H-closed functions

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Abstract. The notion of a Hausdorff function is generalized to the concept of H-closed function and the concept of an H-closed extension of a Hausdorff function is developed. Each Hausdorff function is shown to have an H-closed extension.

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1. Introduction and preliminaries

Functions are more general objects of study than that of spaces, and during the past two decades, there has been an increase in the investigation of functions with certain topological properties. In 1971, H-closed functions were introduced in the class of Hausdorff spaces (i.e., both the domain and range spaces are Hausdorff) by Blaszczyk and Mioduszewski [BM] and characterized by Viglino [V]; additional work was done by Dickman [D] and Friedler [F]. Ul'yanov [U] and, independently, Blaszczyk [B] introduced the notion of Hausdorff functions in a general construction of absolutes for arbitrary spaces (the domain and range spaces of a Hausdorff function are not necessarily Hausdorff). In this paper, we expand the concept of Hausdorff function to H-closed functions and develop the concept of extensions of a Hausdorff function; a special case of our development includes the work of [V], [D], [F]. In particular, for a Hausdorff function $f : X \to Y$ (X and Y are spaces, but not necessarily Hausdorff), we show:

(i) f has an H-closed extension,
(ii) f is compact if f is H-closed and regular, and
(iii) if f is H-closed, $k_X : EX \to X$ is the usual absolute mapping from the absolute EX of X to X, and $f \circ k_X$ is continuous, then $f \circ k_X$ is compact.

Let X, Y, and Z be spaces and $f : X \to Y$ and $g : Z \to Y$ be functions.

(1) $f : X \to Y$ is **Hausdorff** if f is continuous and for $x, y \in X$ such that $x \neq y$ and $f(x) = f(y)$, there are disjoint open sets U and V in X such that $x \in U$ and $y \in V$.

(2) $f : X \to Y$ is **regular** if f is continuous and if F is a closed set in X and $x \in X \backslash F$, there is an open neighborhood V of $f(x)$ such that x and $F \cap f^{\leftarrow}[V]$ can be separated by disjoint open sets of $f^{\leftarrow}[V]$.

(3) g **extends** f if X is a subspace of Z and $g|_X = f$.

(4) If f is Hausdorff, then f is **H-closed** if whenever g is Hausdorff and g extends f, then X is closed in Z.

(5) f is **compact** if for each $y \in Y$, $f^{\leftarrow}(y)$ is compact.

(6) f is **perfect** if f is compact and closed.

1.1 Observations. (1) *If X is Hausdorff (resp. regular), then a continuous function $f : X \to Y$ is Hausdorff (resp. regular).*

(2) *If $f : X \to Y$ is Hausdorff (resp. regular) and Y is Hausdorff (resp. regular), then X is also Hausdorff (resp. regular).*

(3) *If $f : X \to Y$ is continuous, X is H-closed, and Y is Hausdorff, then f is H-closed.*

For a space X, let $\tau(X)$ denote the collection of open sets of X, and for $p \in X$, let $\mathcal{N}_p = \{U \in \tau(X) : p \in U\}$ — the collection of open neighborhoods of p. If $\mathcal{F}$ is a filter base on X, $a_X\mathcal{F}$ is used to denote $\bigcap\{cl_X F : F \in \mathcal{F}\}$,

1.2. *Let $f : X \to Y$ be a Hausdorff function. Then f is H-closed if and only if for each open ultrafilter $\mathcal{U}$ on X and $y \in Y$, if $\mathcal{U} \supseteq f^{\leftarrow}(\mathcal{N}_y)$, then $a_X\mathcal{U} \cap f^{\leftarrow}(y) \neq \varnothing$.*

Proof: Suppose f is H-closed, $\mathcal{U}$ is an open ultrafilter on X and $y \in Y$ such that $\mathcal{U} \supseteq f^{\leftarrow}(\mathcal{N}_y)$. Let $Z = X \cup \{\mathcal{U}\}$ with the simple extension topology, i.e., $\tau(Z) = \{U \subseteq Z : U \cap X \in \tau(X)$ and $\mathcal{U} \in U$ implies $U \cap X \in \mathcal{U}\}$. Define $g : Z \to Y$ by $g|_X = f$ and $g(\mathcal{U}) = y$. Then g is continuous at points of X since X is open in Z; g is continuous at $\mathcal{U}$ since $\mathcal{U} \supseteq f^{\leftarrow}(\mathcal{N}_y)$. Since X is dense in Z and f is H-closed, g is not Hausdorff and there is a point $x \in g^{\leftarrow}(y) \cap X = f^{\leftarrow}(y)$ such that x and $\mathcal{U}$ are not contained in disjoint open neighborhoods of Y. Hence, $x \in a_X\mathcal{U}$, i.e., $a_X\mathcal{U} \cap f^{\leftarrow}(y) \neq \varnothing$. Conversely, suppose there is a Hausdorff function $g : Z \to Y$ such that $g|_X = f$ and X is dense in Z. Let $p \in Z$ and $O^p = \{U \cap X : p \in U, U \in \tau(Z)\}$. Since g is continuous, it follows that $O^p \supseteq f^{\leftarrow}(\mathcal{N}_{g(p)})$. Also, since distinct pair of points of $g^{\leftarrow}(g(p))$ can be separated by disjoint open sets in Z, $g^{\leftarrow}(g(p)) \cap X \cap a_X O^p \subseteq \{p\}$. Let $\mathcal{U}$ be open ultrafilter containing O^p. Clearly, $f^{\leftarrow}(\mathcal{N}_{g(p)}) \subseteq \mathcal{U}$ and $a_X\mathcal{U} \subseteq a_X O^p$. By hypothesis, $f^{\leftarrow}(g(p)) \cap a_X\mathcal{U} \neq \varnothing$. Thus, $a_X O^p = \{p\}$ and $p \in X$. That, is $Z = X$. This completes the proof of the converse. □

