# Asymptotic analysis for a nonlinear parabolic equation on $\mathbb{R}$

Eva Fašangová

*Abstract.* We show that nonnegative solutions of

$$u_t - u_{xx} + f(u) = 0, \quad x \in \mathbb{R}, \quad t > 0,$$
$$u = \alpha\bar{u}, \quad x \in \mathbb{R}, \quad t = 0, \quad \operatorname{supp}\bar{u} \ \text{ compact}$$

either converge to zero, blow up in $\mathrm{L}^2$-norm, or converge to the ground state when $t \to \infty$, where the latter case is a threshold phenomenon when $\alpha > 0$ varies. The proof is based on the fact that any bounded trajectory converges to a stationary solution. The function $f$ is typically nonlinear but has a sublinear growth at infinity. We also show that for superlinear $f$ it can happen that solutions converge to zero for any $\alpha > 0$, provided $\operatorname{supp}\bar{u}$ is sufficiently small.

*Keywords:* parabolic equation, stationary solution, convergence

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## 1. Introduction

In this paper we investigate the asymptotic behaviour of positive (classical) solutions of the equation

$$\begin{aligned} &u_t(t,x) - u_{xx}(t,x) + f(u(t,x)) = 0, \quad x \in \mathbb{R}, \quad t \geq 0, \\ &\lim_{|x|\to\infty} u(t,x) = 0, \quad t \geq 0 \end{aligned} \tag{1.1}$$

with initial condition

$$u(0,x) = u_0(x), \quad x \in \mathbb{R}. \tag{1.2}$$

Equations of this type arise for example in physics in modelling the heat propagation or in biological models of population dynamics.

Let $f : [0,\infty) \mapsto \mathbb{R}$. We denote $F(s) = \int_0^s f(\tau)\,\mathrm{d}\tau$ and

$$\zeta_0 = \inf\{s > 0;\ F(s) \leq 0\}.$$

For the purposes of the paper we will use the following hypotheses:

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(F1) $f \in \mathcal{C}^{1+\mu}([0,\infty))$ for some $\mu > 0$,
(F2) $f(0) = 0$,
(F3) $f'(0) > 0$,
(F4) $0 < \zeta_0 < \infty$ and $f(\zeta_0) < 0$,
(F5) $f(s) \le ks$ for some positive constant $k$,
(F6) $f(s) \ge -ks$ for some positive constant $k$.

We will deal with initial data from the set

$$I = \{\bar u \in \mathrm{W}^{1,2}(\mathbb{R});\ \operatorname{supp} \bar u \ \text{ is compact},\ \bar u \ge 0,\ \bar u \not\equiv 0\}.$$

Then we can state our main results.

**Theorem 1.1.** *For any $f$ satisfying* (F1)–(F6) *and $\bar u \in I$ there exists a critical number $\alpha_c \in (0,\infty)$ such that*

(i) *if $\alpha \in [0,\alpha_c)$ and $u_0 = \alpha \bar u$, then the solution of* (1.1)–(1.2) *satisfies*

$$\lim_{t\to\infty} u(t,.) = 0 \ \ \textit{in} \ \ \mathrm{W}^{1,2}(\mathbb{R}); \tag{1.3}$$

(ii) *if $\alpha \in (\alpha_c,\infty)$ and $u_0 = \alpha \bar u$, then the solution satisfies*

$$\liminf_{t\to\infty}\big(\inf_{|x|\le R} u(t,x)\big) \ge \zeta_0 \ \ \textit{for any} \ \ R > 0; \tag{1.4}$$

(iii) *if $u_0 = \alpha_c \bar u$, then there exists an $\bar x \in \mathbb{R}$ such that the solution satisfies*

$$\lim_{t\to\infty} u(t,.) = w_g(. - \bar x) \ \ \textit{in} \ \ \mathrm{W}^{1,2}(\mathbb{R}), \tag{1.5}$$

*where $w_g$ is the unique positive symmetric ($w(-x) = w(x)$) solution (ground state) of the stationary problem*

$$-w_{xx} + f(w) = 0,\ x \in \mathbb{R}, \quad w \ge 0, \quad w \not\equiv 0, \quad w \in \mathcal{C}(\mathbb{R}), \tag{1.6}$$

$$w(\pm\infty) = 0. \tag{1.7}$$

This theorem is a one-dimensional analogy of the result of Feireisl-Petzeltová [1], where a similar statement is proved for the space domain $\mathbb{R}^N$, $N \ge 3$ and special nonlinearity

$$f(u) = u + \sum_{j=1}^{n} b_j u^{r_j} - \sum_{i=1}^{m} a_i u^{p_i}, \quad a_i, b_j > 0, \quad 1 < r_j < p_i \le \frac{N}{N-2},$$

(which violates (F6)) with the exception that in (ii) of [1] the solution blows up in finite time. The proof of Theorem 1.1 is motivated by [1], namely the use of the method of Zelenyak [2] to show convergence of a trajectory to a single stationary solution. This is possible due to the properties of the linearized problem, which, in our case, is solved directly by methods of ordinary differential equations. Also, in [1] the method of concentrated compactness is involved, but in our paper we can overcome this point and prove directly compactness of bounded trajectories (Proposition 5.5). In this sense the calculation is less sophisticated for $N = 1$ than it is in [1] for $N \geq 3$.

An essential ingredient in the proof of Theorem 1.1 is convergence of relatively compact trajectories. Convergence of relatively compact trajectories to a time-periodic solution have been proved recently by Feireisl-Poláčik [3] under more general assumptions, namely the nonlinear term $f = f(t, u)$ is periodic in time and $u_0 \in \mathcal{C}_0(\mathbb{R})$. In our case, thanks to the energy, there are no nontrivial (i.e. nonconstant in time) time-periodic solutions to (1.1)–(1.2), so convergence would follow from [3], but our proof is simpler and we do not need to investigate the set of zeros of solutions to the linearized problem ("zero numbers"). Moreover, Theorem 1.1 gives a complete characterization of the long-time behaviour of solutions and shows that convergence to the ground state is a threshold phenomenon, which is the main result of the first part of this paper. In the second part we give an example showing that the assumption (F5) is in some sense necessary to obtain the threshold result (ii) (cf. Theorem 1.2).

In the classical paper of Chaffee [4] it was shown that under the hypotheses $f \in \mathcal{C}^3(\mathbb{R})$, (F2), (F3) and

$$f(\zeta_1) > 0 \quad \text{for some} \quad \zeta_1 \in (0, \zeta_0),$$

any solution converges (uniformly on bounded sets of $\mathbb{R}$) to zero as $t \to \infty$, provided the initial condition $u_0 \in L^2(\mathbb{R})$ is uniformly continuous and satisfies

$$0 \leq u_0(x) \leq \zeta_1, \qquad x \in \mathbb{R}.$$

In the setting of Theorem 1.1 it says the following: Suppose that $\zeta_0 = +\infty$ instead of (F4); then for any $\alpha \geq 0$ we have (i). In the present paper it will be shown that if $\zeta_0 < \infty$, then any solution emanating from $u_0 \in I$ and bounded in the $\mathrm{L}^\infty$-norm either converges to a stationary solution (which is 0 if (F4) does not hold, and 0 or $w_g(. - \bar{x})$ if (F4) holds), or satisfies (1.4) (in particular it is unbounded in integral-norm).

