On some fan-tightness type properties

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Abstract. Properties similar to countable fan-tightness are introduced and compared to countable tightness and countable fan-tightness. These properties are also investigated with respect to function spaces and certain classes of continuous mappings.

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Classification: 54A25, 54C35

In this paper, all spaces are assumed to be Hausdorff. We denote by $\mathbb{R}$ the set of real numbers; βX denotes the Stone-Čech compactification of a Tychonoff space X and $C_p(X)$ stands for the space of all real-valued continuous functions on X with the topology of pointwise convergence. A basic open neighborhood of a function $f \in C_p(X)$ is of the form $W(x_1, \dots, x_k;\ f;\ \epsilon) = \{g \in C_p(X) : |f(x_i) - g(x_i)| < \epsilon,\ i = 1, 2, \dots, k\}$, where $k \in \omega$, $x_i \in X$ and $\epsilon > 0$. We denote by $C_p^0(X)$ the set of all bounded continuous functions on X equipped with the topology of pointwise convergence. A cover γ of X is said to be an *ω-cover* if for any finite subset F of X there is a $G \in \gamma$ such that $F \subseteq G$. The notion of countable fan-tightness was introduced in [1]: a space X is said to have *countable fan-tightness* (denoted $vet(X) \leq \omega$) if for each point x in X and any countable family $\{A_n\}_{n\in\omega}$ of subsets of X satisfying $x \in \bigcap\{\overline{A_n} : n \in \omega\}$, there exist finite sets $H_n \subseteq A_n$ such that $x \in \overline{\bigcup\{H_n : n \in \omega\}}$. A space X is said to have *countable strong fan-tightness* if for each $x \in X$ and for each countable family $\{A_n : n \in \omega\}$ of subsets of X such that $x \in \bigcap\{\overline{A_n} : n \in \omega\}$, there exist $a_i \in A_i$ such that $x \in \overline{\{a_i : i \in \omega\}}$.

A space X is said to *have property $vet^*(X) \leq \omega$* if for each point x in X and any countable family $\{A_n\}_{n\in\omega}$ of subsets of X satisfying $x \in \bigcap\{\overline{A_n} : n \in \omega\}$, there exist sets $H_n \subseteq A_n$ with $|H_n| \leq n$ such that $x \in \overline{\bigcup\{H_n : n \in \omega\}}$. Clearly, every space X of countable strong fan-tightness has $vet^*(X) \leq \omega$, and $vet^*(X) \leq \omega$ in turn implies that the fan-tightness of X is countable.

The following theorems were proved in [1] and [5], respectively:

Theorem 1 (Arhangelskii)**.** *For a Tychonoff space X, the following are equivalent:*

(a) *$vet\ C_p(X) \leq \omega$;*
(b) *for each $n \in \omega$, X^n is a Hurewicz space.*

Theorem 2 (Sakai). *For a Tychonoff space X, the following are equivalent:*

(a) *$C_p(X)$ has countable strong fan-tightness;*

(b) *for each sequence $\{\gamma_n : n \in \omega\}$ of open ω-covers of X there exist $U_n \in \gamma_n$ such that $\{U_n : n \in \omega\}$ is an ω-cover of X.*

Lemma 3. *For a topological space X, the following are equivalent:*

(a) *$vet^*(X) \leq \omega$;*

(b) *for each mapping $\phi : \omega \to \omega$ such that $\phi(n) \geq n$ for each $n \in \omega$, for each point $x \in X$ and for each (decreasing) family $\{A_n\}_{n\in\omega}$ of subsets of X satisfying $x \in \bigcap\{\overline{A_n} : n \in \omega\}$, there exist $H_i \subseteq A_i$ such that $x \in \overline{\{H_n : n \in \omega\}}$ and $|H_n| \leq \phi(n)$;*

(c) *for each point $x \in X$ and for each decreasing family $\{A_n\}_{n\in\omega}$ of subsets of X satisfying $x \in \bigcap\{\overline{A_n} : n \in \omega\}$, there exist $a_i \in A_i$ such that $x \in \overline{\{a_n : n \in \omega\}}$.*

Proof: (a) $\Rightarrow$ (b) is trivial.