1.3 Comments. (1) *Let $f : X \to Y$ be H-closed, $y \in Y$, and $\varnothing \neq f^{\leftarrow}(y) \subseteq U \in \tau(X)$. If $\mathcal{U}$ is an open ultrafilter on X and $f^{\leftarrow}(\mathcal{N}_y) \subseteq \mathcal{U}$, then $a_X\mathcal{U} \cap f^{\leftarrow}(y) \neq \varnothing$ by* 1.2. *Thus, U meets $\mathcal{U}$ and, in particular, $U \in \mathcal{U}$. That is, $U \in \bigcap\{\mathcal{U} : \mathcal{U}$ is an open ultrafilter on X and $\mathcal{U} \supseteq f^{\leftarrow}(\mathcal{N}_y)\}$. By* 2.3(k)(1) *in* [PW], *there is some $V \in \mathcal{N}_y$ such that $f^{\leftarrow}[V] \subseteq int_X cl_X U$.*

(2) *A straightforward consequence of* 1.2 *is this statement: A Hausdorff function $f : X \to Y$ is H-closed if and only if for each open filter $\mathcal{G}$ on X and $y \in Y$, if $\mathcal{G} \supseteq f^{\leftarrow}(\mathcal{N}_y)$, then $a_X\mathcal{G} \cap f^{\leftarrow}(y) \neq \varnothing$.*

1.4. *If $f : X \to Y$ is H-closed, σ is a topology on X such that $\sigma \subseteq \tau(X)$ and $f : (X, \sigma) \to Y$ is Hausdorff, then $f : (X, \sigma) \to Y$ is also H-closed.*

Proof: Let $\mathcal{G}$ be an open filter on (X, σ) and $p \in Y$ such that $\mathcal{G} \supseteq f^{\leftarrow}(\mathcal{N}_p)$. Now, $\mathcal{G}$ is an open filter base on X and as $\mathcal{G} \supseteq f^{\leftarrow}(\mathcal{N}_p)$, $a_X\mathcal{G} \cap f^{\leftarrow}(p) \neq \varnothing$. But $a_{(X,\sigma)}\mathcal{G} \supseteq a_X\mathcal{G}$. So, $a_{(X,\sigma)}\mathcal{G} \cap f^{\leftarrow}(p) \neq \varnothing$. □

Let X and Y be spaces. A function $f : X \to Y$ is **regular closed** (see [CN]), if $f[A]$ is closed whenever A is regular closed in X. A subset A of X is an **RD-set** (see [D]) if for every open ultrafilter $\mathcal{U}$ on X, if $a_X\mathcal{U} \cap A = \varnothing$, there is $U \in \mathcal{U}$ such that $cl_X U \cap A = \varnothing$. Also, $A \subseteq X$ is **strongly RD** if for each open filter $\mathcal{F}$ on X, if $a_X\mathcal{F} \cap A = \varnothing$, then for some $F \in \mathcal{F}$, $cl_X F \cap A = \varnothing$. Clearly, strongly RD sets are RD-sets. A set $A \subseteq X$ is an **H-set** if for every cover $\mathcal{U}$ of A by sets open in X, there is a finite $\mathcal{W} \subseteq \mathcal{U}$ such that $A \subseteq \bigcup\{cl_X W : W \in \mathcal{W}\}$; it is straightforward to show that A is an H-set of X if and only if for every open filter $\mathcal{F}$ on X meeting A, $a_X\mathcal{F} \cap A \neq \varnothing$.

1.5. *Let X be a space.*

(a) *If $A \subseteq X$ is an RD-set, then A is an H-set.*
(b) *If $A \subseteq X$ is a RD-set and B is regular closed subset of X, then $A \cap B$ is a RD-set of B.*

Proof: To prove (a), let $\mathcal{F}$ be an open filter on X such that $\mathcal{F}$ meets A. Let $\mathcal{U}$ be an open ultrafilter containing $\mathcal{F} \cup \{U \in \tau(X) : A \subseteq U\}$. In particular, if $V \in \tau(X)$ and $A \subseteq X \backslash cl_X V$, then $X \backslash cl_X V \in \mathcal{U}$ and $V \notin \mathcal{U}$. That is, $A \cap cl_X U \neq \varnothing$ for every $U \in \mathcal{U}$. As A is an RD-set, $a_X\mathcal{U} \cap A \neq \varnothing$. Since $\mathcal{F} \subseteq \mathcal{U}$, $a_X\mathcal{F} \cap A \neq \varnothing$. This completes the proof that A is an H-set.

To verify (b), let V be an open set in X such that $B = cl_X V$, and let $\mathcal{U}$ be an open ultrafilter on B. Then $V \in \mathcal{U}$ and $\mathcal{U}|_V = \{U \cap V : U \in \mathcal{U}\}$ is an open ultrafilter on V. Thus, $\mathcal{U}|_V$ is an open ultrafilter base on X. Suppose $a_B\mathcal{U} \cap (A \cap B) = \varnothing$. Since B is closed in X, $a_B\mathcal{U} = a_X(\mathcal{U}|_V)$ and $a_X\mathcal{U} \cap (A \cap B) = \varnothing$. For each $U \in \mathcal{U}$, $cl_X(U \cap V) \subseteq B$; so, $cl_X(U \cap V) \cap A = cl_X(U \cap V) \cap A \cap B$. Hence, $a_X(\mathcal{U}|_V) \cap A = \varnothing$. As A is an RD-set in X, there is some $W \in \mathcal{U}|_V$ such that $cl_X W \cap A = \varnothing$. In particular, $cl_X W \cap (A \cap B) = \varnothing$. □

1.6. *Let $f : X \to Y$ be a Hausdorff function. The following are equivalent:*

(a) *f is H-closed,*
(b) *f is regular closed and each point-inverse is an RD-set, and*
(c) *f is regular closed and each point-inverse is strongly RD.*