The hypothesis (F2) ensures that zero is a stationary solution and (F4) is a necessary and sufficient condition for the existence of the ground state. (F3) implies stability of the zero solution. In particular, instead of (F3) one could assume

$$f > 0 \quad \text{on} \quad (0, \varepsilon), \quad \text{for some} \quad \varepsilon > 0$$

in order to obtain stability of 0 in $\mathcal{C}_0(\mathbb{R})$, but for the method used in the proof of convergence of bounded trajectories (Proposition 5.1) (F3) is essential. (F6) guaranties that the solution is defined for any $t > 0$. The hypothesis on the compactness of $\operatorname{supp} \bar{u}$ is technical and it is used to prove convergence of bounded trajectories emanating from $\alpha\bar{u}$. This hypothesis can be replaced by

$$\bar{u} > 0, \quad \bar{u}(x) = \bar{u}(-x) \leq w_g(x-R) \text{ for } x > R, \quad \bar{u} \text{ is nonincreasing on } (R, \infty),$$

for some $R > 0$ (Lemma 5.4 works if this replaces the assumption "$\operatorname{supp} \bar{u}$ compact"). The next theorem shows that the condition (F5) is natural in the sense that if $f$ is superlinear, then $\alpha_c$ can be $+\infty$.

**Theorem 1.2.**

(a) *Let $f$ be defined on some interval $[0, a_1]$ and satisfy* (F1)–(F4) *with $\zeta_0 < a_1$, $f(a_1) = 0$. Let $\alpha > 1$ and $s_0 > a_1$ be fixed.*

*Then the following assertion holds: There exist positive constants $\bar{M}$, $\bar{r}$ such that if $f$ is prolonged to $[0, \infty)$ in such a way that $f \in \mathcal{C}^{1+\mu}$, $\mu > 0$, $f > 0$ on $(a_1, s_0)$ and*

(F7) $\qquad f(s) \geq M s^\alpha, \quad s \geq s_0,$

*for some $M \geq \bar{M}$, then the solution of* (1.1)–(1.2) *converges to* 0 *in* $\mathrm{W}^{1,2}(\mathbb{R})$ *as $t \to \infty$, provided* meas $\operatorname{supp} u_0 \leq \bar{r}$.

(b) *Let $f$ be defined on $[0, \infty)$, satisfy* (F1)–(F4) *and* (F7) *for some $M > 0$, $s_0 > \zeta_0$ and, in addition, $\alpha > 3$.*

*Then there exists a positive number $\bar{r}$ such that if* meas $\operatorname{supp} u_0 \leq \bar{r}$, *then the solution of* (1.1)–(1.2) *converges to* 0 *in* $\mathrm{W}^{1,2}(\mathbb{R})$ *as $t \to \infty$.*

The paper is organized as follows. Section 2 is a review of the existence theory for the problem. In Section 3 the stationary problem is solved. Section 4 contains some spectral properties of the corresponding linear operator. In Section 5 we show that bounded trajectories are always convergent. Here we use energy estimates, symmetry arguments and we take advantage of the fact that 0 is a simple isolated eigenvalue of the linearized (at the ground state) operator. In Section 6 the unbounded trajectories are studied (the word "unbounded" refers to unboundedness in integral norm), using the result of Fife-McLeod [5] on stability of travelling fronts. Finally, Sections 7 and 8 contain the proofs of Theorems 1.1 and 1.2. The comparison principle is used throughout the paper.

## 2. Existence theory

In this section we assume that $f$ is locally Lipschitz continuous and $f(0) = 0$. The evolution problem (1.1)–(1.2) can be solved using the theory of analytic semigroups, cf. [6].

**Proposition 2.1.** *Suppose $f$ is locally Lipschitz continuous and $f(0) = 0$. For any nonnegative $u_0 \in \mathrm{W}^{1,2}(\mathbb{R})$ there exists a unique solution of* (1.1)–(1.2)

$$u \in \mathcal{C}([0, T_{max}), \mathrm{W}^{1,2}(\mathbb{R})) \cap \mathcal{C}^1((0, T_{max}), \mathrm{W}^{1,2}(\mathbb{R}))$$

*where $[0, T_{max})$ is the maximal interval of existence. This solution is a classical solution in the sense that all derivatives appearing in* (1.1) *are continuous in* $(0,\infty)\times\mathbb{R}$ *and $u$ is continuous in* $[0,\infty)\times\mathbb{R}$. *If $T_{max} < \infty$, then*

$$\sup_{t\in[0,T_{max})} \|u(t)\|_{\mathrm{W}^{1,2}} = \infty. \tag{2.1}$$

*The energy functional associated to* (1.1),

$$E_u(t) = \frac{1}{2}\int_{\mathbb{R}} u_x^2(t,x)\,\mathrm{d}x + \int_{\mathbb{R}} F(u(t,x))\,\mathrm{d}x \tag{2.2}$$

*is nonincreasing along any trajectory and*

$$\frac{\mathrm{d}}{\mathrm{d}t}E_u(t) = -\int_{\mathbb{R}} u_t^2(t,x)\,\mathrm{d}x. \tag{2.3}$$

*If $u_0 \not\equiv 0$, then $u(t,x) > 0$ for any $t \in (0, T_{max})$ and $x \in \mathbb{R}$.*

The first part of the proposition may be deduced from [6, Theorems 3.3.3 and 3.5.2] applied on the basic space $\mathrm{L}^2(\mathbb{R})$, the nonlinearity $f$ being locally Lipschitz continuous from $\mathrm{W}^{1,2}(\mathbb{R})$ into $\mathrm{L}^2(\mathbb{R})$. Since $u_0$ is Hölder continuous, this solution is a classical solution. A-priori estimates show that the solution is unbounded if its existence interval is bounded. The relation (2.3) can be obtained by multiplying the equation (1.1) by $u_t$ and integrating by parts over $\mathbb{R}$. The last statement is an application of the strong comparison principle (cf. Proposition 2.2).

**Proposition 2.2.** *Let $f, g$ be locally Lipschitz continuous, $f(0) = g(0) = 0$.*

(i) *Let $u$ and $v$ be solutions from Proposition* 2.1 *satisfying the inequalities*

$$u_t - u_{xx} + f(u) \geq v_t - v_{xx} + f(v), \quad (t,x) \in \operatorname{int}\mathcal{M}, \tag{2.4}$$

$$u \geq v, \quad (t,x) \in \partial\mathcal{M}, \tag{2.5}$$

*the derivatives appearing in* (2.4) *being continuous, where $\mathcal{M} = [0,\infty)\times(a,b)$, $-\infty \leq a < b \leq \infty$. Then $u \geq v$ in $\mathcal{M}$. If moreover $u(0,x) > v(0,x)$ in an open subinterval of $(a,b)$, then $u > v$ in $\operatorname{int}\mathcal{M}$.*

(ii) *The same assertion holds if we suppose instead of (2.4) that $f \leq g$ and*

$$u_t - u_{xx} + f(u) = v_t - v_{xx} + g(v) = 0, \quad (t,x) \in \operatorname{int}\mathcal{M}. \tag{2.6}$$

This comparison principle can be deduced from the strong maximum principle for linear parabolic inequalities (cf. [7]) applied to the function $w = u - v$ satisfying

$$w_t - w_{xx} + \frac{f(u)-f(v)}{u-v}w \geq 0 \ \text{ in } \ \operatorname{int}\mathcal{M}, \quad w \geq 0 \ \text{ on } \ \partial\mathcal{M}. \tag{2.7}$$

Using standard a-priori estimates and the comparison principle to estimate the solution of (1.1) with the solution of the ordinary differential equation

$$z_t + f(z) = 0, \quad t > 0 \tag{2.8}$$

we can prove the following lemma.