(b) $\Rightarrow$ (c). Assume (b) and fix $x \in X$ and a decreasing family $\{A_n\}_{n\in\omega}$ of subsets of X such that $x \in \bigcap\{\overline{A_n} : n \in \omega\}$. Consider the subset $\{n_k : k \in \omega\}$ of ω defined as follows: $n_1 = 1$ and $n_k = n_{k-1} + \phi(k)$. Since $x \in \overline{A_{n_k}}$ for each k, select $H_k \subseteq A_{n_k}$ with $|H_k| \leq \phi(k)$ and $x \in \overline{\bigcup\{H_n : n \in \omega\}}$. Without loss of generality it may be assumed that $H_k = \{x^k_1, x^k_2, \dots, x^k_{\phi(k)}\}$. For $i \in \omega$ such that $n_{k-1} < i \leq n_k$, put $a_i = x^k_{i-n_{k-1}}$. Clearly, $a_i \in A_i$ and $x \in \overline{\{a_i : i \in \omega\}} = \overline{\bigcup\{H_n : n \in \omega\}}$.

(c) $\Rightarrow$ (a). Assume (c) and fix $\{B_n\}_{n\in\omega}$ with $x \in \bigcap\{\overline{B_n} : n \in \omega\}$. Put $A_i = \bigcup\{B_k : k \geq i\}$. The family $\{A_i : i \in \omega\}$ satisfies (c); select $a_i \in A_i$ with $x \in \overline{\{a_n : n \in \omega\}}$ and put $H_i = B_i \cap \{a_n : 1 \leq n \leq i\}$. Clearly, $|H_i| \leq i$ and $x \in \overline{\bigcup\{H_n : n \in \omega\}}$. □

Proposition 4. *Let X be a Fréchet space of countable fan-tightness. Then $vet^*(X) \leq \omega$.*

Proof: Fix a decreasing sequence $\{A_n : n \in \omega\}$ of subsets of X and a point $x \in X$ such that $x \in \bigcap\{\overline{A_n} \setminus A_n : n \in \omega\}$. There exist finite sets $H_n \subseteq A_n$ with $x \in \overline{\bigcup\{H_n : n \in \omega\}}$. Choose a sequence $\{a_n : n \in \omega\} \subseteq \bigcup\{H_n : n \in \omega\}$ converging to x and define a countable subset of A_1 as follows: for each $i \in \omega$, put $k_i = \max\{n : a_n \in H_i\}$ if $\{a_n\}_{n\in\omega} \cap H_i \neq \emptyset$ and put $b_i = a_{k_i}$, if $\{a_n\}_{n\in\omega} \cap H_i \neq \emptyset$ and $b_i = b_{i+1}$ otherwise. Since each H_i is finite, the sequence $\{b_i\}_{i\in\omega}$ is well-defined and $b_i \in A_i$ for each i. Since $\{b_i\}_{i\in\omega}$ contains a subsequence of $\{a_n\}_{n\in\omega}$, we have $x \in \overline{\{b_i : i \in \omega\}}$ and therefore $vet^*(X) \leq \omega$. □

Theorem 5. *Let X be a Tychonoff space. Then the following are equivalent:*

(a) *$vet^*C_p(X) \leq \omega$;*

(b) *for every sequence $\{\gamma_n : n \in \omega\}$ of open ω-covers of X there exist $\lambda_n \subseteq \gamma_n$ such that $|\lambda_n| \leq n$ and $\bigcup\{\lambda_n : n \in \omega\}$ is an ω-cover of X.*

Proof: Assume (a). Fix a sequence $\{\gamma_n : n \in \omega\}$ of open ω-covers of X and for each natural number n put $A_n = \{f \in C_p(X) : \exists U \in \gamma_n$ such that $f(X \setminus U) = \{0\}\}$. Put $f^*(x) = 1$ for each $x \in X$. Clearly, $f^* \in \overline{A_n}$ for each n. Choose $H_n \subseteq A_n$ such that $f^* \in \overline{\bigcup\{H_n : n \in \omega\}}$ and $|H_n| \leq n$. For each n and for each $f \in H_n$ fix $U_f \in \gamma_n$ such that $f(X \setminus U_f) = \{0\}$ and put $\lambda_n = \{U_f : f \in H_n\}$. To show that $\bigcup\{\lambda_n : n \in \omega\}$ is an ω-cover of X, fix $x_1, \dots, x_k \in X$. There exist $n \in \omega$ and $f \in H_n$ such that $f \in W(x_1, \dots, x_k;\ f^*;\ 1/2)$. Thus for each $i = 1, \dots, k$ we have $f(x_i) > \frac{1}{2}$ and $x_i \in U_f$.