Proof: Suppose $f : X \to Y$ is H-closed. To show that f is regular closed, let U be an open subset of X and $y \notin f[cl_X U]$. So, $f^{\leftarrow}(y) \subseteq X \backslash cl_X U$. By 1.3(1), there is $V \in \mathcal{N}_y$ such that $f^{\leftarrow}[V] \subseteq int_X cl_X(X \backslash cl_X U) = X \backslash cl_X U$. Thus, $V \cap f[cl_X U] = \varnothing$ and $f[cl_X U]$ is closed. Let $p \in Y$. To show that $f^{\leftarrow}(p)$ is strongly RD, let $\mathcal{F}$ be an open filter on X such that $cl_X U \cap f^{\leftarrow}(p) \neq \varnothing$ for all $U \in \mathcal{F}$. In particular, it follows that $U \cap f^{\leftarrow}[T] \neq \varnothing$ for all $T \in \mathcal{N}_{f(p)}$. So,

$\mathcal{F} \cup f^{\leftarrow}(\mathcal{N}_{f(p)})$ is contained in some open ultrafilter $\mathcal{U}$ on X. Since $\mathcal{U}$ is an open ultrafilter on X and $\mathcal{U} \supseteq f^{\leftarrow}(\mathcal{N}_{f(p)})$, it follows that $a_X\mathcal{U} \cap f^{\leftarrow}(p) \neq \varnothing$ as f is H-closed. As $a_X\mathcal{U} \subseteq a_X\mathcal{F}$, $a_X\mathcal{F} \cap f^{\leftarrow}(p) \neq \varnothing$

Clearly (c) implies (b). To show (b) implies (a), suppose f is regular closed and each point-inverse is an RD-set. Let $\mathcal{U}$ be an open ultrafilter on X and $y \in Y$ such that $\mathcal{U} \supseteq f^{\leftarrow}(\mathcal{N}_y)$. To show that $a_X\mathcal{U} \cap f^{\leftarrow}(y) \neq \varnothing$, since $f^{\leftarrow}(y)$ is a RD-set, it suffices to show that $cl_XU \cap f^{\leftarrow}(y) \neq \varnothing$ for each $U \in \mathcal{U}$. As f is regular closed, $f[cl_XU]$ is closed. If $y \in Y\backslash f[cl_XU] \in \mathcal{N}_y$, $f^{\leftarrow}[Y\backslash f[cl_XU]] \in \mathcal{U}$. But $f^{\leftarrow}[Y\backslash f[cl_XU]] \cap cl_XU = \varnothing$ which is impossible. So, $y \in f[cl_XU]$ or $cl_XU \cap f^{\leftarrow}(y) \neq \varnothing$. □

1.7 Corollary. (a) *If $f : X \to Y$ is an H-closed function and B is a regular closed subset of X, then $f|_B : B \to Y$ is also H-closed.*

(b) *A perfect, Hausdorff function is H-closed.*

1.8 Examples. (1) If X is a space, then an immediate application of 1.6 is that $id_X : X \to X$ is H-closed.

(2) Here is an example to show that the point-inverse conditions of 1.6 (b), (c) cannot be improved to H-closed. Let $X = \{a\} \cup \omega \times \omega$. A subset $U \subseteq X$ is defined to be open whenever $(n, 0) \in U$ implies $\{(n, m) : m \geq k\} \subseteq U$ for some $k \in \omega$ and $a \in U$ implies $\{(n, m) : n \geq k, m \geq 1\} \subseteq U$ for some $k \in \omega$. It is easy (see 4.8 (b), (d) and 4N in [PW]) to show that X is H-closed, $D = \{(n, m) : m \geq 1\}$ is a dense set of isolated points and $X\backslash D$ is an H-set of X. Let Y be the one-point compactification of the subspace D. Define $f : X \to Y$ by $f|_D = id_D$ and $f[X\backslash D] = Y\backslash D$. By 1.1 (3), f is H-closed. However, the point-inverse of $Y\backslash D$ is $X\backslash D$ and $X\backslash D$ is not H-closed.

1.9. *If $f : X \to Y$ is H-closed and X is regular (not necessarily Hausdorff), then f is perfect and $f[X]$ is regular.*

Proof: Since H-sets of regular spaces (see 4N (10) in [PW]) are compact, point-inverses of f are compact. Let $A \subseteq X$ be a closed set and $y \in Y\backslash f[A]$. Since $f^{\leftarrow}(y)$ is compact and X is regular, there is an open set V in X such that $A \subseteq V$ and $f^{\leftarrow}(y) \cap cl_XV = \varnothing$. But $f[A] \subseteq f[cl_XV]$, $f[cl_XV]$ is closed, and $y \notin f[cl_XV]$. This completes the proof that $f[A]$ is closed. It is well-known (see 1.8 (h) in [PW]) that the perfect continuous image of a regular space is regular. □

1.10. *If $f : X \to Y$ is H-closed and regular, then f is perfect.*

Proof: To show that f is compact, let $p \in Y$, $\mathcal{F}$ be a closed filter on $f^{\leftarrow}(p)$, and $\mathcal{G}$ be the open filter $\{W \in \tau(X) : W \supseteq F$ for some $F \in \mathcal{F}\}$. For each $q \in f^{\leftarrow}(p)\backslash \bigcap \mathcal{F}$, there is $F \in \mathcal{F}$ such that $q \notin cl_XF$. By regularity of f, there are open sets $U, V \in \tau(X)$ and $O \in \mathcal{N}_p$ such that $q \in U$, $cl_XF \cap f^{\leftarrow}[O] \subseteq V$, and $U \cap V = \varnothing$. It follows that $a_X\mathcal{G} \cap f^{\leftarrow}(p) \subseteq \cap\mathcal{F}$. As $\mathcal{G}$ meets $f^{\leftarrow}(p)$ and $f^{\leftarrow}(p)$ is an H-set, then $a_X\mathcal{G} \cap f^{\leftarrow}(p) \neq \varnothing$. Thus, $\bigcap \mathcal{F} \neq \varnothing$ and this completes the proof that f is compact.