**Lemma 2.3.** *If, in addition to the hypotheses of Proposition* 2.1, $f$ *satisfies also* (F6)*, then any solution is global.*

Continuous dependence on initial data is also a standard result, see [6, Theorem 3.4.1].

**Proposition 2.4.** *Suppose $f$ is locally Lipschitz continuous and $f(0) = 0$. Assume $u_0^n \in \mathrm{W}^{1,2}(\mathbb{R})$, $n = 1, 2 \dots$ are nonnegative and $u_0^n \to u_0$ in $\mathrm{W}^{1,2}(\mathbb{R})$. Let $u^n$ be the solutions of* (1.1) *defined on $[0, T_{max}^n)$ corresponding to initial data $u_0^n$.*

*Then the solution $u$ of* (1.1)–(1.2) *with initial datum $u_0$ exists on $[0, T_{max})$ with*

$$\liminf_{n\to\infty} T_{max}^n \geq T_{max},$$

*and for any $T < T_{max}$ we have*

$$\lim_{n\to\infty} u^n = u \quad in \quad \mathcal{C}([0,T], \mathrm{W}^{1,2}(\mathbb{R})).$$

## 3. The stationary problem

In this section we assume that $f$ is locally Lipschitz continuous, $f(0) = 0$ and $\zeta_0 > 0$. The following statements can be proved by standard techniques for ordinary differential equations found for example in [8] (the first of them is adopted from [9]).

**Lemma 3.1.** *The problem* (1.6)–(1.7) *admits a solution if and only if $f$ satisfies* (F4)*. Moreover, if* (F4) *is satisfied, then the solution is unique up to a translation of the origin and after a suitable translation satisfies*

(i) $w(x) = w(-x), \quad x \in \mathbb{R}$,
(ii) $w(x) > 0, \quad x \in \mathbb{R}$,
(iii) $w(0) = \zeta_0$,
(iv) $w'(x) < 0, \quad x > 0$,
(v) *if* $f'(0) > 0$, *then* $w \in \mathrm{W}^{2,2}(\mathbb{R})$.

We denote by $w_g$ the solution which satisfies (i), (ii) and call it the *ground state.*

**Lemma 3.2.**

(i) *If $\zeta_0 = \infty$, then the problem* (1.6) *has a unique* (*up to spatial shift*) *solution $w$ satisfying $w(\infty) = 0$, $w(-\infty) = \infty$ and $w$ is decreasing.*
(ii) *If $0 < \zeta_0 < \infty$ and $f(\zeta_0) = 0$, then* (1.6) *has a unique* (*up to spatial shift*) *solution $w$ satisfying $w(-\infty) = \zeta_0$, $w(\infty) = 0$ and $w$ is decreasing.*

**Corollary 3.3.** *The set of stationary solutions* (*i.e. solutions of* (1.6)–(1.7)) $S = \{0\}$ *if $f$ does not satisfy* (F4)*, and* $S = \{0, w_g(. - \bar{x}); \bar{x} \in \mathbb{R}\}$ *if $f$ satisfies* (F4).

## 4. The linear problem

In this section we assume that $f$ satisfies the hypotheses (F1), (F2), (F3) and (F4).

**Lemma 4.1.** *Suppose* (F1)–(F4). *The problem*

$$-v''(x) + f'(w_g(x))v(x) = 0,\ x \in \mathbb{R}, \quad v \in \mathrm{W}^{1,2}(\mathbb{R}) \tag{4.1}$$

*admits a unique solution $w_g'$ up to a multiplicative constant.*

PROOF: By differentiating the equation (1.6) we see that $\varphi = w_g'$ is a solution of (4.1). The substitution of variables $z = (\frac{v}{\varphi})'$ leads to the equation $z'\varphi + 2z\varphi' = 0$ which has an explicit solution $z(x) = \varphi^{-2}(x)$ and hence

$$v(x) = c_1\varphi(x)\int_0^x \varphi^{-2}(\tau)\,\mathrm{d}\tau + c_2\varphi(x)$$

is a general solution of the equation in $(0,\infty)$. Applying l'Hospitals rule we find

$$\lim_{x\to\infty} v(x) = c_1 \lim_{x\to\infty} \frac{\int_0^x \varphi^{-2}(\tau)\,\mathrm{d}\tau}{\varphi^{-1}(x)} + 0 = -c_1 \lim_{x\to\infty} \frac{1}{\varphi'(x)},$$

where the last limit is 0 only for $c_1 = 0$. □

We can characterize the spectrum of the linear operator

$$L_{w_g} : \mathrm{L}^2(\mathbb{R}) \mapsto \mathrm{L}^2(\mathbb{R}), \quad D(L_{w_g}) = \mathrm{W}^{2,2}(\mathbb{R}),$$
$$L_{w_g}v = -v_{xx} + f'(w_g)v, \quad v \in D(L_{w_g}).$$

**Proposition 4.2.** *Suppose* (F1)–(F4). *The spectrum of $L_{w_g}$ consists of the essential spectrum $\sigma_e = [f'(0),\infty)$ and the simple eigenvalues* 0 (*the corresponding eigenfunction is $w_g'$*) *and* $-\lambda < 0$ (*the corresponding eigenfunction is strictly positive*).

PROOF: Since $\lim\limits_{|x|\to\infty} f'(w_g(x)) = f'(0)$, it is a classical result about the spectrum of the Schrödinger operator that $\sigma = [f'(0),\infty) \cup \{-\lambda_k, -\lambda_{k-1}, \dots, \lambda_0\}$, where $-\lambda_k < -\lambda_{k-1} < \dots < \lambda_0 = 0$ are eigenvalues (cf. [10, Section XIII.4]). By Lemma 4.1, 0 is a simple eigenvalue. By [10, Theorem XIII.44], $-\lambda_k$ is a simple eigenvalue and the corresponding eigenfunction is positive. Since $w_g'$ changes sign, 0 is not the smallest eigenvalue. By the Sturm-Liouville theory the eigenfunction $e_i$ corresponding to $-\lambda_i$ has $k - i + 2$ zeros (together with $\pm\infty$), (see for example [11, Chapter VIII, Section 1]). Since $w_g'$ has 3 zeros, $k = 1$. □

**Proposition 4.3.** *Suppose* (F1)–(F4) *and in addition* (F6). *Then the stationary solutions of the evolution problem* (1.1)–(1.2) *have the following properties:*

(i) 0 *is locally asymptotically stable in* $\mathrm{L}^\infty(\mathbb{R})$*;*

(ii) $w_g$ *is unstable in the following sense: Let* $u$ *be a global solution of* (1.1)–(1.2) *such that* $\lim_{t\to\infty} u(t) = w_g(. - \bar{x})$ *for some* $\bar{x} \in \mathbb{R}$. *Let* $v_0 \in I$ *be such that* $v_0 \geq u_0$, $v_0 \not\equiv u_0$. *Then the solution* $v$ *of* (1.1) *with the initial value* $v(0) = v_0$ *satisfies*

$$\lim_{t\to\infty} \|u(t)\|_{\mathrm{L}^2(\mathbb{R})} = \infty.$$

The part (i) is an immediate consequence of the stability of 0 for (2.8) and the comparison principle. The proof of (ii) is based on the existence of a negative eigenvalue of the corresponding linear operator and is postponed into Section 5.

## 5. Convergence of bounded trajectories

We denote $S = \{0, w_g(. - \bar{x});\ \bar{x} \in \mathbb{R}\}$, the set of stationary solutions (i.e. solutions of (1.6)–(1.7)) when (F4) is satisfied. The main result of this section is the following.

