On the cardinality of Hausdorff spaces

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Abstract. The aim of this paper is to show, using the reflection principle, three new cardinal inequalities. These results improve some well-known bounds on the cardinality of Hausdorff spaces.

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Two of the most known inequalities in the theory of cardinal functions are the Hajnal-Juhàsz's inequality [7]: "For $X \in T_2$, $|X| \le 2^{c(X)\chi(X)}$" and the Arhangel'skii's inequality [5]: "For $X \in T_2$, $|X| \le 2^{L(X)t(X)\psi(X)}$".

In this paper we will use the language of elementary submodels (see [4], [10], [1] and [2]) to establish three new cardinal inequalities which generalize the results mentioned above. We refer the reader to [3], [5], [7] for notations and terminology not explicitly given. χ, c, ψ, t, L and π_χ denote character, cellularity, pseudocharacter, tightness, Lindelöf degree and π-character respectively.

Definitions. (i) Let X be a Hausdorff space.

The closed pseudocharacter of X, denoted $\psi_c(X)$, is the smallest infinite cardinal κ such that for every $x \in X$ there is a collection $\mathcal{U}_x$ of open neighbourhoods of x such that $\bigcap\{\overline{U} : U \in \mathcal{U}_x\} = \{x\}$ and $|\mathcal{U}_x| \le \kappa$ ([7]).

The Hausdorff pseudocharacter of X, denoted $H\psi(X)$, is the smallest infinite cardinal κ such that for every $x \in X$ there is a collection $\mathcal{U}_x$ of open neighbourhoods of x with $|\mathcal{U}_x| \le \kappa$ such that if $x \ne y$, there exist $U \in \mathcal{U}_x$, $V \in \mathcal{U}_y$ with $U \cap V = \emptyset$ ([6]).

Clearly $\psi_c(X) \le H\psi(X) \le \chi(X)$ for every Hausdorff space X.

(ii) Let X be a topological space, $ac(X)$ is the smallest infinite cardinal κ such that there is a subset S of X such that $|S| \le 2^\kappa$ and for every open collection $\mathcal{U}$ in X there is a $\mathcal{V} \in [\mathcal{U}]^{\le\kappa}$ with $\bigcup\mathcal{U} \subset S \cup \overline{\bigcup\{\overline{V} : V \in \mathcal{V}\}}$.

Observe that $ac(X) \le c(X)$ for every space X.

Theorem 1. *If X is a T_2-space then $|X| \le 2^{ac(X)H\psi(X)}$.*

Proof: Let $\lambda = ac(X)H\psi(X)$, $\kappa = 2^\lambda$, let τ be the topology on X and let S be an element of $[X]^{\le\kappa}$ witnessing that $ac(X) \le \lambda$. For every $x \in X$ let $\mathcal{B}_x$ be a collection of open neighbourhoods of x with $|\mathcal{B}_x| \le \lambda$ such that if $x \ne y$ then there exist $U \in \mathcal{B}_x$, $V \in \mathcal{B}_y$ such that $U \cap V = \emptyset$, and let $f : X \to \mathcal{P}(\tau)$ be the map defined by $f(x) = \mathcal{B}_x$ for every $x \in X$.

Let $A = \kappa \cup \{S, X, \tau, \kappa, f\}$ and take a set $\mathcal{M}$ such that $\mathcal{M} \supset A$, $|\mathcal{M}| = \kappa$ and which reflects enough formulas to carry out our argument. To be more precise we ask that $\mathcal{M}$ reflects enough formulas so that the following conditions are satisfied:

(i) $C \in \mathcal{M}$ for every $C \in [\mathcal{M}]^{\leq\kappa}$;
(ii) $\mathcal{B}_x \in \mathcal{M}$ for every $x \in X \cap \mathcal{M}$;
(iii) if $B \subset X$ and $B \in \mathcal{M}$ then $\overline{B} \in \mathcal{M}$;
(iv) if $\mathcal{A} \in \mathcal{M}$ then $\bigcup \mathcal{A} \in \mathcal{M}$;
(v) if B is a subset of X such that $X \cap \mathcal{M} \subset B$ and $B \in \mathcal{M}$ then $X = B$;
(vi) if $E \in \mathcal{M}$ and $|E| \leq \kappa$ then $E \subset \mathcal{M}$.

Observe that by (ii) and (vi) $\mathcal{B}_y \subset \mathcal{M}$ for every $y \in X \cap \mathcal{M}$.