To show that f is closed, let A be a closed subset of X and $p \in X\backslash f[A]$. Now, $f^{\leftarrow}(p) \cap A = \varnothing$. For each $q \in f^{\leftarrow}(p)$, there are open sets $U_q, V_q \in \tau(X)$ and $O_q \in \mathcal{N}_p$ such that $U_q \cup V_q \subseteq f^{\leftarrow}[O_q]$, $q \in U_q$, $A \cap f^{\leftarrow}[O_q] \subseteq V_q$, and $U_q \cap V_q = \varnothing$. By compactness of $f^{\leftarrow}(p)$, there is a finite subset $Q \subseteq f^{\leftarrow}[O_q]$ such that $f^{\leftarrow}(p) \subseteq U$ where $U = \bigcup\{U_q : q \in Q\}$. Let $V = \bigcap\{V_q : q \in Q\}$ and $O = \bigcap\{O_q : q \in Q\}$. Then $A \cap f^{\leftarrow}[O] \subseteq V$ and $U \cap V = \varnothing$. By 1.3(1), there is an open set $T \in \mathcal{N}_p$ such that $f^{\leftarrow}[T] \subseteq int_X cl_X U$ and we can assume that $T \subseteq O$. As $f^{\leftarrow}[T] \cap A \subseteq int_X cl_X U \cap A \cap f^{\leftarrow}[O] \subseteq int_X cl_X U \cap V = \varnothing$, $T \cap f[A] = \varnothing$. This shows that f is closed. □

1.11 Example. This is an example of a Hausdorff and regular function $f : X \to Y$ such that X is H-closed but f is not H-closed. In particular, by 1.1(3), Y is not Hausdorff. Let $Z = \{(\frac{1}{n}, 0) : n \in \mathbb{N}\} \cup \{(\frac{1}{n}, \frac{1}{m}) : n, |m| \in \mathbb{N}\}$ be a subset of the plane with the usual topology from the plane. Let $X = \{a, b\} \cup Z$. A set $U \subseteq X$ is **open** if $U \cap Z$ is open in Z and $a \in U$ (resp. $b \in U$) implies there is $k \in \mathbb{N}$ such that $\{(\frac{1}{n}, \frac{1}{m}) : n \geq k, m \in \mathbb{N}\}$ (resp. $\{(\frac{1}{n}, -\frac{1}{m}) : n \geq k, m \in \mathbb{N}\} \subseteq U$. The space X is the well-known example of a semiregular, H-closed space which is not compact (see 4.8(d) in [PW]). Let $Y = X\backslash\{(\frac{1}{n}, 0) : n \in \mathbb{N}\} \cup \{c\}$. Define $f : X \to Y$ by $f(x) = x$ for $x \in X\backslash\{(\frac{1}{n}, 0) : n \in \mathbb{N}\}$ and $f((\frac{1}{n}, 0)) = c$.

Place the quotient topology on Y. Now, Y is not Hausdorff as a and c are not contained in disjoint open sets, but f is a regular and Hausdorff function. Also, $f^{\leftarrow}(c) = \{(\frac{1}{n}, 0) : n \in \mathbb{N}\}$ is not an H-set; so, f is not H-closed. □

An absolute (see [U], [S], [PS]) of a space X is a pair (aX, p) where aX is extremally disconnected and $p : aX \to X$ is Hausdorff, perfect, irreducible and onto. (A function $f : Y \to Z$ is **irreducible** if for $\varnothing \neq U \in \tau(Y)$, there is $z \in Z$ such that $f^{\leftarrow}(z) \subseteq U$.) Let $EX = (aX)(s)$ (i.e., the underlying set of aX with the topology generated by the regular open subsets of aX) and define $k_X : EX \to X$ by $k_X(y) = p(y)$. The space EX is extremally disconnected and completely regular; the function $k_X : EX \to X$ is perfect, θ-continuous, irreducible, and onto. (A function $f : Y \to Z$ is **θ-continuous** if for each $y \in Y$ and $f(y) \in V \in \tau(Z)$, there is an open set $U \in \tau(Y)$ such that $y \in U$ and $f[cl_Y U] \subseteq cl_Z V$.) If $\varnothing \neq U \in \tau(EX)$, let $k_X^{\#}[U] = \{x \in X : k_X^{\leftarrow}(x) \subseteq U\}$; it is easy to verify that $k_X^{\#}[U] = X\backslash k_X[EX\backslash U]$ is a nonempty open set. Except for the continuity requirement, k_X is regular and Hausdorff.

1.12. *Let $f : X \to Y$ be H-closed and $f \circ k_X$ be continuous. Then $f \circ k_X$ is perfect.*

Proof: Since EX is regular, it suffices, by 1.9, to show that $f \circ k_X$ is H-closed. Let $\mathcal{U}$ be an open ultrafilter on EX and $y \in Y$ such that $(f \circ k_X)^{\leftarrow}(\mathcal{N}_y) \subseteq \mathcal{U}$. Now, $k_X^{\#}(\mathcal{U})$ $(= \{k_X^{\#}[U] : U \in \mathcal{U}\})$ is contained in an unique open ultrafilter $\mathcal{W}$ on X. If $V \in \mathcal{N}_y$, then $k_X^{\leftarrow}[f^{\leftarrow}[V]] \in \mathcal{U}$. But $k_X^{\#}[k_X^{\leftarrow}[f^{\leftarrow}[V]]] = f^{\leftarrow}[V] \in \mathcal{W}$. Thus, $f^{\leftarrow}(\mathcal{N}_y) \subseteq \mathcal{W}$. As f is H-closed, there is some $x \in a_X \mathcal{W} \cap f^{\leftarrow}(y)$. Then $\mathcal{N}_x \subseteq \mathcal{W}$. So, for $U \in \mathcal{U}$, $x \in cl_X k_X^{\#}[U] \subseteq cl_X k_X[U] \subseteq k_X[cl_{EX} U]$ as k_X is closed. Thus,

