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# Homotopy properties of curves

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*Abstract.* Conditions are investigated that imply noncontractibility of curves. In particular, a plane noncontractible dendroid is constructed which contains no homotopically fixed subset. A new concept of a homotopically steady subset of a space is introduced and its connections with other related concepts are studied.

*Keywords:* continuum, contractible, curve, deformation, dendroid, fixed, homotopy, steady
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## 1. Introduction

Various concepts of points (or sets) of stability or fixation play an important role in investigation of homotopy properties of topological spaces. The reader is referred to concepts introduced and results obtained by H. Hopf and E. Pannwitz in [8], developed and modified by K. Borsuk and J.W. Jaworowski in [1], and also by A. Lelek in [9]. Some of the above mentioned concepts were local ones, connected either with local structure of the investigated continuum or with local homotopy properties. In the present paper we investigate some other homotopy properties of continua (especially of curves), ones which are global, and thereby are more suitable to study contractibility of curves in the whole.

After preliminaries, homotopically fixed sets are studied in the third section. It is shown that the existence of a homotopically fixed subset is not necessary for a dendroid to be contractible: namely in Example 3.4 a noncontractible plane dendroid is constructed that contains no homotopically fixed subset. Concepts of homotopically steady and strongly homotopically steady subsets of a space are introduced in the next section, and their connections with other related notions, mostly for dendroids, are studied. However, a problem concerning the existence of a homotopically steady point in any noncontractible dendroid remains open (Question 4.5).

## 2. Preliminaries

All considered spaces are assumed to be metric. We denote by $\mathbb{N}$ the set of all positive integers, by $\mathbb{R}$ the space of real numbers, and by $\mathbb{I}$ the closed unit interval $[0,1]$ of reals. Given two points $a$ and $b$ in the plane $\mathbb{R}^2$, let $ab$ denote the straight line segment with end points $a$ and $b$. The symbols $\mathrm{Li}$, $\mathrm{Ls}$ and $\mathrm{Lim}$ mean the lower limit, the upper limit and the topological limit of a sequence of sets. The reader is referred to e.g. [10] for definitions of various concepts used in the paper.

By a *mapping* we always mean a continuous function. Given a space $X$, a mapping $H : X \times \mathbb{I} \to X$ is called a *homotopy.* A homotopy $H$ such that $H(x, 0) = x$ for each point $x \in X$ is called a *deformation.* A space $X$ is said to be *contractible* if there exists a deformation $H : X \times \mathbb{I} \to X$ such that $H(X \times \{1\})$ is a singleton. It is well-known that each contractible curve is a uniformly arcwise connected dendroid (see e.g. [2, Propositions 1 and 4, p. 73]). Therefore, investigation of some obstructions of contractibility for curves can be restricted to uniformly arcwise connected dendroids only. One of such obstructions is the existence of an $R^i$-continuum. A nonempty proper subcontinuum $A$ of a dendroid $X$ is called an $R^i$*-continuum* (where $i \in \{1, 2, 3\}$) if there exist an open set $U$ containing $A$ and two sequences $\{C_n^1 : n \in \mathbb{N}\}$ and $\{C_n^2 : n \in \mathbb{N}\}$ of components of $U$ such that

$$A = \begin{cases} (\operatorname{Ls} C_n^1) \cap (\operatorname{Ls} C_n^2) & \text{for } \ i = 1, \\ (\operatorname{Lim} C_n^1) \cap (\operatorname{Lim} C_n^2) & \text{for } \ i = 2, \\ \operatorname{Li} C_n^1 & \text{for } \ i = 3. \end{cases} \tag{2.1}$$

If an $R^1$-continuum $A$ is a singleton $\{p\}$, then $p$ is called an *R-point.* If a dendroid $X$ contains an $R^i$-continuum, for some $i \in \{1, 2, 3\}$, then $X$ is not contractible, [6, Theorem 3, p. 300].

For more information on this and related concepts the reader is referred to [3] and [5].

## 3. Homotopically fixed sets

The following proposition is known (see [4, Proposition 1, p. 239]).