**Proposition 5.1.** *Suppose* (F1), (F2), (F3) *and* $u_0 \in I$. *Suppose that the solution* $u$ *of* (1.1)–(1.2) *satisfies*

$$0 \leq u(t,x) \leq c,\ x \in \mathbb{R},\ t \geq 0; \tag{5.1}$$

$$u(t, \pm r) \leq \zeta_1,\ t \geq 0; \quad u(0,x) = 0,\ |x| \geq r, \tag{5.2}$$

*for some* $c > 0$, $\zeta_1 < \zeta_0$, $r > 0$. *Then there exists a stationary solution* $w \in S$ *such that*

$$u(t) \to w \ \text{ in } \ \mathcal{C}_0(\mathbb{R}) \ \text{ as } \ t \to \infty. \tag{5.3}$$

Here we use the notation $\mathcal{C}_0(\mathbb{R}) = \{w \in \mathcal{C}(\mathbb{R});\ w(\pm\infty) = 0\}$ with the topology of uniform convergence. The proof consists of three steps: proving relative compactness, investigating possible limits and proving convergence to a single element.

**Lemma 5.2.** *Suppose* (F1), (F2), (F3). *Let* $u$ *be a solution of* (1.1)–(1.2) *satisfying* (5.1), (5.2). *Then the set* $\{u(t),\ t \geq 0\}$ *is relatively compact in* $\mathcal{C}_0(\mathbb{R})$.

Proof: There exists a solution $w$ on $[r, \infty)$ of (1.6) with $w(r) = \zeta_1$, $w(\infty) = 0$ which is decreasing. Using the comparison principle for $u$ and $w$ on $\mathcal{M} = [0, \infty) \times [r, \infty)$ we get $u(t,x) \leq w(x)$, $t \geq 0$, $x \geq r$ (and analogously for $x \leq -r$).

Since $F > 0$ on $(0, \zeta_0)$, the energy can be estimated in the following way:

$$(5.4)\qquad \begin{aligned} E_u(0) \geq E_u(t) &= \frac{1}{2}\int_{\mathbb{R}} u_x^2(t) + \int_{|x|\leq r} F(u(t)) + \int_{|x|>r} F(u(t)) \\ &\geq \frac{1}{2}\int_{\mathbb{R}} u_x^2(t) + 2r \inf_{[0,c]} F, \end{aligned}$$

hence $\{u_x(t);\ t \geq 0\}$ is bounded in $\mathrm{L}^2(\mathbb{R})$.

The assertion follows from the theorem of Arzela-Ascoli ($u(t)$ are bounded, equicontinuous on every bounded subinterval of $\mathbb{R}$ and $u(t,x) \to 0$ as $|x| \to \infty$ uniformly in $t$). $\square$

**Lemma 5.3.** *Suppose* (F1), (F2), (F3). *Let $u_0 \in I$ and suppose that the solution $u$ of* (1.1)–(1.2) *satisfies* (5.1), (5.2). *Then $u(t) \to S$ as $t \to \infty$.*

Proof: Since the energy is nonincreasing, from (5.4) we get that $E_u(t)$ is bounded. So by (2.3) we have

$$(5.5)\qquad \int_0^\infty \int_{\mathbb{R}} u_t^2 = E_u(0) - E_u(\infty) < \infty.$$

We want to conclude that $u_t(t) \to 0$ in $\mathrm{L}^2(\mathbb{R})$ as $t \to \infty$. By differentiating (1.1) with respect to $t$ we find that the function $v = u_t$ satisfies

$$(5.6)\qquad v_t - v_{xx} + f'(u)v = 0.$$

Multiplying (5.6) by $v$ and integrating by parts over $\mathbb{R}$ we get

$$(5.7)\qquad \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} v^2 = 2\int_{\mathbb{R}} vv_t = -2\int_{\mathbb{R}} v_x^2 + v^2 f'(u) \leq 2\sup_{[0,c]}(f')\int_{\mathbb{R}} v^2.$$

Hence the function $\varphi(t) = \int_{\mathbb{R}} u_t^2(t)$ satisfies $\varphi(t) \leq e^{c_1(t-s)}\varphi(s),\ t \geq s$ for some $c_1 > 0$ and $\varphi \in \mathrm{L}^1(0,\infty)$. It is an exercise that then $\varphi(t) \to 0$ as $t \to \infty$.

Let $u(t_n) \to w$ in $\mathcal{C}_0(\mathbb{R})$ for $t_n \to \infty$. Then passing to the limit in the equation (1.1) in the sense of distributions we find that $w \in S$. $\square$

Proof of Proposition 5.1: The case when $u(t_n) \to 0$ for some $t_n \to \infty$ is straightforward by Proposition 4.3 (i). The nontrivial part is that in the case when

$$u(t) \to \{w_g(. - \bar{x}),\ \bar{x} \in \mathbb{R}\}$$

we have also convergence to a single element. To this end one can use the method of Zelenyak [2] as in [1], using the results of Section 4 (namely that the eigenvalue 0 is an isolated point of the spectrum of $L_{w_g(.-\bar{x})}$ and is simple). The assumption $f'(0) > 0$ is crucial here. $\square$

The main consequence of the compactness of the support of the initial value is the content of the following lemma (cf. [1] for the more dimensional case).

**Lemma 5.4.** *Let* $u_0 \in I$ *and let* $\operatorname{supp} u_0 \subset [-r, r]$. *If* $u$ *is the solution of* (1.1)–(1.2) *from Proposition* 2.1, *then for any* $\lambda \geq r$ *we have*

$$u(t, \lambda - x) \geq u(t, \lambda + x), \quad x \geq 0, \quad t \geq 0,$$

*and similarly for any* $\lambda \leq -r$ *we have*

$$u(t, \lambda - x) \leq u(t, \lambda + x), \quad x \geq 0, \quad t \geq 0.$$

*In particular,* $u(t, .)$ *is nonincreasing in* $[r, \infty)$ *and nondecreasing in* $(-\infty, -r]$, *for any* $t \geq 0$.

Proof: Let $\lambda \geq r$ be fixed. We define the function $v(t, x) = u(t, 2\lambda - x)$ for $t \geq 0$, $x \geq \lambda$. Then $v$ is a solution of (1.1) in $\mathcal{M} = \{[t, x],\ x \geq \lambda,\ t \geq 0\}$ satisfying the boundary condition $v \geq u$ on $\partial\mathcal{M}$. Hence $v \geq u$ in $\mathcal{M}$. □

**Proposition 5.5.** *Suppose* (F1), (F2), (F3), (F6) *and* $u_0 \in I$. *Let* $u$ *be the solution of* (1.1)–(1.2). *If* $u$ *satisfies* (5.2) *with some* $\zeta_1 < \zeta_0$, $r > 0$, *then the conclusion of Proposition* 5.1 *remains true. If* (5.2) *does not hold, then*

$$\lim_{t\to\infty} \|u(t)\|_{\mathrm{L}^2(\mathbb{R})} = \infty. \tag{5.8}$$

Proof: Suppose first that (F5) holds too. We will show that then the sublinearity of $f$ and (5.2) imply (5.1). We note first that since $f$ is sublinear, by the comparison principle $u$ can be estimated from below and from above by the solutions $v$ and $w$ of

$$v_t - v_{xx} - kv = 0, \quad x \in \mathbb{R}, \quad t > t_0, \tag{5.9}$$

$$w_t - w_{xx} + kw = 0, \quad x \in \mathbb{R}, \quad t > t_0, \tag{5.10}$$

namely, if $w(t_0, x) \leq u(t_0, x) \leq v(t_0, x)$ for any $x \in \mathbb{R}$, then $w(t, x) \leq u(t, x) \leq v(t, x)$ for any $x \in \mathbb{R}$, $t \geq t_0$.