Claim: $X \subset \mathcal{M}$ (and hence $|X| \leq 2^{ac(X)H\psi(X)}$). Suppose not and take $p \in X \setminus \mathcal{M}$. Let $\mathcal{B}_p = \{B_\alpha\}_{\alpha<\lambda}$, clearly $\bigcap\{\overline{B}_\alpha : \alpha < \lambda\} = \{p\}$. Now for every $\alpha < \lambda$ let $(X \cap \mathcal{M})_\alpha = \{y \in X \cap \mathcal{M} : \exists B \in \mathcal{B}_y \text{ for which } B \cap B_\alpha = \emptyset\}$. For every $y \in (X \cap \mathcal{M})_\alpha$ choose a $B_{y,\alpha} \in \mathcal{B}_y$ such that $B_{y,\alpha} \cap B_\alpha = \emptyset$, clearly $\mathcal{U}_\alpha = \{B_{y,\alpha} : y \in (X \cap \mathcal{M})_\alpha\}$ covers $(X \cap \mathcal{M})_\alpha$. Since $ac(X) \leq \lambda$ it follows that there is a $\mathcal{V}_\alpha \in [\mathcal{U}_\alpha]^{\leq\lambda}$ such that $(X \cap M)_\alpha \subset S \cup \overline{\bigcup\{\overline{V} : V \in \mathcal{V}_\alpha\}}$. Observe that $p \notin S \cup \overline{\bigcup\{\overline{V} : V \in \mathcal{V}_\alpha\}}$ ($S \in \mathcal{M}$ and $|S| \leq \kappa$ so by (vi) $S \subset \mathcal{M}$, moreover $\overline{\bigcup\{\overline{V} : V \in \mathcal{V}_\alpha\}} \subset X \setminus B_\alpha$). We have also $\overline{\bigcup\{\overline{V} : V \in \mathcal{V}_\alpha\}} \in \mathcal{M}$ ($V \in \mathcal{M}$ for every $V \in \mathcal{V}_\alpha$, so by (iii) $\overline{V} \in \mathcal{M}$, therefore $\{\overline{V} : V \in \mathcal{V}_\alpha\} \subset \mathcal{M}$ and $\{\overline{V} : V \in \mathcal{V}_\alpha\} \in \mathcal{M}$ by (i), hence by (iv) $\bigcup\{\overline{V} : V \in \mathcal{V}_\alpha\} \in \mathcal{M}$, so $\overline{\bigcup\{\overline{V} : V \in \mathcal{V}_\alpha\}} \in \mathcal{M}$ by (iii)). Set $C_\alpha = S \cup \overline{\bigcup\{\overline{V} : V \in \mathcal{V}_\alpha\}}$ for every $\alpha < \lambda$ and observe that $C_\alpha \in \mathcal{M}$ (recall that $S, \overline{\bigcup\{\overline{V} : V \in \mathcal{V}_\alpha\}} \in \mathcal{M}$). Now $X \cap \mathcal{M} \subset \bigcup\{C_\alpha : \alpha < \lambda\}$, since $\bigcup\{C_\alpha : \alpha < \lambda\} \in \mathcal{M}$ ($\{C_\alpha : \alpha < \lambda\} \subset \mathcal{M}$, so by (i) $\{C_\alpha : \alpha < \lambda\} \in \mathcal{M}$, hence by (iv) $\bigcup\{C_\alpha : \alpha < \lambda\} \in \mathcal{M}$) it follows by (v) that $\bigcup\{C_\alpha : \alpha < \lambda\} = X$. This is a contradiction ($p \notin \bigcup\{C_\alpha : \alpha < \lambda\}$). $\square$

Corollary 2 ([7]). *If X is a T_2-space then $|X| \leq 2^{c(X)\chi(X)}$.*

Remark 3. The above result of Hajnal and Juhàsz has been improved also by Hodel, in fact in [6] it is shown that $|X| \leq 2^{c(X)H\psi(X)}$ for every Hausdorff space X. It is clear that Theorem 1 generalizes also this result of Hodel.

Now let X be the Michael line, i.e. let X be $\mathbb{R}$ topologized by isolating the points of $\mathbb{R} \setminus \mathbb{Q}$ and leaving the points of $\mathbb{Q}$ with their usual neighbourhoods. Then X is a normal space such that $|X| = 2^{ac(X)H\psi(X)} < 2^{c(X)H\psi(X)}$.

Observe that in Theorem 1 $H\psi(X)$ cannot be replaced by $\psi_c(X)$, in fact for every infinite cardinal κ there is a T_3-space X with $|X| = \kappa$ and $\psi(X) = c(X) = ac(X) = \omega$ (see e.g. [5]).