**3.1 Proposition.** *If a space $X$ contains two subsets $A$ and $B$ such that $\emptyset \neq A \subset B \neq X$ and*

$$\text{for every deformation} \quad H : X \times \mathbb{I} \to X \quad \text{we have} \quad H(A \times \mathbb{I}) \subset B, \tag{3.2}$$

*then $X$ is not contractible.*

A nonempty proper subset $A$ of a space $X$ is said to be *homotopically fixed* provided that condition (3.2) is satisfied with $B = A$. Examples of homotopically fixed subsets can be seen from the next result (see [6, Theorem 3, p. 300]).

**3.3 Proposition.** *Each $R^i$-continuum of a dendroid $X$ (where $i \in \{1, 2, 3\}$) is a homotopically fixed subset of $X$.*

Questions were asked in [2, Questions 1.4 and 1.5, p. 561] whether converse implication to that of Proposition 3.1 is true for dendroids; in particular, whether does each noncontractible dendroid contain any homotopically fixed subset? Recently these questions have been answered in the negative. The example is presented below.

**3.4 Example.** *There exists a noncontractible uniformly arcwise connected dendroid in the plane that contains no homotopically fixed subset.*

Proof: In the Cartesian coordinates $(x, y)$ in the plane $\mathbb{R}^2$ put $p = (0, 0)$, $q = (1, 0)$, and, for each $n \in \mathbb{N}$, let $p_n = (1, 1/n)$ and $q_n = (0, -1/n)$. Then the needed dendroid $X$ is defined by $X = pq \cup \bigcup\{pp_n \cup qq_n : n \in \mathbb{N}\}$.

Indeed, $X$ is noncontractible because it is of type $N$ between points $p$ and $q$, [11, Corollary 2.2, p. 839]. If a nonempty proper subset $A$ of $X$ is distinct from the limit segment $pq$, then one can find a deformation $H : X \times \mathbb{I} \to X$ such that $H(A \times \mathbb{I}) \setminus A \neq \emptyset$ in a routine way. Thus only the case of $A = pq$ needs a proof. We will prove that there exists a deformation $H$ such that $H(q, 1) = p_1$. To this aim we introduce an auxiliary notation. Given points $a$, $b$, $c$ and $d$ in $\mathbb{R}^2$ (not necessarily distinct), we put $abcd = ab \cup bc \cup cd$, and define a mapping $\beta(a, b, c, d) : [0, 3] \to abcd$ by

$$\beta(a,b,c,d)(t) = \begin{cases} (1-t)a + tb, & \text{if } t \in [0,1], \\ (2-t)b + (t-1)c & \text{if } t \in [1,2], \\ (3-t)c + (t-2)d & \text{if } t \in [2,3]. \end{cases}$$

Notice that $\beta$ is a continuous function depending on five variables $a$, $b$, $c$, $d$ and $t$. Then define the needed deformation $H : X \times \mathbb{I} \to X$ by the formula

$$H((x,y),t) = \begin{cases} \beta(p, p_n, p, p_1)((2t+1)x) & \text{if } (x,y) \in pp_n, \\ \beta(p, q, p, p_1)((2t+1)x) & \text{if } (x,y) \in pq, \\ \beta(q_n, q, p, p_1)((2t+1)x) & \text{if } (x,y) \in qq_n. \end{cases}$$

Clearly $H$ has the required properties, and thereby $A = pq$ is not a homotopically fixed subset of $X$. □

## 4. Homotopically steady sets

A nonempty subset $A$ of a space $X$ is said to be *homotopically steady* provided that

(4.1) for every deformation $H : X \times \mathbb{I} \to X$ we have $A \subset H(X \times \{1\})$.

In other words, $A$ is homotopically steady if and only if for every deformation $H$ as above and for every number $t \in \mathbb{I}$ the set $A$ is contained in the image $H(X \times \{t\})$ of the whole space $X$. In particular, a point $p \in X$ is called a *homotopically steady point* of $X$ if the singleton $\{p\}$ is a homotopically steady subset of $X$. For example, if $X$ is an arc, then no point (and no subset) of $X$ is homotopically steady, while for $X$ being a simple closed curve each subset of $X$, in particular each point, as well as the whole space $X$, is homotopically steady.