We distinguish two cases:

(a) Suppose that there exists a sequence of times $t_n \to \infty$ such that

$$\lim_{n\to\infty} \int_{-r}^{r} u(t_n, x)\, \mathrm{d}x = \infty. \tag{5.11}$$

Let $w$ be the solution of (5.10) with initial value $w(t_n) = u(t_n)$. Using the fundamental solution of the heat equation, $w$ can be explicitly calculated:

$$w(t_n + t, x) = \frac{e^{-kt}}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} w(t_n, y)\, \mathrm{d}y, \quad t > 0. \tag{5.12}$$

So for $t = 1$ and $x = r$ we can estimate

$$\begin{aligned} u(t_n+1,r) \geq w(t_n+1,r) &\geq \frac{e^{-k}}{\sqrt{4\pi}} \int_{-r}^{r} e^{-\frac{(r-y)^2}{4}} u(t_n,y)\,\mathrm{d}y \\ &\geq \frac{e^{-k}e^{-r^2}}{\sqrt{4\pi}} \int_{-r}^{r} u(t_n,y) \to \infty, \end{aligned}$$

which contradicts (5.2).

(b) Suppose now that there exists a constant $c_1 < \infty$ such that

$$\int_{-r}^{r} u(t,x)\,\mathrm{d}x \leq c_1, \quad t > 0. \tag{5.13}$$

Again, the solution of (5.9) with initial value $v(t_0) = u(t_0)$ can be written explicitly and for $t \in [\frac{1}{2}, 1]$ and $|x| \leq r$ can be estimated as follows:

$$\begin{aligned} &u(t_0+t,x) \leq v(t_0+t,x) \\ &\quad\leq \frac{e^{kt}}{\sqrt{4\pi t}}\left(\int_{-r}^{r} e^{-\frac{(x-y)^2}{4t}} u(t_0,y)\,\mathrm{d}y + \int_{|y|\geq r} e^{-\frac{(x-y)^2}{4t}} u(t_0,y)\,\mathrm{d}y\right) \\ &\quad\leq \frac{e^{k}}{\sqrt{2\pi}}\left(\int_{-r}^{r} u(t_0,y)\,\mathrm{d}y + \zeta_1 \int_{|y|\geq r} e^{-\frac{(x-y)^2}{4}}\,\mathrm{d}y\right) \\ &\quad\leq \frac{e^{k}}{\sqrt{2\pi}}\left(c_1 + \zeta_1 \int_{\mathbb{R}} e^{-\frac{y^2}{4}}\,\mathrm{d}y\right) \leq \frac{e^{k}}{\sqrt{2\pi}}(c_1 + 2\zeta_1\sqrt{\pi}), \end{aligned}$$

where we used (5.2) and (5.13). Since $t_0 > 0$ was arbitrary, we get (5.1).

Now, if (F5) does not hold, then we can find an $s_0 > \|u_0\|_{\mathrm{L}^\infty}$ such that $f(s_0) > 0$, then by comparing $u$ to the solution of (2.8) with $z(0) = s_0$ we get (5.1).

The second assertion follows from Lemma 5.4. □

**Lemma 5.6.** *Assume* (F1), (F2), (F3). *Then* (5.3) *implies convergence in the* $\mathrm{W}^{1,2}$*-norm.*

Proof: If $u$ is a solution of (1.1)–(1.2), then

$$u_t - u_{xx} + f'(0)u = f'(0)u - f(u) \equiv g, \quad x \in \mathbb{R}, \quad t > 0, \tag{5.14}$$

and since $f \in \mathcal{C}^{1+\mu}$, we can write

$$\|g\|_{\mathrm{W}^{1,2}(\mathbb{R})} \leq c_1 \|u(t)\|^{\mu}_{\mathrm{L}^\infty(\mathbb{R})} \|u(t)\|_{\mathrm{W}^{1,2}(\mathbb{R})} = h(t)\|u(t)\|_{\mathrm{W}^{1,2}(\mathbb{R})}, \tag{5.15}$$

with $\lim\limits_{t\to\infty} h(t) = 0$. Then using the variation-of-constants formula and the stability of 0 for the homogeneous equation (2.8) and for the linear equation (5.14) in the space $\mathrm{W}^{1,2}$ (which is a consequence of $f'(0) > 0$) one can show $u(t) \to 0$ as $t \to \infty$ in $\mathrm{W}^{1,2}(\mathbb{R})$. $\square$

Proof of Proposition 4.3 (ii) (cf. [1]): From symmetry we can restrict ourselves to $\bar{x} = 0$. By contradiction, suppose the contrary is true. Then Proposition 5.5 yields that $v(t)$ converges to a stationary solution. By the maximum principle $v(t,x) > u(t,x),\ t > 0,\ x \in \mathbb{R}$, so necessarily $\lim_{t\to\infty} v(t) - u(t) = 0$. The function $w = v - u$ satisfies the equation

$$w_t - w_{xx} + f'(w_g)w + h = 0, \tag{5.16}$$

where $h = f(v) - f(u) - f'(w_g)w$. Let $-\lambda < 0$ be a negative eigenvalue of $L_{w_g}$ with the strictly positive eigenfunction $e$ and set $\varphi(t) = \int_{\mathbb{R}} w(t,x)e(x)\,\mathrm{d}x$. Taking the $\mathrm{L}^2$-scalar product of (5.16) with $e$ we get

$$\varphi'(t) - \lambda\varphi(t) + \int_{\mathbb{R}} h(t,x)e(x)\,\mathrm{d}x = 0. \tag{5.17}$$

From the smoothness of $f$ and from the convergence of $w$ to 0 we get

$$|h(t,x)| \le \frac{\lambda}{2} w(t,x) \quad \text{for} \quad x \in \mathbb{R}, \quad t \ge t_0, \tag{5.18}$$

if $t_0$ is sufficiently large depending on $\lambda$. Hence $\varphi'(t) \ge \frac{\lambda}{2}\varphi(t),\ t \ge t_0$. This implies $\varphi(t) \to \infty$ which contradicts $w(t) \to 0$. $\square$

## 6. Characterization of unbounded trajectories

We report the result of Fife-McLeod [5, Theorem 3.2] characterizing the initial data which give rise to unbounded (in integral-norm) solutions.

**Theorem 6.1.** *Let $f \in \mathcal{C}^1[0,1]$ satisfy*

$$\begin{aligned} & f(0) = f(1) = 0, \quad f'(0) > 0, \quad f'(1) > 0, \\ & f(s) > 0, \quad 0 < s < \alpha_0, \\ & f(s) < 0, \quad \alpha_1 < s < 1, \\ & \int_0^1 f(s)\,\mathrm{d}s < 0, \end{aligned} \tag{6.1}$$

*where $0 < \alpha_0 \le \alpha_1 < 1$. Suppose that there exists a travelling front solution $U$ of* (1.1) *(i.e. $v(t,x) = U(x - ct)$ is a solution of* (1.1) *and $U(-\infty) = 0,\ U(\infty) = 1$). Let $u_0$ satisfy*

$$\begin{aligned} & \limsup_{|x|\to\infty} u_0(x) < \alpha_0, \\ & u_0(x) > \alpha_1 + \eta, \quad x \in (a,b), \quad b - a > L, \end{aligned} \tag{6.2}$$

*where $\eta$ and $L$ are some positive numbers. Then if $L$ is sufficiently large (depending on $\eta$ and $f$), we have for some constants $x_0, x_1, K > 0$, $\omega > 0$ and $c < 0$ ($c > 0$ if $\int_0^1 f > 0$) that the solution of* (1.1)–(1.2) *satisfies*

$$|u(x,t) - U(x - ct - x_0)| < Ke^{-\omega t}, \quad x < 0,$$
$$|u(x,t) - U(-x - ct - x_1)| < Ke^{-\omega t}, \quad x > 0.$$

Note that in particular the solution from the above theorem satisfies

$$\liminf_{t\to\infty}\{\inf_{|x|\le R} u(t,x)\} \ge \alpha_1 \quad \text{for any} \quad R > 0. \tag{6.3}$$

The existence of a travelling wave $U$ for (1.1)–(1.2) is guaranteed by the existence of the ground state $w_g$ (the proof in [12, Theorem 4.76] can be carried out).