Assume (b) and fix $f \in C_p(X)$ and a sequence $\{A_n\}_{n\in\omega}$ of subsets of X such that $f \in \bigcap\{\overline{A_n} : n \in \omega\}$. Put $\gamma_n = \{(g - f)^{-1}(-\frac{1}{n},\ \frac{1}{n}) : g \in A_n\}$. To show that γ_n is an ω-cover of X, fix $x_1, \dots, x_k \in X$. Since $W(x_1, \dots, x_k;\ f;\ \frac{1}{n}) \cap A_n \neq \emptyset$, there exists $g \in A_n$ such that $x_i \in (g - f)^{-1}(-\frac{1}{n},\ \frac{1}{n})$ for each $i = 1, \dots, k$.
Case 1. There exists a subsequence $\{n_k\}_{k\in\omega}$ such that $X \in \gamma_{n_k}$ for each k. Fix $g_{n_k} \in A_{n_k}$ such that $X = (g_{n_k} - f)^{-1}(-\frac{1}{n_k}, \frac{1}{n_k})$. It is easy to see that $f \in \overline{\{g_{n_k} : k \in \omega\}}$.
Case 2. X is an element of finitely many members of $\{\gamma_n\}$. Without loss of generality we may assume that $X \notin \gamma_n$ for each n. Choose $\lambda_n \subseteq \gamma_n$ with $|\lambda_n| \leq n$ and for each $U \in \lambda_n$, fix $g_U \in A_n$ with $U = (g_U - f)^{-1}(-\frac{1}{n}, \frac{1}{n})$. Put $H_n = \{g_U : U \in \lambda_n\}$. Fix a basic open neighborhood $W(x_1, \dots, x_k;\ f;\ \frac{1}{n})$ of f. Since $X \notin \bigcup\{\lambda_n : n \in \omega\}$, $|\{U \in \bigcup\{\lambda_n : n \in \omega\} : x_i \in U$, for each $i = 1, \dots, k\}| = \omega$ and there exists $N \geq n$ such that for some $U \in \lambda_N$, $x_i \in (g_U - f)^{-1}(-\frac{1}{N}, \frac{1}{N})$ for all i. Hence $g_U \in W(x_1, \dots, x_k;\ f;\ \frac{1}{n}) \cap H_N$ and $f \in \overline{\bigcup\{H_n : n \in \omega\}}$. This completes the proof. □

Question 1. Does $vet^*(X) \leq \omega$ imply that X has countable strong fan-tightness? In particular, are these two properties equivalent for function spaces (equivalently, are condition (b) of Theorem 2 and condition (b) of Theorem 5 equivalent)?

Corollary 6. *Condition (b) of Theorem 5 is preserved by t-equivalence.*

Remark. It can be shown that a space X satisfies condition (b) of Theorem 5 if and only if for each finite power X^k of X and for each sequence $\{\gamma_n : n \in \omega\}$ of open covers of X^k there exist $\lambda_n \subseteq \gamma_n$ such that $|\lambda_n| \leq n$ and $\bigcup\{\lambda_n : n \in \omega\}$ is a cover of X^k. It can also be shown that every Tychonoff space X satisfying condition (b) of Theorem 5 is zero-dimensional.

Example 7. Countable fan-tightness does not imply that $vet^*(X) \leq \omega$: Consider $X = C_p(0,1)$, where $(0,1)$ is the open unit interval. By Arhangelskii's theorem, $vet\ C_p(0,1) \leq \omega$. It is easy to see, however, that the sequence $\{\gamma_n\}$ of open covers of $(0,1)$, where $\gamma_n = \{\bigcup\{(a_i,\ b_i) : 1 \leq i \leq k\} : k \in \omega,\ a_i, b_i \in (0,1)$, and $\sum_{i=1}^{k}(b_i - a_i) < \frac{1}{n3^n}\}$, does not admit the choice of $\lambda_n \subseteq \gamma_n$ satisfying condition (b) of Theorem 5 and, therefore, $vet^* C_p(0,1) \not\leq \omega$.

Denote by S_c the space obtained by identifying the limit points of continuum many convergent sequences.