$k_X^{\leftarrow}(x) \cap cl_{EX} U \neq \varnothing$. Since $k_X^{\leftarrow}(x)$ is compact, it follows that $a_{EX}\mathcal{U} \cap k_X^{\leftarrow}(x) \neq \varnothing$. This implies that $a_{EX}\mathcal{U} \cap (f \circ k_X)^{\leftarrow}(y) \neq \varnothing$ and $f \circ k_X$ is H-closed. □

2. Construction of an H-closed extension of a Hausdorff function

In this section, it is shown that each Hausdorff function can be extended to an H-closed function. Let $f : X \to Y$ be a continuous function. Let $\kappa(X, f) = X \cup \{(\mathcal{U}, y) : \mathcal{U}$ is an open ultrafilter on X, $y \in Y$, $\mathcal{U} \supseteq f^{\leftarrow}(\mathcal{N}_y)$, and $a_X\mathcal{U} \cap f^{\leftarrow}(y) = \varnothing\}$. Define $U \subseteq \kappa(X, f)$ to be open if $U \cap X \in \tau(X)$ and whenever $(\mathcal{U}, y) \in U$, $U \cap X \in \mathcal{U}$. Define $\kappa f : \kappa(X, f) \to Y$ by $\kappa f|_X = f$ and $\kappa f(\mathcal{U}, y) = y$. Note that $\kappa(X, f)$ is a simple extension of X and that κf is continuous since X is open in $\kappa(X, f)$ and $\mathcal{U} \supseteq f^{\leftarrow}(\mathcal{N}_y)$ for $(\mathcal{U}, y) \in \kappa(X, f)\backslash X$.

2.1. *Let $f : X \to Y$ be Hausdorff. Then $\kappa f : \kappa(X, f) \to Y$ is H-closed and X is dense in $\kappa(X, f)$.*

Proof: It is clear that X is dense in $\kappa(X, f)$. To show κf is Hausdorff, let $y \in Y$ and consider distinct points $a, b \in (\kappa f)^{\leftarrow}(y)$. If $a, b \in (\kappa f)^{\leftarrow}(y) \cap X$, the open sets which separate a and b in X are also open in $\kappa(X, f)$. If $a = (\mathcal{U}, y)$ and $b \in (\kappa f)^{\leftarrow}(y) \cap X$, then $b \notin a_X\mathcal{U}$. So, there are open sets $U \in \mathcal{U}$ and V in X such that $b \in V$ and $U \cap V = \varnothing$. Now $\{(\mathcal{U}, y)\} \cup U$ and V are disjoint open sets in $\kappa(X, f)$ of $(\mathcal{U}, y)$ and b, respectively. Finally, if $a = (\mathcal{U}, y)$ and $b = (\mathcal{V}, y)$, then $\mathcal{U} \neq \mathcal{V}$ and there are $U \in \mathcal{U}$, $V \in \mathcal{V}$ such that $U \cap V = \varnothing$. $\{(\mathcal{V}, y)\} \cup U$ and $\{(\mathcal{V}, y)\} \cup V$ are disjoint open sets in $\kappa(X, f)$ of a and b, respectively.

To show κf is H-closed, let $\mathcal{W}$ be an open ultrafilter on $\kappa(X, f)$ and $y \in Y$ such that $(\kappa f)^{\leftarrow}(\mathcal{N}_y) \subseteq \mathcal{W}$. Let $\mathcal{U} = \{W \cap X : W \in \mathcal{W}\}$. Then $\mathcal{U}$ is an open ultrafilter on X. As $(\kappa f)^{\leftarrow}(\mathcal{N}_y) \cap X = f^{\leftarrow}(\mathcal{N}_y)$, $\mathcal{U} \supseteq f^{\leftarrow}(\mathcal{N}_y)$. Also, $a_{\kappa(X,f)}\mathcal{W} \cap (\kappa f)^{\leftarrow}(y) \supseteq a_X\mathcal{U} \cap f^{\leftarrow}(y)$. If $a_X\mathcal{U} \cap f^{\leftarrow}(y) \neq \varnothing$, then $a_{\kappa(X,f)}\mathcal{W} \cap (\kappa f)^{\leftarrow}(y) \neq \varnothing$. On the other hand, if $a_X\mathcal{U} \cap f^{\leftarrow}(y) = \varnothing$, then $(\mathcal{U}, y) \in \kappa(X, f)\backslash X$ and $(\mathcal{U}, y) \in (\kappa f)^{\leftarrow}(y)$. But $\mathcal{U}$ meets $\mathcal{W}$; so, $(\mathcal{U}, y) \in a_{\kappa(X,f)}\mathcal{W}$ and $(\kappa f)^{\leftarrow}(y) \cap a_{\kappa(X,f)}\mathcal{W} \neq \varnothing$. This completes the proof. □

Let $f : X \to Y$ be Hausdorff. An H-closed function $g : Z \to Y$ is an **H-closed extension** of f if X is a dense subspace of Z and $g|_X = f$. By 2.1, note that each Hausdorff function $f : X \to Y$ has an H-closed extension, namely, $\kappa f : \kappa(X, f) \to Y$.

2.2 Lemma. *Let $f : X \to Y$ be Hausdorff, $g_1 : Z_1 \to Y$ be an H-closed extension of f, $g_2 : Z_2 \to Y$ be an Hausdorff extension of f, and $h : Z_1 \to Z_2$ be continuous such that $h|_X = id_X$ and $g_2 \circ h = g_1$. Then h is Hausdorff, onto, H-closed, and g_2 is an H-closed extension of f.*

Proof: For $p \in Z_2$, since $h^{\leftarrow}(p) \subseteq h^{\leftarrow}(g_2^{\leftarrow}(g_2(p)) = g_1^{\leftarrow}(g_2(p))$, it is immediate that h is Hausdorff.