**Corollary 6.2.** *Suppose* (F1)–(F4). *Let $\zeta_1 < \zeta_0$ be such that $f < 0$ on $[\zeta_1, \zeta_0)$. Then there exists an $L > 0$ such that if $u_0$ satisfies*

$$u_0(x) > \zeta_1, \quad x \in (a,b), \quad b - a > L, \tag{6.4}$$

*then the solution of* (1.1)–(1.2) *satisfies* (1.4). *In particular, if the solution does not satisfy* (5.2), *then* (1.4) *holds.*

PROOF: We choose $0 < \alpha_0 < \alpha_1 < \zeta_1$ so that $f > 0$ on $(0, \alpha_0)$, $f < 0$ on $(\alpha_1, \zeta_0)$ and define a function $\bar{f}$ so that $\bar{f}$ satisfies (6.1) on an interval $[0, \zeta_2]$, $\zeta_2 > \zeta_0$ for $\alpha + \eta = \zeta_1$ and also $\bar{f} \ge f$. Let $L$ be given by Theorem 6.1 for $\bar{f} \in \mathcal{C}^1([0, \zeta_2])$ (after rescaling). Let $u$ be the solution of (1.1)–(1.2). Then for a function $u_0$ satisfying (6.4) we can choose a function $v_0 \le u_0$ such that $v_0$ satisfies (6.2). Let $v$ be the solution of (1.1) with initial value $v(0) = v_0$ and let $\bar{v}$ be the solution of

$$\bar{v}_t - \bar{v}_{xx} + \bar{f}(\bar{v}) = 0, \quad \bar{v}(0) = v_0.$$

Then by the comparison principle $u \ge v \ge \bar{v}$ and by Lemma 6.1 $\bar{v}$ satisfies (1.4).
For the additional assertion use Lemma 5.4. $\square$

## 7. Proof of Theorem 1.1

We define two sets:

$$\mathcal{A}_0 = \{u_0 \in \mathrm{W}^{1,2}(\mathbb{R});\ u_0 \ge 0, \ \text{ the solution of (1.1)–(1.2) converges to 0} \\ \text{in } \mathcal{C}_0(\mathbb{R}) \text{ as } t \to \infty\}$$
$$\mathcal{A}_\infty = \{u_0 \in \mathrm{W}^{1,2}(\mathbb{R});\ u_0 \ge 0, \ \text{ the solution of (1.1)–(1.2) satisfies (1.4) }\}.$$

If (F3) holds, it follows from the local asymptotic stability of 0 and the continuous dependence on initial data that $\mathcal{A}_0$ is a nonempty (containing 0) open subset of $\mathrm{W}^{1,2}(\mathbb{R}) \cap \{v \ge 0\}$.

**Lemma 7.1.** *Suppose* (F1)–(F4). $\mathcal{A}_\infty$ *is an open subset of* $\mathrm{W}^{1,2}(\mathbb{R}) \cap \{v \geq 0\}$.

PROOF: Let $u_0 \in \mathcal{A}_\infty$. Let $0 < \zeta_1 < \zeta_0$ and $t_0 > 0$ be such that $u(t_0)$ satisfies (6.4). Then from continuous dependence on initial data there exists a neighborhood $\mathcal{U}$ of $u_0$ such that if $v_0 \in \mathcal{U}$, then the corresponding solution satisfies (6.4) at time $t_0$, hence by Corollary 6.2 we get $v_0 \in \mathcal{A}_\infty$. □

**Lemma 7.2.** *Suppose* (F1)–(F6). *Given* $u_0 \in I$, *then* $\alpha u_0 \in \mathcal{A}_\infty$ *provided* $\alpha$ *is sufficiently large.*

PROOF: Let $u$ be the solution of (1.1) with $u(0) = \alpha u_0$. Since (F5) holds, the comparison principle yields $u \geq w$, where $w$ is the solution of (5.10) with $w(0) = u(0)$. For $|x| \leq L$, $t = 1$ we can estimate as follows:

$$\begin{aligned} u(1,x) \geq w(t,x) &= \frac{e^{-kt}}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} \alpha u_0(y)\,\mathrm{d}y \\ &\geq \alpha \frac{e^{-k} e^{-L^2}}{\sqrt{4\pi}} \int_{-L}^{L} u_0(y)\,\mathrm{d}y > \zeta_1, \end{aligned}$$

for $\alpha$ large enough, depending on $\zeta_1$, $L$. Then we use Corollary 6.2. □

PROOF OF THEOREM 1.1: For $\bar{u} \in I$ we set

$$\alpha_c = \sup\{\alpha > 0,\ \alpha\bar{u} \in \mathcal{A}_0\}.$$

Lemma 7.2 yields $\alpha_c < \infty$. By the comparison principle $\{\alpha \geq 0;\ \alpha\bar{u} \in \mathcal{A}_0\} = [0, \alpha_c)$. Since $\mathcal{A}_\infty$ is open, $\alpha_c \bar{u} \notin \mathcal{A}_\infty$, so by Corollary 6.2, for the solution $u$ with $u(0) = \alpha_c \bar{u}$ necessarily (5.2) holds. Then by Proposition 5.5 $u(t)$ converges (the convergence being in the $\mathrm{W}^{1,2}$-norm by Lemma 5.6) to a stationary solution $w$ and $w \neq 0$, hence (ii). Finally, Proposition 4.3 (ii) ends the proof. □

## 8. Proof of Theorem 1.2

Throughout this section we assume that $f$ satisfies the hypotheses of Theorem 1.2 (a) resp. (b) and we denote by $a_0$ a positive constant satisfying

$$f(s) > 0, \qquad s \in (0, a_0].$$

Note that (F6) is automatically fulfilled. In parts (a) resp. (b) of the following statements we suppose the hypotheses of part (a) resp. (b) of Theorem 1.2. The other statements hold in both cases.

**Lemma 8.1.** *There exists a* $\delta > 0$ (*depending on* $k$, $a_0$) *such that if* $\|u(0)\|_{\mathrm{L}^1(\mathbb{R})} \leq \delta$, *then* $u(t) \to 0$ *in* $\mathrm{W}^{1,2}(\mathbb{R})$ *as* $t \to \infty$.