Theorem 8. *Let X be a topological space such that $t(S_c \times X) \leq \omega$. Then $vet^*(X) \leq \omega$.*

Proof: Enumerate the convergent sequences of S_c by the elements of $\mathbb{R}$: $S_c = \{C_\alpha : \alpha \in \mathbb{R}\} \cup \{O\}$, where $C_\alpha = \{\frac{\alpha}{n} : n \in \omega\}$ and O is the only non-isolated point of S_c.

Fix $x^* \in X$ and a countable family $\{A_n : n \in \omega\}$ of subsets of X such that $x^* \in \bigcap\{\overline{A_n} : n \in \omega\}$. Since $t(X) \leq t(S_c \times X) = \omega$, we may assume without loss of generality that $|A_n| = \omega$.

Consider $K = \{(a_i)_{i\in\omega} : a_i \in A_i \ \forall\, i \in \omega\}$. Since $|K| = 2^\omega$, $K = \{\xi_\alpha : \alpha \in \mathbb{R}\}$, where each $\xi_\alpha = (a_i^\alpha)_{i\in\omega}$ and $\xi_\alpha \neq \xi_{\alpha'}$ whenever $\alpha \neq \alpha'$.

For each $\alpha \in \mathbb{R}$, put $\zeta_\alpha = \{(\frac{\alpha}{n}, a_n^\alpha) : n \in \omega\}$. Let $B = \bigcup\{\zeta_\alpha : \alpha \in \mathbb{R}\} \subseteq S_c \times X$.

Claim 1: $\overline{B} \ni (O, x^*)$. Fix a neighborhood O_{x^*} of x^* in X and a neighborhood V of O in S_c. For each $n \in \omega$ there exists an $a_n^* \in O_{x^*} \cap A_n$. Also, there is a real number α^* such that $\xi_{\alpha^*} = (a_i^*)_{i\in\omega}$. Since V contains all but finitely many points of C_{α^*}, $\zeta_{\alpha^*} \cap (V \times O_{x^*}) \neq \emptyset$.

Choose a countable subset M of B such that $\overline{M} \ni (O, x^*)$. Without loss of generality, it may be assumed that $M = \bigcup\{\zeta_{\alpha_k} : k \in \omega\}$. Put $H_i = \{a_i^{\alpha_k} : 1 \leq k \leq i\}$. Clearly, each H_i is a subset of A_i and $|H_i| \leq i$.

Claim 2: $x^* \in \overline{\bigcup\{H_i : i \in \omega\}}$. Fix a neighborhood O_{x^*} of x^* in X. Put $V = S_c \setminus \{\frac{\alpha_k}{n} : k \in \omega, n \in \omega, n < k\}$. Clearly, V is an open neighborhood of O, and, consequently, $M \cap (V \times O_{x^*}) \neq \emptyset$. Fix natural numbers k and n such that $(\frac{\alpha_k}{n},\ a_n^{\alpha_k}) \in (V \times O_{x^*}) \cap M$. Since $\frac{\alpha_k}{n} \in V$, we have $n \geq k$ and, therefore, $a_n^{\alpha_k} \in H_n \cap O_{x^*}$. This completes the proof. □

The following two corollaries provide answers to questions posed in [3]:

Corollary 9. *Let X be a Hausdorff space such that $t(S_c \times X) \leq \omega$. Then $vet\ (X) \leq \omega$.*

Remark. Example 7 shows that the last corollary cannot be reversed: It follows from theorem (on product) that $t(C_p(0,1) \times S_c) > \omega$. In fact, it follows from Example 2 in [6] that $t(C_p(0,1) \times S_c) \geq 2^\omega$.

Question 2. Can Theorem 8 be reversed? Also, is it true that for a space X of countable strong fan-tightness we have $t(S_c \times X) \leq \omega$?

Corollary 10. *Let X be a Tychonoff space such that $t(C_p(X) \times S_c) \leq \omega$. Then X is a Hurewicz space.*

It was shown in [3] that for a regular countably compact space X of countable tightness, the tightness of the product space $S_c \times X$ is countable. It was also proved that for a regular countably compact space, countable fan-tightness and countable tightness are equivalent. The following corollary improves the last result:

Corollary 11. *Let X be a regular countably compact space of countable tightness. Then $vet^*(X) \leq \omega$.*

Question 3. Let X be a regular pseudocompact space of countable tightness. Is it true that $t(S_c \times X) \leq \omega$? In particular, is it true that $vet^*(X) \leq \omega$?