Denote by $\mathcal{D}(X)$ the family of all deformations on $X$, and put

$$\Sigma(X) = \bigcap\{H(X \times \{1\}) : H \in \mathcal{D}(X)\},$$

calling the above intersection the *kernel of steadiness* of $X$. Then condition (4.1) can be reformulated as $A \subset \Sigma(X)$, which leads to the following observation.

**4.2 Observation.** *A subset of a space $X$ is homotopically steady if and only if it is contained in the kernel $\Sigma(X)$ of steadiness of $X$.*

The concept of homotopical steadiness is related to contractibility by the following result.

**4.3 Theorem.** *Each contractible space has empty kernel of steadiness.*

Proof: Let a space $X$ be contractible. Thus there are a point $a \in X$ and a deformation $H_0 : X \times \mathbb{I} \to X$ with $H_0(X \times \{1\}) = \{a\}$. Let $b \in X \setminus \{a\}$. Since $X$ is arcwise connected, there is an arc $ab \subset X$. Let $h : \mathbb{I} \to ab$ be a homeomorphism with $h(0) = a$ and $h(1) = b$. Define a deformation $H : X \times \mathbb{I} \to X$ by $H(x,t) = H_0(x, 2t)$ for $t \in [0, 1/2]$, and $H(x,t) = h(2t-1)$ for $t \in [1/2, 1]$. Then $H(X \times \{1\}) = \{b\}$. Thus $\Sigma(X) \subset H_0(X \times \{1\}) \cap H(X \times \{1\}) = \emptyset$. □

A proof of the next result is left to the reader.

**4.4 Propositions.** *Let a continuum $X$ be hereditarily unicoherent.* (a) *If a subset $A$ of $X$ is homotopically steady, then the continuum $I(A)$ irreducible about $A$ is homotopically steady as well.* (b) *Each homotopically steady point of $X$ belongs to the maximal homotopically steady subset of $X$ which equals the kernel $\Sigma(X)$ of steadiness of $X$, and which is, therefore, a continuum.*

For the annulus $X = \{(x,y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 4\}$ we have $\Sigma(X) = \emptyset$, while $X$ is not contractible. Thus the converse to Theorem 4.3 is not true for arbitrary continua. It would be interesting to know whether this converse is true for a dendroid. Thus we have the following question.

**4.5 Question.** Does every noncontractible dendroid have nonempty kernel of steadiness (i.e. does every noncontractible dendroid contain a homotopically steady point)?

Below a result is shown that can be considered as a partial answer to Question 4.5.

**4.6 Proposition.** *If a noncontractible dendroid $X$ contains a point $p$ such that each component of its complement is contractible, then $p$ is a homotopically steady point of $X$.*

Proof: Suppose that $p$ is not homotopically steady. Thus there exists a deformation $H_0 : X \times \mathbb{I} \to X$ such that $p \in X \setminus H_0(X \times \{1\})$. Let $C$ be a component of $X \setminus \{p\}$ containing the continuum $H_0(X \times \{1\})$. Since $C$ is contractible, there is $q \in C$ and a deformation $H_1 : C \times \mathbb{I} \to C$ such that $H_1(C \times \{1\}) = \{q\}$. Then the mapping $H : X \times \mathbb{I} \to X$ defined by $H(x,t) = H_0(x, 2t)$ for $t \in [0, 1/2]$, and $H(x,t) = H_1(H_0(x,1), 2t-1)$ for $t \in [1/2, 1]$ is a deformation with $H(x,1) = H_1(x,1) = \{q\}$. Thus $X$ is contractible, a contradiction. □

The dendroid $X$ described in Example 3.4 has a nondegenerate kernel of steadiness, namely $\Sigma(X) = pq$, while no subset of $X$ is homotopically fixed. Thus the existence of a nondegenerate homotopically steady continuum in a dendroid does not imply the existence of a homotopically fixed subset in the considered dendroid.

The next example shows that a homotopically fixed subset of a dendroid need not be homotopically steady.