(*) Before showing h is onto and H-closed, we show that if $\mathcal{U}$ is an open ultrafilter on Z_1 and $p \in Z_2$ such that $h^{\leftarrow}(\mathcal{N}_p) \subseteq \mathcal{U}$, there is $q \in a_{Z_1}\mathcal{U} \cap g_1^{\leftarrow}(g_2(p))$ such that $h(q) = p$. Now $\mathcal{N}_p \supseteq g_2^{\leftarrow}(\mathcal{N}_{g_2(p)})$ since g_2 is continuous. So, $\mathcal{U} \supseteq h^{\leftarrow}(\mathcal{N}_p) \supseteq$

$h^{\leftarrow}(g_2^{\leftarrow}(\mathcal{N}_{g_2(p)})) = g_1^{\leftarrow}(\mathcal{N}_{g_2(p)})$. By H-closedness of g_1, there is some $q \in a_{Z_1}\mathcal{U} \cap g_1^{\leftarrow}(g_2(p))$. Assume that $h(q) \neq p$. But $h(q)$ and $p \in g_2^{\leftarrow}(g_2(p))$. Using that g_2 is Hausdorff, there are disjoint open sets U and V in Z_2 such that $h(q) \in U$ and $p \in V$. So, $h^{\leftarrow}[U] \cap h^{\leftarrow}[V] = \varnothing$. But $h^{\leftarrow}[U] \in h^{\leftarrow}(\mathcal{N}_{h(q)}) \subseteq \mathcal{N}_q \subseteq \mathcal{U}$ and $h^{\leftarrow}[V] \in h^{\leftarrow}(\mathcal{N}_p) \subseteq \mathcal{U}$, a contradiction. So, $h(q) = p$. To show that h is onto, let $p \in Z_2$. Then $h^{\leftarrow}(\mathcal{N}_p)$ is contained in some ultrafilter $\mathcal{U}$ on Z_1 (this uses that X is dense in Z_2 and $X \subseteq h[Z_1]$). By (*), there is $q \in a_{Z_2}\mathcal{U} \cap g_1^{\leftarrow}(g_2(p))$ such that $h(q) = p$. So, h is onto. To show that h is H-closed, let $\mathcal{U}$ be an open ultrafilter on Z_1 and $p \in Z_2$ such that $h^{\leftarrow}(\mathcal{N}_p) \subseteq \mathcal{U}$. By (*), there is some $q \in a_{Z_1}\mathcal{U} \cap g_1^{\leftarrow}(g_2(p))$ such that $h(q) = p$. In particular, $q \in a_{Z_1}\mathcal{U} \cap h^{\leftarrow}(p)$. By 1.2, h is H-closed.

Finally, to show that g_2 is H-closed, let $\mathcal{U}$ be an open ultrafilter on Z_2 and $y \in Y$ such that $\mathcal{U} \supseteq g_2^{\leftarrow}(\mathcal{N}_y)$. Now, $h^{\leftarrow}(\mathcal{U}) \supseteq h^{\leftarrow}(g_2^{\leftarrow}(\mathcal{N}_y)) = g_1^{\leftarrow}(\mathcal{N}_y)$. Let $\mathcal{W}$ be an open ultrafilter on Z_1 such that $\mathcal{W} \supseteq h^{\leftarrow}(\mathcal{U})$. By H-closedness of g_1, there is some point $q \in a_{Z_1}\mathcal{W} \cap g_1^{\leftarrow}(y)$. Let $U \in \mathcal{U}$. Then $q \in cl_{Z_1}h^{\leftarrow}[U]$ and $h(q) \in h[cl_{Z_1}h^{\leftarrow}[U]] \subseteq cl_{Z_2}hh^{\leftarrow}[U] \subseteq cl_{Z_2}U$. This show that $h(q) \in a_{Z_2}\mathcal{U}$. Since $q \in g_1^{\leftarrow}(y) = h^{\leftarrow}(g_2^{\leftarrow}(y))$, it follows that $h(q) \in g_2^{\leftarrow}(y)$. This completes the proof that g_2 is H-closed. □

2.3 Theorem. *Let $f : X \to Y$ be Hausdorff and $g : Z \to Y$ be an H-closed extension of f. Then there is an H-closed surjection $h : \kappa(X, f) \to Z$ such that $g \circ h = \kappa f$ and $h|_X = id_X$.*

Proof: By 2.2, it suffices to find a continuous function $h : \kappa(X, f) \to Z$ such that $g \circ h = \kappa f$ and $h|_X = id_X$. Start by defining $h(x) = x$ and $x \in X$. Let $(\mathcal{U}, y) \in \kappa(X, f)\backslash X$. So $f^{\leftarrow}(\mathcal{N}_y) \subseteq \mathcal{U}$ and $a_X\mathcal{U} \cap f^{\leftarrow}(y) = \varnothing$. Then $\mathcal{W} = \{W \in \tau(Z) : W \cap X \in \mathcal{U}\}$ is an open ultrafilter on Z and $a_X\mathcal{W} \cap f^{\leftarrow}(y) = a_X\mathcal{U} \cap f^{\leftarrow}(y) = \varnothing$. Since $f^{\leftarrow}(\mathcal{N}_y) \subseteq \mathcal{U}$, it follows that $g^{\leftarrow}(\mathcal{N}_y) \subseteq \mathcal{W}$. As g is H-closed, there is some point $p \in a_Z\mathcal{W} \cap g^{\leftarrow}(y)$. So, $\mathcal{N}_p \subseteq \mathcal{W}$ and $h(\mathcal{U}, y)$ is defined to be p (note that h is well-defined since g is Hausdorff implies $|a_Z\mathcal{W} \cap g^{\leftarrow}(y)| = 1$). So, h is defined such that $h|_X = id_X$ and $g \circ h = \kappa f$. To show h is continuous, let $h(\mathcal{U}, y) = p$. Then, as above, $\mathcal{N}_p \subseteq \mathcal{W}$. If $U \in \mathcal{N}_p$, then $U \cap X \in \mathcal{U}$ and $h[(U \cap X) \cup \{(\mathcal{U}, y)\}] = (U \cap X) \cup \{p\} \subseteq U$ is continuous at $(\mathcal{U}, y)$ as $(U \cap X) \cup \{(\mathcal{U}, y)\}$ is an open neighborhood of $(\mathcal{U}, y)$ in $\kappa(X, f)$. So, h is continuous at points of $\kappa(X, f)\backslash X$. Also, h is continuous at points of X since X is open in $\kappa(X, f)$ and $h|_X = id_X$. Hence, h is continuous. Applying 2.2, the desired conclusions follow. □