Definition. A mapping $f : X \to Y$ is said to be *countably biquotient*, if for each point $y \in Y$ and for each increasing open cover $\{U_n : n \in \omega\}$ of $f^{-1}(y)$ there is a number n such that $y \in Int(f(U_n))$.

Proposition 12. *Let X be a space such that $vet^*(X) \leq \omega$ and let $f : X \to Y$ be a continuous countably biquotient mapping onto Y. Then $vet^*(Y) \leq \omega$.*

Proof: Use condition (c) of Lemma 3. Fix $y \in Y$ and a countable decreasing family $\{A_n\}_{n\in\omega}$ of subsets of Y such that $y \in \bigcap \overline{A_n}$. Put $B_n = f^{-1}(A_n)$. There is a point $x \in f^{-1}(y)$ such that $x \in \bigcap\{\overline{B_n} : n \in \omega\}$. Indeed, otherwise $\{X \setminus \overline{B_n} : n \in \omega\}$ be an increasing cover of $f^{-1}(y)$ and for some n, we would have $y = f(x) \in Int(f(X \setminus \overline{f^{-1}(A_n)})) \subseteq Y \setminus A_n$, a contradiction to $y \in \overline{A_n}$.

Fix $x \in f^{-1}(y)$ such that $x \in \overline{B_n}$ for each n and choose $b_i \in B_i$ with the property $x \in \overline{\{b_i : i \in \omega\}}$. It is easy to see that $f(b_i) \in A_i$ and $y \in \overline{\{f(b_i) : i \in \omega\}}$. By Lemma 3, $vet^*(Y) \leq \omega$. □

Corollary 13. *If X is a topological space such that $vet^*(X) \leq \omega$ and Y is an image of X under a continuous open mapping, then $vet^*(Y) \leq \omega$.*

We shall say that a space X has *countable omega-fan-tightness* ($vet_\omega(X) \leq \omega$) if for each point $x \in X$ and any countable family $\{A_n\}_{n\in\omega}$ of countable subsets of X satisfying $x \in \bigcap\{\overline{A_n} : n \in \omega\}$, there exist finite sets $H_n \subseteq A_n$ such that $x \in \overline{\bigcup\{H_n : n \in \omega\}}$.

Theorem 14. *Let X be a Tychonoff space such that for every finite k and for every sequence $\{\gamma_n\}_{n\in\omega}$ of countable open covers of X^k there exist finite subfamilies $\lambda_n \subseteq \gamma_n$ such that $\bigcup\{\lambda_n : n \in \omega\}$ is a cover of X^k. Then $vet_\omega C_p(X) \leq \omega$.*

Proof: Fix a family $\{A_n : n \in \omega\}$ of countable subsets of $C_p(X)$ and a function $f \in C_p(X)$ such that $f \in \bigcap \overline{A_n}$. Fix natural numbers n and k and for each $g \in A_k$, put $V_n(g) = (g - f)^{-1}(-\frac{1}{n}, \frac{1}{n})$. Put $\gamma_k^n = \{V_n(g)^n : g \in A_k\}$. The family γ_k^n is an open cover of X^n. Indeed, for $(x_1, x_2, \dots, x_n) \in X^n$ there is $h \in A_k$ such that $h \in W(x_1, x_2, \dots, x_n; f; \frac{1}{n})$ and hence $(x_1, x_2, \dots, x_n) \in V_n(h)^n$.

Consider a sequence $\{\gamma_k^n : k \geq n\}$ of open countable covers of X^n and select finite families $\lambda_k^n = \{V_n(g) : g \in H_k^n\} \subseteq \gamma_k^n$, where H_k^n is a finite subset of A_n and $\bigcup\{\lambda_k^n : k \geq n\} \supseteq X$. Put $H_i = \bigcup\{H_i^n : n \leq i\}$. Clearly, H_i is a finite subset of A_i.

Claim: $f \in \overline{\bigcup\{H_i : i \in \omega\}}$. Fix $x_1, x_2, \dots, x_n \in X$ and $\epsilon > 0$. It may be assumed without loss of generality that $\frac{1}{n} < \epsilon$. For some natural number $k \geq n$, we have $\bigcup \lambda_k^n \ni (x_1, x_2, \dots, x_n)$ and there is an $h \in H_k^n \subseteq H_k$ such that

$(x_1, x_2, \dots, x_n) \in V_n(h)^n$, i.e. $h \in W(x_1, x_2, \dots, x_n; f; \epsilon) \cap H_k$. The proof is complete. $\square$

The following two theorems were proved by Professor Arhangelskii, who kindly permitted me to include them in this paper.