**4.7 Example.** *There exists a plane dendroid $X$ and a subcontinuum $A$ of $X$ such that $A$ is an $R^1$-, $R^2$-, and $R^3$-continuum in $X$, so it is homotopically fixed, while not homotopically steady.*

Proof: In the Cartesian coordinates $(x, y)$ in $\mathbb{R}^2$ put $c = (0,0)$, $p = (-1,0)$, $q = (1,0)$, $r = (0,1)$ and, for each $n \in \mathbb{N}$, let $p_n = (-1/n, 1/n)$, $q_n = (1/n, 1/n)$, $a_n = (-1/n, 1)$, and $b_n = (1/n, 1)$. Then the needed dendroid $X$ is defined by

$$X = pq \cup cr \cup \bigcup \{pp_n \cup p_n a_n : n \in \mathbb{N}\} \cup \bigcup \{qq_n \cup q_n b_n : n \in \mathbb{N}\}.$$

Put $A = cr$, $U = X \setminus \{p, q\}$, and for each $n \in \mathbb{N}$ let $C_n^1$ and $C_n^2$ denote components of $U$ that contain the points $a_n$ and $b_n$, respectively. Then equality (2.1) for $i = 1$ and $i = 2$ is satisfied. Putting $C'_{2n-j} = C_n^j$ for $j \in \{1, 2\}$ we get $A = \operatorname{Li} C'_n$, so $A$ is an $R^i$-continuum in $X$ for each $i \in \{1, 2, 3\}$, and by Proposition 3.3 it is a homotopically fixed subset of $X$.

Let a deformation $H : X \times \mathbb{I} \to X$ be defined by

$$H((x, y), t) = \begin{cases} (x, y) & \text{if } y \le |x|, \\ (x, t(x - y) + y) & \text{if } y > |x|. \end{cases}$$

Then $H(X \times \{1\}) \subset \{(x, y) \in X : y \le |x|\}$, and we see that $A$ is not contained in $H(X \times \{1\})$. Therefore $A$ is not a homotopically steady subset of $X$. The proof is then complete. □

**4.8 Remarks.** (a) Example 4.7 shows that an analog of Proposition 3.3 is not true, i.e., the term "homotopically fixed" cannot be replaced by "homotopically steady" in that result. (b) In the dendroid $X$ constructed in Example 4.7 the kernel $\Sigma(X)$ of steadiness is nondegenerate (it equals the segment $pq$). Thus the following questions, which are related to Question 4.5 and to Proposition 3.3, are natural and interesting.

**4.9 Questions.** (a) Does the existence of a homotopically fixed subset in a dendroid imply the existence of a homotopically steady subset? (b) What are interrelations between $R^i$-continua and homotopically steady subsets of dendroids? More precisely, let an $R^i$-continuum $A$ (for some $i \in \{1, 2, 3\}$) be contained in a dendroid $X$. Must $A$ contain a nonempty homotopically steady subset of $X$?

The concept of a homotopically steady subset of a space can be strengthened by demanding that for each deformation it remains not only in the image of the whole space, but even in the image of itself. More precisely, a nonempty subset $A$ of a space $X$ is said to be *strongly homotopically steady* provided that

(4.10) for every deformation $H : X \times \mathbb{I} \to X$ we have $A \subset H(A \times \{1\})$.

In particular, a point $p \in X$ is called a *strongly homotopically steady point* of $X$ if the singleton $\{p\}$ is a strongly homotopically steady subset of $X$. In other words, $p \in X$ is a strongly homotopically steady point of $X$ if and only if

(4.11) for every deformation $H : X \times \mathbb{I} \to X$ we have $H(\{p\} \times \mathbb{I}) = \{p\}$.

Obviously, it follows from 4.1 and 4.10 that each strongly homotopically steady set is homotopically steady. The opposite implication does not hold, as it can be seen from Example 4.7. Namely the middle point $c = (0, 0)$ of the straight line segment $pq$ is a homotopically steady point, while not a strongly homotopically steady point of the dendroid, that can be seen using a deformation of $X$ which pushes $c$ a little bit along $cr$ in the direction to $r$, forcing to move the points $p_n$ along $p_n a_n$ as well as $q_n$ along $q_n b_n$ in the directions to $a_n$ and to $b_n$, respectively. Furthermore, using the above mentioned deformation and the deformation $H$ defined in Example 4.7, we can observe that no point of the dendroid $X$ of Example 4.7 is strongly homotopically steady. However, the following questions are unanswered.