Comment. Let X be a space and $\kappa X = X \cup \{\mathcal{U} : \mathcal{U}$ is an open ultrafilter on X such that $a_X\mathcal{U} = \varnothing\}$. A set $U \subseteq \kappa X$ is defined as open if $U \cap X \in \tau(X)$ and whenever $\mathcal{U} \in U, U \cap X \in \mathcal{U}$. The space κX is a simple extension of X with the property that every open filter on κX has nonempty adherence. In particular, κX is the Katětov extension of an arbitrary space X. If Y is a space and $f : X \to Y$ is continuous, then $X \subseteq \kappa(X, f) \subseteq \kappa X$ (where $(\mathcal{U}, f) \in \kappa(X, f)\backslash X$ is identified with $\mathcal{U} \in \kappa X\backslash X$). Also, if $\mathcal{U} \in \kappa X\backslash X$ and f has a continuous extension to $F : X \cup \{\mathcal{U}\} \to Y$, then $(\mathcal{U}, f) \in \kappa(X, f)$. That is, $\kappa(X, f)$ is the largest subspace $M_f \subseteq \kappa X$ such that $X \subseteq M_f$ and f has a continuous extension to M_f.

3. Semiregular H-closed functions

In this section, a necessary and sufficient condition is derived for an H-closed function to be perfect and regular.

A continuous function $f : X \to Y$ is **semiregular** ([CN]) if for $p \in U \in \tau(X)$, there are $V \in \tau(X)$ and $O \in \mathcal{N}_{f(p)}$ such that $p \in (int_{f^{\leftarrow}[O]} cl_{f^{\leftarrow}[O]} V) \subseteq f^{\leftarrow}[O] \cap U$. Cammaroto and Nordo [CN] have derived these useful properties of semiregular functions.

3.1. (a) A continuous function $f : X \to Y$ is semiregular if and only if for $p \in U \in \tau(X)$, there are $V \in \tau(X)$ and $O \in \mathcal{N}_{f(p)}$ such that $p \in (int_X cl_X V) \cap f^{\leftarrow}[O] \subseteq U$.

(b) If $f : X \to Y$ is Hausdorff and σ is the topology generated by $\{int_X cl_X U \cap f^{\leftarrow}[O] : U \in \tau(X) \text{ and } O \in \tau(Y)\}$, then $f : (X, \sigma) \to Y : x \mapsto f(x)$ is semiregular and Hausdorff.

A Hausdorff function $f : X \to Y$ is **minimal Hausdorff** if σ is a topology on X and $\sigma \subsetneq \tau(X)$, then $f : (X, \sigma) \to Y$ is not a Hausdorff function.

3.2. *A function $f : X \to Y$ is minimal Hausdorff if and only if f is H-closed and semiregular.*

PROOF: Suppose f is minimal Hausdorff. By 3.1, f is semiregular. To show that f is H-closed, let $\mathcal{F}$ be an open filter on X and $p \in Y$ such that $\mathcal{F} \supseteq f^{\leftarrow}(\mathcal{N}_p)$. Assume that $a_X \mathcal{F} \cap f^{\leftarrow}(p) = \varnothing$. Let $q \in f^{\leftarrow}(p)$. Define a new topology σ on X by $U \in \sigma$ if and only if $U \in \tau(X)$ and $q \in U$ implies there is $F \in \mathcal{F}$ such that $F \subseteq U$. As $q \notin a_X \mathcal{F}$, $\sigma \subsetneq \tau(X)$. Since $f^{\leftarrow}(\mathcal{N}_p) \subseteq \mathcal{F}$ and $f^{\leftarrow}(\mathcal{N}_p) \subseteq \mathcal{N}_q^{\tau(X)}$ (the open neighborhoods of q in X), it follows that $f : (X, \sigma) \to Y$ is continuous. To show $f : (X, \sigma) \to Y$ is Hausdorff, since the only open neighborhoods which are changed are those of q, it suffices to show that q and $r \in f^{\leftarrow}(p) \setminus \{q\}$ can be separated by open sets in (X, σ). There are open sets $U \in \mathcal{N}_q^{\tau(X)}$ and $V \in \mathcal{N}_r^{\tau(X)}$ such that $U \cap V = \varnothing$. As $a_X \mathcal{F} \cap f^{\leftarrow}(p) = \varnothing$, there are $F \in \mathcal{F}$ and $W \in \mathcal{N}_r^{\tau(X)}$ such that $F \cap W = \varnothing$. Now, $U \cup F \in \mathcal{N}_q^{\sigma}$ and $V \cap W \in \mathcal{N}_r^{\sigma}$. Also, $(U \cup F) \cap (V \cap W) = \varnothing$. So, $f : (X, \sigma) \to Y$ is Hausdorff. This is a contradiction as $f : X \to Y$ is minimal Hausdorff. Hence, $f : X \to Y$ is H-closed. Conversely, suppose $f : X \to Y$ is both H-closed and semiregular. Let σ be a topology on X such that $\sigma \subseteq \tau(X)$ and $f : (X, \sigma) \to Y$ is Hausdorff. Let $V \in \tau(X)$. Then $int_X cl_X V \in \tau(X)$ and $B = X \setminus int_X cl_X V$ is regular closed in X. By 1.7, $f|_B : B \to Y$ is H-closed. By 1.4, $f|_B : (B, \sigma_B) \to Y$ is also H-closed. However, by the definition of H-closed functions, B is closed in (X, σ), i.e., $X \setminus B \in \sigma$. That is, $int_X cl_X V \in \sigma$. Let $O \in \tau(Y)$. Since $f : (X, \sigma) \to Y$ is continuous, $int_X cl_X V \cap f^{\leftarrow}[O] \in \sigma$. By 3.1(a), $\tau(X)$ is generated by $\{int_X cl_X V \cap f^{\leftarrow}[O] : V \in \tau(X) \text{ and } O \in \tau(Y)\}$; so, $\tau(X) \subseteq \sigma$. This completes the proof that $\tau(X) = \sigma$ and shows that f is minimal Hausdorff. □