PROOF: The solution $u$ is dominated by the solution of the linear equation (5.9) satisfying $v(0) = u(0)$, which can be explicitly calculated, hence we get

$$\|u(1)\|_{\mathrm{L}^\infty(\mathbb{R})} \leq \frac{e^k}{\sqrt{4\pi}} \|u(0)\|_{\mathrm{L}^1(\mathbb{R})}. \tag{8.1}$$

Then from the stability of 0 in $L^\infty$ we get $\lim_{t\to\infty} \|u(t)\|_{L^\infty(\mathbb{R})} = 0$, provided $\|u(0)\|_{L^1}$ is sufficiently small. Finally, convergence in $L^\infty$ implies convergence in $W^{1,2}$ as in Lemma 5.6. □

Before proceeding, we describe the idea of the proof of Theorem 1.2. By Lemma 8.1, it is enough to find a $T_1 > 0$ such that $\|u(T_1)\|_{L^1} < \delta$. Over a small space-interval, we will estimate the $L^1$-norm of $u(T_1)$ by the $L^\infty$-norm (Lemma 8.3) by comparing $u$ with the steady state solution $z$ of $z_t + f(z) = 0$. On the complement of this small interval, we compare $u$ with the solution of $v_t - v_{xx} - kv = 0$, $t > 0$, $x > 0$ with zero initial condition at $t = 0$ and bounded boundary condition at $x = 0$. To this end we will control $u$ on the space-boundary with the solution of $-w_{xx} + f(w) = 0$ (more precisely with $w$ in the stable manifold at the greatest equilibrium point; Lemma 8.4). Here we need a sufficiently large $f$.

**Lemma 8.2.** *For any $\varepsilon > 0$ and $A < \infty$ there exists a $T = T(k, A, \varepsilon) > 0$ such that if a function $v$ defined on some set $\{(t, x) \in \mathbb{R}^2,\ t \ge 0,\ x \ge x_0\}$ satisfies*

$$\begin{aligned} &v_t - v_{xx} - kv = 0, \quad x > x_0, \quad t > 0, \\ &0 \le v(t, x_0) \le A, \quad t > 0, \\ &v(0, x) = 0, \quad x \ge x_0, \end{aligned} \tag{8.2}$$

*then*

$$\|v(t)\|_{L^1(x_0,\infty)} \le \varepsilon, \quad t \in [0, T]. \tag{8.3}$$

*One can choose $T = C_k \frac{\varepsilon^2}{A^2}$, where $C_k$ is a constant depending on $k$.*

Proof: Without loss of generality we can suppose $x_0 = 0$. Let $\varepsilon > 0$ be given and define the function $\varphi(x) = \frac{A(\delta-x)^3}{\delta^3}$ for $x \in [0, \delta]$, and $\varphi(x) = 0$ for $x > \delta$, $\delta > 0$ to be chosen later. By the comparison principle, $v$ is dominated by the solution $y$ of

$$\begin{aligned} &y_t - y_{xx} - ky = 0, \quad x > 0, \quad t > 0, \\ &y(t, 0) = A, \quad t \ge 0, \\ &y(0, x) = \varphi(x), \quad x \ge 0. \end{aligned} \tag{8.4}$$

Then $w = y - \varphi$ is the solution of

$$\begin{aligned} &w_t - w_{xx} - kw = k\varphi + \varphi'' \equiv g, \quad x > 0, \quad t > 0, \\ &w(t, 0) = 0, \quad t \ge 0, \\ &w(0, x) = 0, \quad x \ge 0, \end{aligned} \tag{8.5}$$

and can be estimated using the variation-of-constants formula (note that the operator $L(w) = -w_{xx}$ generates a semigroup of contractions on $\mathrm{L}^1(0,\infty)$)

$$\|w(t)\|_{\mathrm{L}^1(0,\infty)} \le \int_0^t e^{k(t-s)} \|g\|_{\mathrm{L}^1(0,\infty)} \,\mathrm{d}s = \frac{e^{kt}-1}{k} \|g\|_{\mathrm{L}^1(0,\infty)}. \tag{8.6}$$

So $v$ can be estimated for $t \le T$ as follows

$$\begin{aligned}
\|v(t)\|_{\mathrm{L}^1(0,\infty)} &\le \|y(t)\|_{\mathrm{L}^1(0,\infty)} \le \|w(t)\|_{\mathrm{L}^1(0,\infty)} + \|\varphi\|_{\mathrm{L}^1(0,\infty)} \\
&\le \frac{e^{kt}-1}{k} (k\|\varphi\|_{\mathrm{L}^1(0,\infty)} + \|\varphi''\|_{\mathrm{L}^1(0,\infty)}) + \|\varphi\|_{\mathrm{L}^1(0,\infty)} \\
&= e^{kt} \|\varphi\|_{\mathrm{L}^1(0,\infty)} + \frac{e^{kt}-1}{k} \|\varphi''\|_{\mathrm{L}^1(0,\infty)} \\
&\le e^{kT} \|\varphi\|_{\mathrm{L}^1(0,\infty)} + \frac{e^{kT}-1}{k} \|\varphi''\|_{\mathrm{L}^1(0,\infty)} \\
&\le e^{kT} \frac{A\delta}{4} + T \frac{3A}{\delta} \frac{e^{kT}-1}{kT}\,.
\end{aligned}$$

If we choose $T$ such that

$$T \frac{e^{kT}-1}{kT} e^{kT} \le \frac{\varepsilon^2}{3A^2} \tag{8.7}$$

and $\delta = \frac{2\varepsilon}{Ae^{kT}}$, then we get

$$\|v(t)\|_{\mathrm{L}^1(0,\infty)} \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad t \in [0,T].$$

□

**Lemma 8.3.**

(a) *For any $a > 0$ and $T > 0$ there exists an $M_0 = M_0(\alpha, s_0, a, T) > 0$ such that for any $M \ge M_0$ and for any solution of* (1.1) *independently on $u_0$ we have*

$$\|u(T)\|_{\mathrm{L}^\infty(\mathbb{R})} \le s_0 + a. \tag{8.8}$$

(b) *If $\alpha > 3$, then for any $\varrho > 0$ there exists an $a = a(\alpha, k, M, s_0, \varrho) > 0$ such that* (8.8) *holds for the specially chosen time $T = \frac{\varrho}{(s_0+a)^2}$.*

Proof: By the comparison theorem, any solution is bounded by the solution of the ordinary differential equation

$$z_t + f(z) = 0, \quad t > 0, \quad z(0) = z_0, \tag{8.9}$$

where $z_0 > \max\{\|u(0)\|_{L^\infty}, s_0+a\}$. Integrating the equation we find that $z = z(t)$ is implicitly given by

$$\int_z^{z_0} \frac{1}{f(y)}\,\mathrm{d}y = t. \tag{8.10}$$

$z$ is nonincreasing. We want to show that $z(T) \le s_0 + a$, which is equivalent to

$$\int_{s_0+a}^{z_0} \frac{1}{f} \le T. \tag{8.11}$$

We can estimate

$$\int_{s_0+a}^{z_0} \frac{1}{f} \le \int_{s_0+a}^{\infty} \frac{1}{f} \le \int_{s_0+a}^{\infty} \frac{1}{Ms^\alpha}\,\mathrm{d}s = \frac{1}{M(\alpha-1)}(s_0+a)^{1-\alpha}. \tag{8.12}$$

(a) For given $a$ and $T$ the last term can be dominated by $T$ provided $M$ is sufficiently large.