Definition 4. Let X be a topological space, $lc(X)$ is the smallest infinite cardinal κ such that there is a closed subset F of X such that $|F| \leq 2^\kappa$ and for every open collection $\mathcal{U}$ in X there is a $\mathcal{V} \in [\mathcal{U}]^{\leq\kappa}$ with $\bigcup\mathcal{U} \subset F \cup \overline{\bigcup\{\overline{V} : V \in \mathcal{V}\}}$.

Clearly $ac(X) \leq lc(X) \leq c(X)$ for every space X.

Theorem 5. *If X is a Hausdorff space then $|X| \leq 2^{lc(X)\pi_\chi(X)\psi_c(X)}$.*

Proof: Let $\lambda = lc(X)\pi_\chi(X)\psi_c(X)$ and let $\kappa = 2^\lambda$, let τ be the topology on X and let F be a closed subset of X with $|F| \leq \kappa$ and witnessing that $lc(X) \leq \lambda$. For every $x \in X$ let $\mathcal{B}_x$ be a local π-base at x such that $|\mathcal{B}_x| \leq \lambda$, and let $f : X \to \mathcal{P}(\tau)$ be the map defined by $f(x) = \mathcal{B}_x$ for every $x \in X$. Let $A = \kappa \cup \{F, X, \tau, \kappa, f\}$ and take a set $\mathcal{M} \supset A$ such that $|\mathcal{M}| = \kappa$ and which reflects enough formulas so that the conditions (i)-(vi) listed in Theorem 1 are satisfied.

Claim: $X \subset \mathcal{M}$ (and hence $|X| \leq 2^{lc(X)\pi_\chi(X)\psi_c(X)}$). Suppose not and take $p \in X \setminus \mathcal{M}$. Let $\{G_\alpha : \alpha \in \lambda\}$ be a family of open neighbourhoods of p such that $\bigcap\{\overline{G_\alpha} : \alpha \in \lambda\} = \{p\}$. Set $V_\alpha = X \setminus \overline{G_\alpha}$ and $S_\alpha = X \cap \mathcal{M} \cap V_\alpha$ for every $\alpha \in \lambda$. Now let $\mathcal{W}_\alpha = \{B : B \in \mathcal{B}_y, y \in S_\alpha \wedge B \subset V_\alpha\}$, since $lc(X) \leq \lambda$ it follows that there is a $\mathcal{V}_\alpha \in [\mathcal{W}_\alpha]^{\leq\lambda}$ such that $\bigcup \mathcal{W}_\alpha \subset \overline{F \cup \bigcup\{\overline{V} : V \in \mathcal{V}_\alpha\}}$. Since $S_\alpha \subset \overline{\bigcup \mathcal{W}_\alpha}$ (let $y \in S_\alpha$ and U be an open neighbourhood of y, $y \notin \overline{G_\alpha}$ so there is an open neighbourhood V of y such that $V \cap G_\alpha = \emptyset$, let $B \in \mathcal{B}_y$ such that $B \subset U \cap V$, $\emptyset \neq B \subset (\bigcup \mathcal{W}_\alpha) \cap U$ and $y \in \overline{\bigcup \mathcal{W}_\alpha}$) it follows that $S_\alpha \subset \overline{F \cup \bigcup\{\overline{V} : V \in \mathcal{V}_\alpha\}}$; moreover $\overline{\bigcup\{\overline{V} : V \in \mathcal{V}_\alpha\}} \in \mathcal{M}$ and $p \notin \overline{F \cup \bigcup\{\overline{V} : V \in \mathcal{V}_\alpha\}}$. Set $C_\alpha = \overline{\bigcup\{\overline{V} : V \in \mathcal{V}_\alpha\}}$, since $X \cap \mathcal{M} \subset F \cup \bigcup\{C_\alpha : \alpha < \lambda\}$ and $F \cup \bigcup\{C_\alpha : \alpha < \lambda\} \in \mathcal{M}$ it follows that $F \cup \bigcup\{C_\alpha : \alpha < \lambda\} = X$, a contradiction. $\square$

By Theorem 5 it follows again that $|X| \leq 2^{c(X)\chi(X)}$ for every T_2-space X. Moreover we have the following

Corollary 6 ([6]). *If X is a T_3-space then $|X| \leq 2^{c(X)\pi_\chi(X)\psi(X)}$.*

Remark 7. A generalization of the inequality in corollary 6 has also been obtained by Sun in [8]: "$|X| \leq 2^{c(X)\pi_\chi(X)\psi_c(X)}$ for every Hausdorff space X". Note that even this result is a corollary of Theorem 5. Moreover if X is the Michael line then $|X| = 2^{lc(X)\pi_\chi(X)\psi_c(X)} < 2^{c(X)\pi_\chi(X)\psi_c(X)}$. Observe also that the π-character cannot be omitted in Theorem 5 (see the comment at the end of Remark 3).