**4.12 Questions.** (a) Does the existence of a homotopically steady point (nondegenerate subcontinuum) imply the existence of a strongly homotopically steady nondegenerate subcontinuum in a dendroid? (b) In particular, is $\Sigma(X)$ a strongly homotopically steady subcontinuum of a dendroid $X$?

The following result is an analog of Proposition 4.4 (a). Its proof is omitted.

**4.13 Proposition.** *If a subset $A$ of a hereditarily unicoherent continuum $X$ is strongly homotopically steady, then the continuum $I(A)$ irreducible about $A$ is strongly homotopically steady as well.*

A compact space is said to be *rational* provided it has an open basis composed entirely of sets whose boundaries are countable. In [7, Example 3.2, p. 192], a rational uniformly arcwise connected dendroid $X$ in the plane is constructed (called below the *Fitzpatrick-Lelek dendroid*) such that each nonempty connected open subset of $X$ is dense in $X$. Thus, if we denote by $L(X)$ the set of all points of $X$ at which it is locally connected, we have $L(X) = \emptyset$. On the other hand, the set $K(X)$ of all points of $X$ at which it is connected im kleinen is known to be a dense subset of $X$ [7, Theorem 3.1, p. 191]. The Fitzpatrick-Lelek dendroid has a very strong homotopical property. To formulate it, we recall two concepts related to mappings. A mapping $f : A \to B$ between topological spaces $A$ and $B$ is said to be:

— *interior at a point* $p \in A$ provided that $f(p) \in \operatorname{int} U$ for each open subset $U \subset A$ containing $p$;

— *strongly homotopically stabile at a point* $p \in A$ provided that for each homotopy $H : A \times \mathbb{I} \to B$ with $H(a, 0) = f(a)$ for $a \in A$ we have $H(p, t) = f(p)$ for each $t \in \mathbb{I}$, i.e., $H(\{p\} \times \mathbb{I}) = \{f(p)\}$.

The following theorem is shown in [9, Example 1.2, p. 195].

**4.14 Theorem.** *The Fitzpatrick-Lelek dendroid $X$ contains a countable dense subset $D \subset X$ such that each mapping $f$ of a topological space $T$ into $X$ which is interior at a point $p \in T$, where $f(p) \in D$, is strongly homotopically stabile at $p$.*

Taking $T = X$ and $f : X \to X$ as the identity mapping we conclude from Theorem 4.14 that (4.11) holds for each point $p \in D \subset \operatorname{cl} D = X$, so we get the following result.

**4.15 Statement.** *Each point of the Fitzpatrick-Lelek dendroid $X$ is strongly homotopically steady; thus $\Sigma(X) = X$.*

**4.16 Remark.** By a modification of the construction of the Fitzpatrick-Lelek dendroid another plane dendroid $E'$ is shown in [9, Remarks, p. 196] having all the above mentioned homotopy properties of $X$ and such that not only $K(E')$ but also $L(E')$ is a dense subset of $E'$. By the same argument as for $X$ one can see that $E'$ has the property that its kernel of steadiness equals the whole dendroid.

The above remark, as well as Statement 4.15, motivate the following problems.

**4.17 Problems.** (a) Give an internal characterization of dendroids $X$ for which $\Sigma(X) = X$. (b) Give an internal characterization of dendroids $X$ such that each point of $X$ is strongly homotopically steady.

A point $e$ of a dendroid $X$ is called an *end point* of $X$ provided that it is an end point of every arc contained in $X$ and containing $e$. Observation 4.2 and Proposition 4.4 imply the next result, that is related to Problem 4.17 (a).

**4.18 Statement.** *For each dendroid $X$ the following conditions are equivalent:*

(a) $\Sigma(X) = X$;
(b) *the set $E(X)$ of all end points of $X$ is homotopically steady;*
(c) *each end point of $X$ is a homotopically steady point of $X$.*

The following question (asked by W.J. Charatonik) which is related to Question 4.5, seems to be interesting by itself. Let $X$ be a dendroid having a nondegenerate kernel $\Sigma(X)$ of steadiness. Shrink $\Sigma(X)$ to a point. The resulting space, being a monotone image of a dendroid, is a dendroid, too.

**4.19 Question.** Given a dendroid $X$ with a nondegenerate kernel $\Sigma(X)$ of steadiness, is the dendroid $X/\Sigma(X)$ always contractible?

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