A function $f : X \to Y$ is **Urysohn** ([CN]) if f is continuous and for $x, y \in X$ such that $x \neq y$ and $f(x) = f(y)$, there are open sets U and $V \in \tau(X)$ and $W \in$

$\mathcal{N}_{f(x)}$ such that $U \cup V \subseteq f^{\leftarrow}[W]$, $x \in U$, $y \in V$, and $cl_{f^{\leftarrow}[W]}U \cap cl_{f^{\leftarrow}[W]}V = \varnothing$.

3.3 Lemma. *If $f : X \to Y$ is H-closed and Urysohn, $A \subseteq X$ is regular closed, and $p \in X \backslash A$, there are open sets U and $W \in \tau(X)$ and $O \in \mathcal{N}_{f(p)}$ such that $p \in U \subseteq f^{\leftarrow}[O]$, $A \cap f^{\leftarrow}(f(p)) \subseteq cl_{f^{\leftarrow}[O]}W \subseteq f^{\leftarrow}[O]$, and $cl_{f^{\leftarrow}[O]}U \cap cl_{f^{\leftarrow}[O]}W = \varnothing$.*

Proof: For each $q \in f^{\leftarrow}(f(p)) \cap A$, there are open sets $U_q, W_q \in \tau(X)$ and $O_q \in \mathcal{N}_{f(p)}$ such that $U_q \cup W_q \subseteq f^{\leftarrow}[O_q]$, $p \in U_q$, $q \in W_q$, and $cl_{f^{\leftarrow}[o_p]}U \cap cl_{f^{\leftarrow}[o_p]}W = \varnothing$. By 1.7, $f^{\leftarrow}(f(p)) \cap A$ is an RD-set (and hence, an H-set). There is a finite subset $Q \subseteq f^{\leftarrow}(f(p)) \cap A$ such that $f^{\leftarrow}(f(p)) \cap A \subseteq cl_X(\bigcup\{W_q : q \in Q\})$. Let $O = \bigcap\{O_q : q \in Q\}$, $U = \bigcap\{U_q : q \in Q\} \cap f^{\leftarrow}[O]$ and $W = (\bigcup\{W_q : q \in Q\}) \cap f^{\leftarrow}[O]$. Thus, we have that $p \in U \subseteq f^{\leftarrow}[O]$, $f^{\leftarrow}(f(p)) \cap A \subseteq cl_{f^{\leftarrow}[0]}W$, and $cl_{f^{\leftarrow}[0]}U \cap cl_{f^{\leftarrow}[0]}W = \varnothing$. □

3.4 Theorem. *Let $f : X \to Y$ be minimal Hausdorff and Urysohn. Then f is perfect and regular.*

Proof: By 1.10, it suffices to show that f is regular. Let A be a closed subset of X and $p \in X \backslash A$. As f is semiregular, there are $V \in \tau(X)$ and $P \in \mathcal{N}_{f(p)}$ such that $p \in int_X cl_X V \cap f^{\leftarrow}[P] \subseteq X \backslash A$. Applying 3.3 to $f^{\leftarrow}(f(p)) \backslash int_X cl_X V$, there are open sets $U, V \in \tau(X)$ and $O \in \mathcal{N}_{f(p)}$ such that $p \in U \subseteq f^{\leftarrow}[O]$, $O \subseteq P$, $f^{\leftarrow}(f(p)) \backslash int_X cl_X V \subseteq cl_{f^{\leftarrow}[O]}W \subseteq f^{\leftarrow}[O]$ and $cl_{f^{\leftarrow}[O]}U \cap cl_{f^{\leftarrow}[O]}W = \varnothing$. So, $f^{\leftarrow}(f(p)) \cap (X \backslash int_X cl_X V) \cap cl_X(X \backslash cl_X W) = \varnothing$. This implies that $f^{\leftarrow}(f(p)) \cap cl_X(X \backslash cl_X V \cap X \backslash cl_X W) = \varnothing$. By 1.3(1), there is $R \in \mathcal{N}_{f(p)}$ such that $R \subseteq O \subseteq P$ and $f^{\leftarrow}[R] \cap cl_X(X \backslash cl_X V \cap X \backslash cl_X W) = \varnothing$. Note that $p \in U \cap f^{\leftarrow}[R]$ and that $cl_X U \cap f^{\leftarrow}[R] \cap cl_X(X \backslash cl_X V) \subseteq f^{\leftarrow}[R] \cap cl_{f^{\leftarrow}[O]}U \cap cl_X(X \backslash cl_X V) \subseteq f^{\leftarrow}[R] \cap X \backslash cl_X W \cap cl_X(X \backslash cl_X V) = \varnothing$ (as $f^{\leftarrow}[R] \cap X \backslash cl_X W \cap X \backslash cl_X V = \varnothing$). Thus, $cl_X U \cap f^{\leftarrow}[R] \subseteq X \backslash cl_X(X \backslash cl_X V) \subseteq int_X cl_X V$ and $cl_X U \cap f^{\leftarrow}[R] \subseteq X \backslash A$. Therefore, $cl_X U \cap f^{\leftarrow}[R] \cap A = \varnothing$. That is, $A \cap f^{\leftarrow}[R] \subseteq X \backslash cl_X U$. Since $p \in U \cap f^{\leftarrow}[R]$, it follows that f is regular. □

3.5 Corollary. *A Hausdorff function is perfect and regular if and only if it is H-closed, semiregular, and Urysohn.*

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