Theorem 15. *Let X be a Tychonoff pseudocompact space. Then $vet_\omega C_p(X) \leq \omega$.*

Proof: The restriction mapping $r : C_p(\beta X) \to C_p(X)$ is a continuous bijection. Fix a countable subset $A \subseteq C_p(X)$. Then the restriction of the inverse mapping $r^{-1}|_A : A \to C_p(\beta X)$ is continuous. Indeed, for each $z \in \beta X \setminus X$ the set $F = \bigcap\{\tilde{g}^{-1}(g(z)) : g \in A\}$, where $\tilde{g}$ is an extension of function g to a continuous function on βX, is a G_δ-set in βX and, therefore, there exists a $y_z \in X \cap F$, i.e. $g(z) = g(y_z)$ for each $g \in A$. From here, $r(W(z, r^{-1}(f), \epsilon) \cap r^{-1}(A)) = W(y_z, f, \epsilon) \cap A$ for any $f \in A$. Clearly, for each $x \in X$ and each $f \in A$ we have $r(W(x, r^{-1}(f), \epsilon) \cap r^{-1}(A)) = W(x, f, \epsilon) \cap A$.

Fix $f \in C_p(X)$ and a sequence $\{A_n\}_{n \in \omega}$ of subsets of $C_p(X)$ such that $f \in \bigcup \overline{A_n}$. Then $r^{-1}(f) \in \bigcap \overline{r^{-1}(A_n)}$ and by Arhangelskii's Theorem, $vet(C_p(\beta X)) \leq \omega$. Fix finite $H_n \subseteq r^{-1}(A_n)$ such that $r^{-1}(f) \in \overline{\bigcup\{H_n : n \in \omega\}}$. It follows that $f \in \overline{\bigcup\{r(H_n) : n \in \omega\}}$ and each $r(H_n)$ is a finite subset of A_n. $\square$

Remark. The last theorem shows that Theorem 14 cannot be reversed: a pseudocompact Tychonoff space not satisfying the assumptions of Theorem 14 would be a counterexample.

It is known that countable fan-tightness is preserved by continuous open surjective mappings. The next theorem shows that it is not true for countable omega-fan-tightness.

Theorem 16. *Let Y be a Tychonoff space. Then there exist a Tychonoff space X with $vet_\omega(X) \leq \omega$ and a continuous open surjection $f : X \to Y$.*

Proof: Consider the space $Z = \big((\omega_1 + 1) \times \beta(C_p(Y))\big) \setminus \big(\{\omega_1\} \times (\beta(C_p(Y)) \setminus C_p(Y))\big)$. Since Z contains a dense countably compact space $\omega_1 \times \beta(C_p(Y))$, the space Z is pseudocompact, and therefore $vet_\omega(C_p(Z)) \leq \omega$. It is easy to see that $\{\omega_1\} \times C_p(Y)$ is closed in Z and every bounded continuous function on $\{\omega_1\} \times C_p(Y)$ can be extended to a continuous function on Z. Thus the restriction mapping $r : C_p(Z) \to C_p(\{\omega_1\} \times C_p(Y)) = C_p(C_p(Y))$ is an open mapping; a topological copy of Y is contained in $C_p^0(C_p(Y)) \subseteq r(C_p(Z))$. Put $X = r^{-1}(Y)$ and put $f = r|_X$. It is easy to see that f is a continuous open mapping onto Y and $vet_\omega(X) \leq \omega$. $\square$

Remark. After this paper had been submitted, the author proved independently from S. Garcia-Ferreira and A. Tamariz-Mascarua that for a Tychonoff space X, $vet^* C_p(X) \leq \omega$ implies that $C_p(X)$ has countable strong fan-tightness. In the article "Some generalizations of rapid ultrafilters in topology and id-fan tightness", Tsukuba J. Math, 19 (1) (1995), 173–185, the two authors also showed that $vet^*(X) \leq \omega$ does not imply in general that X has countable strong fan-tightness. This provides a complete answer to Question 1. It is not clear, however, whether the two properties coincide for topological groups.

Also, A. Bella noticed that the answer to Question 2 is negative.

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