(b) If $\alpha > 3$ and $M,\ \varrho > 0$ are given, then by choosing $a$ large enough, the last term in (8.12) can be dominated by $\frac{\varrho}{(s_0+a)^2}$. □

**Lemma 8.4.**

(a) *For any $\varepsilon > 0$ there exists a constant $M_1 = M_1(\alpha, s_0, a, \varepsilon)$ such that if $M \ge M_1$, then the stationary problem*

$$-w_{xx} + f(w) = 0, \quad x \in \mathbb{R} \tag{8.13}$$

*has a solution such that*

$$\begin{gathered} \lim_{x\to 0} w(x) = +\infty, \quad \lim_{x\to\infty} w(x) = a_1 < s_0, \\ w'(x) < 0, \quad x > 0; \quad w(x) \le s_0 + a, \quad x \ge \varepsilon. \end{gathered} \tag{8.14}$$

(b) *If $\alpha > 3$, then for any $\varrho > 0$ there exists an $a = a(\alpha, M, s_0, \varrho)$ such that the problem* (8.13) *possesses a solution satisfying* (8.14) *for the specially chosen $\varepsilon = \frac{\varrho}{s_0+a}$.*

Proof: If $w$ is a solution of (8.13), then

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(-\frac{1}{2}w_x^2 + F(w)\Big) = 0, \tag{8.15}$$

so $-w_x^2 + 2F(w) = -c$ for some constant $c$. From (8.14) we have $c = -2F(a_1)$. Hence the desired solution should satisfy $w_x = -\sqrt{2(F(w) - F(a_1))}$, which implies that $w = w(x)$ is implicitly given by

$$x = \int_w^\infty \frac{1}{\sqrt{2(F(y) - F(a_1))}}\,\mathrm{d}y. \tag{8.16}$$

$w$ decreases to $a_1$ as $x$ tends to $\infty$. So in order to satisfy (8.14), we need only to show that $w(\varepsilon) \le s_0 + a$, which is equivalent to

$$\int_{s_0+a}^{\infty} \frac{1}{\sqrt{2(F(y)-F(a_1))}}\,\mathrm{d}y \le \varepsilon. \tag{8.17}$$

Since $f > 0$ on $(a_1, s_0)$, we deduce from (F7) that

$$F(s) \ge F(a_1) + \int_{s_0}^{s} f \ge F(a_1) + \frac{M}{1+\alpha}(s^{1+\alpha} - s_0^{1+\alpha}), \quad s > s_0.$$

Hence we can estimate

$$\begin{aligned}
\int_{s_0+a}^{\infty} \frac{1}{\sqrt{2(F(y)-F(a_1))}}\,\mathrm{d}y &\le \int_{s_0+a}^{\infty} \sqrt{\frac{\alpha+1}{2M}} \frac{1}{\sqrt{y^{\alpha+1} - s_0^{\alpha+1}}}\,\mathrm{d}y \\
&= \sqrt{\frac{\alpha+1}{2M}} \int_{s_0+a}^{\infty} \frac{\mathrm{d}y}{y^{\frac{\alpha+1}{2}}\sqrt{1-(\frac{s_0}{y})^{1+\alpha}}} \\
&\le \sqrt{\frac{\alpha+1}{2M}} \frac{1}{\sqrt{1-(\frac{s_0}{s_0+a})^{1+\alpha}}} \int_{s_0+a}^{\infty} y^{-\frac{\alpha+1}{2}}\,\mathrm{d}y \\
&= \sqrt{\frac{\alpha+1}{2M}} \frac{1}{\sqrt{1-(\frac{s_0}{s_0+a})^{1+\alpha}}} \frac{2}{\alpha-1}(s_0+a)^{\frac{1-\alpha}{2}}.
\end{aligned} \tag{8.18}$$

(a) For $\varepsilon$ given, the last term can be made smaller than $\varepsilon$, provided $M$ is large enough.

(b) If $\alpha > 3$ and $\varrho$ is given, then the last term in (8.18) is less than $\frac{\varrho}{s_0+a}$, provided $a$ is large enough. □

Proof of Theorem 1.2: Let $a_0$, $k$ be fixed and let $\delta$ be given by Lemma 8.1.

(a) Let $\alpha$, $s_0$ and $a > 0$ be given. Let $T_1 = T(k, s_0 + a, \frac{\delta}{10})$ be given by Lemma 8.2. Let $M_0 = M_0(\alpha, s_0, a, T_1)$ be given by Lemma 8.3 and $M_1 = M_1(\alpha, s_0, a, \frac{\delta}{5(s_0+a)})$ be given by Lemma 8.4. If $M \ge \max\{M_0, M_1\}$ (which depends on $\alpha$, $k$, $a_0$, $s_0$, $a$), and if $\operatorname{supp} u_0 \subset [-r, r]$, with

$$r < \frac{\delta}{5(s_0+a)}, \tag{8.19}$$

then we can estimate $\|u(T_1)\|_{\mathrm{L}^1(\mathbb{R})}$ by the comparison principle as follows.

First of all we have from Lemma 8.3

$$0 \le u(T_1, x) \le s_0 + a, \quad x \in \mathbb{R}. \tag{8.20}$$

Let $w$ be given in Lemma 8.4. We will compare $u$ with $w(.-r)$ on the set $\mathcal{M}=\{(t,x),\ x\geq r',\ t\in[0,T_1]\}$, where $r'>r$ is chosen close to $r$ so that $w(r'-r)\geq\sup\{u(t,x);\ t\in[0,T_1],\ x\in\mathbb{R}\}$. We get $u(t,x)\leq w(x-r)$ in $\mathcal{M}$. Due to (8.14), $r'$ can be chosen arbitrarily close to $r$, so we get

$$0\leq u(t,x)\leq w(x-r),\quad x>r,\quad t\in[0,T_1]. \tag{8.21}$$

Similarly, comparing $u$ with the function $x\mapsto w(-x-r)$ on the set $\{(t,x),\ x\leq -r',\ t\in[0,T_1]\}$, we get

$$0\leq u(t,x)\leq w(-x-r),\quad x<-r,\quad t\in[0,T_1]. \tag{8.22}$$

Using (8.21), (8.22), (8.14) and Lemma 8.2 on the set $\{(t,x),\ x\geq r+\frac{\delta}{5(s_0+a)},\ t\in[0,T_1]\}$, and, by symmetry, on $\{(t,x),\ x\leq -r-\frac{\delta}{5(s_0+a)},\ t\in[0,T_1]\}$, we obtain

$$\begin{aligned}\|u(T_1)\|_{\mathrm{L}^1(\mathbb{R})}&=\int_{|x|\leq r+\frac{\delta}{5(s_0+a)}}u(T_1)+\int_{r+\frac{\delta}{5(s_0+a)}<|x|}u(T_1)\\&\leq 2(s_0+a)(r+\frac{\delta}{5(s_0+a)})+2\|u(T_1)\|_{\mathrm{L}^1(r+\frac{\delta}{5(s_0+a)},\infty)}\\&\leq\frac{4\delta}{5}+2\frac{\delta}{10}=\delta,\end{aligned}$$

where we have used (8.20), (8.19) and (8.3) (with $\varepsilon=\frac{\delta}{10}$). Finally, by Lemma 8.1 we get $\lim\limits_{t\to\infty}\|u(t)\|_{\mathrm{W}^{1,2}(\mathbb{R})}=0$. The problem being translation invariant, the part (a) is proved in the case when $\operatorname{supp}u_0$ is contained in an interval of a sufficiently small length. If this is not the case, we can find a finite number of closed intervals $I_k$ such that $\operatorname{supp}u_0\subset\bigcup I_k$ and $\sum|I_k|$ is small, and we make the above construction on any $I_k$.

(b) Let $\alpha,\ s_0,\ M$ be given. Let $T_1=C_k\left(\frac{\delta}{10}\right)^2\frac{1}{(s_0+a)^2}$ be given by Lemma 8.2, where $a$ is chosen large enough to satisfy Lemma 8.3 (b) for $T=T_1$ and also Lemma 8.4 (b) for $\varepsilon=\frac{\delta}{5(s_0+a)}$. Then we follow the same procedure as in part (a). □

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Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Prague, Czech Republic

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