Now let us turn our attention to the Arhangel'skii's inequality: "For $X \in T_2$, $|X| \leq 2^{L(X)t(X)\psi(X)}$".

Definitions. Let X be a topological space.

(i) ([8]) A subset A of X with $|A| \leq 2^\kappa$ is said to be κ-quasi-dense if for each open cover $\mathcal{U}$ of X there exist a $\mathcal{V} \in [\mathcal{U}]^{\leq\kappa}$ and a $B \in [A]^{\leq\kappa}$ such that $(\bigcup \mathcal{V}) \cup \overline{B} = X$; $qL(X)$ is the smallest infinite cardinal κ such that X has a κ-quasi dense subset.

(ii) $aqL(X)$ is the smallest infinite cardinal κ such that there is a subset S of X with $|S| \leq 2^\kappa$ such that for every open cover $\mathcal{U}$ of X there is a $\mathcal{V} \in [\mathcal{U}]^{\leq\kappa}$ with $X = S \cup (\bigcup \mathcal{V})$.

Clearly $aqL(X) \leq L(X)$ for every space X.

Theorem 8. *If X is a Hausdorff space then $|X| \leq 2^{aqL(X)t(X)\psi_c(X)}$.*

PROOF: Let $\lambda = aqL(X)t(X)\psi_c(X)$, $\kappa = 2^\lambda$, let τ be the topology on X and let S be an element of $[X]^{\leq\kappa}$ witnessing that $aqL(X) \leq \lambda$. For every $x \in X$ let $\mathcal{B}_x$ be a family of open neighbourhoods of x with $|\mathcal{B}_x| \leq \lambda$ and $\bigcap\{\overline{B} : B \in \mathcal{B}_x\} = \{x\}$, and let $f : X \to \mathcal{P}(\tau)$ be the map defined by $f(x) = \mathcal{B}_x$ for every $x \in X$. Let $A = \kappa \cup \{S, X, \tau, \kappa, f\}$ and take a set $\mathcal{M} \supset A$ such that $|\mathcal{M}| = \kappa$ and which reflects enough formulas so that the conditions (i)–(vi) listed in Theorem 1 are satisfied. First observe that $X \cap \mathcal{M}$ is a closed subset of X, although this fact follows from a general result which can be found in [4] we give a proof of it for the sake of completeness: let $x \in \overline{X \cap \mathcal{M}}$, since $t(X) \leq \lambda$ there is a $C \in [X \cap \mathcal{M}]^{\leq\lambda}$ such that $x \in \overline{C}$. Since $C \in \mathcal{M}$ (by (i)), it follows that $\overline{C} \in \mathcal{M}$ (by (iii)). Now it remains to observe that $|\overline{C}| \leq \kappa$ (recall that $t(X)\psi_c(X) \leq \lambda$) and hence by (vi) $x \in \overline{C} \subset X \cap \mathcal{M}$.

We have done if we show that $X \subset \mathcal{M}$. Suppose there is a $p \in X \setminus \mathcal{M}$, for every $y \in X \cap \mathcal{M}$ let $B_y \in \mathcal{B}_y$ such that $p \notin B_y$. Since $\mathcal{U} = \{B_y : y \in X \cap \mathcal{M}\} \cup \{X \setminus \mathcal{M}\}$ is an open cover of X and $aqL(X) \leq \lambda$ there is a $\mathcal{V} \in [\mathcal{U}]^{\leq\lambda}$ such that $X = S \cup (\bigcup \mathcal{V})$. Let $\mathcal{W} = \{B_y : B_y \in \mathcal{V}\}$, since $X \cap \mathcal{M} \subset S \cup (\bigcup \mathcal{W})$ and $S \cup (\bigcup \mathcal{W}) \in \mathcal{M}$ it follows that $X = S \cup (\bigcup \mathcal{W})$, a contradiction ($p \notin S \cup (\bigcup \mathcal{W})$). □

A consequence of Theorem 8 is the following generalization of the Arhangel'skii's inequality.

Corollary 9 ([8])**.** *If X is a Hausdorff space then $|X| \leq 2^{qL(X)t(X)\psi_c(X)}$.*

PROOF: It is enough to note that $aqL(X) \leq qL(X)t(X)\psi_c(X)$. □

Remark 10. Let κ be an infinite cardinal number and let X be the discrete space of cardinality 2^κ. This space shows that Theorem 8 can give a better estimation than the one in Corollary 9.

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