some $\phi\}$. Let us define the functions:

$$G : (\kappa^\lambda)^+ \longrightarrow 2^{\mathcal{X}} : \phi \longmapsto G(\phi)$$

and

$$F : \mathcal{X} \longrightarrow 2^{(\kappa^\lambda)^+} : T \longmapsto \{\beta : \sim (TrS_\beta)\}.$$

We shall show that $(F, G)$ are twin functions. Let $\beta < \alpha < (\kappa^\lambda)^+$, then there exists $T \in G(\beta)$ such that $\sim (TrS_\alpha)$. Hence $\alpha \in F(T)$. By the theorem there exists a selector $g : K \longrightarrow \mathcal{X}$, $\lambda^+ \leq card(K)$ such that

$1^o$ for each $\beta \in K$ we have $g(\beta) \in G(\beta)$,

$2^o$ for each $\alpha, \beta \in K$; $\beta < \alpha$ implies $\alpha \in F(g(\beta))$.

By $1^o$ we have that $g(\beta) \in [H_\beta]^{<\omega}$. By $2^o$ we have that for $\alpha > \beta$, $\sim (S_\alpha r g(\beta))$. Hence $(S_\alpha, g(\alpha)); \alpha \in K$ is a strong sequence. □

In [4] the following theorem has been proved.

**Theorem** (The Bolzano-Weierstrass Method)**.** *Suppose* $\lambda$ *and* $\kappa$ *are cardinal numbers such that* $\kappa > 1$ *and* $\lambda$ *is infinite. Assume that* $Y = \{y_\alpha : \alpha < (\kappa^\lambda)^+\}$ *is a set of different indexed points. If for any* $\alpha < (\kappa^\lambda)^+$ *the family*

$$F_{y_\alpha} = \{F_{y_\alpha}(\beta) : \beta < \kappa\}$$

*consists of pairwise disjoint subsets of* $X$ *such that*

$$(*) \qquad \bigcup F_{y_\alpha} \cup \{y_\alpha\} \subset \bigcap \{\bigcup F_{y_\gamma} : \gamma < \alpha\},$$

*then there exist a function* $f : \lambda^+ \longrightarrow \kappa$ *and an indexed subset* $\{p_\gamma : \gamma < \lambda^+\} \subset Y$ *such that any condition* $\beta < \tau < \lambda^+$ *implies* $p_\tau \in F_{p_\beta}(f(\beta))$*.*

Proof: Let us define the set

$$X = \{F_{y_\alpha}(\beta) : \alpha < (\kappa^\lambda)^+ \text{ and } \beta < \kappa\}.$$

Let $G : (\kappa^\lambda)^+ \longrightarrow 2^X : \alpha \longmapsto \{F_\gamma(\alpha) : \gamma < \kappa\}$ and let $F : X \longrightarrow 2^{(\kappa^\lambda)^+} : F_{y_\alpha}(\beta) \longmapsto \{\gamma : y_\gamma \in F_{y_\gamma}(\beta)\}$.
We shall show that $(F, G)$ are twin functions. By $(*)$ we have that for each $\beta < \alpha$, $y_\alpha \in \bigcup F_{y_\beta}$. Hence $y_\alpha \in F_{y_\beta}(\gamma)$ for some $\gamma < \kappa$. Then $\alpha \in F(F_{y_\beta}(\gamma))$. We have $card(G(\alpha)) \leq \kappa$ for each $\alpha < (\kappa^\lambda)^+$. Then, by the theorem, there exists a selector $g : K \longrightarrow X$, $\lambda^+ \leq card(K)$ such that

$1^o$ for each $\beta \in K$ there is $g(\beta) \in G(\beta)$

and

$2^o$ for each $\alpha, \beta \in K$ the condition $\beta < \alpha$ implies $\alpha \in F(g(\beta))$.

From this it follows that

$$\text{for each } \alpha \in K,\ \alpha \in \bigcap F(g(\beta)), \quad \text{where } \beta \in K, \beta < \alpha.$$

The selector $g : K \longrightarrow X$ and any increasing map $h$ from $\lambda^+$ into $K$ define a map $f : \lambda^+ \longrightarrow \kappa$ in the following way: $f(\beta) = \gamma$ if $g(h(\beta)) = F_{y_{h(\beta)}}(\gamma)$ and a set $\{p_\gamma : \gamma < \lambda^+$, where $p_\gamma = y_{h(\gamma)}\}$. $\square$

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