# Uniformly $\mu$-continuous topologies on Köthe-Bochner spaces and Orlicz-Bochner spaces

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*Abstract.* Some class of locally solid topologies (called uniformly $\mu$-continuous) on Köthe-Bochner spaces that are continuous with respect to some natural two-norm convergence are introduced and studied. A characterization of uniformly $\mu$-continuous topologies in terms of some family of pseudonorms is given. The finest uniformly $\mu$-continuous topology $\mathcal{T}_I^\varphi(X)$ on the Orlicz-Bochner space $L^\varphi(X)$ is a generalized mixed topology in the sense of P. Turpin (see [11, Chapter I]).

*Keywords:* Orlicz spaces, Orlicz-Bochner spaces, Köthe-Bochner spaces, locally solid topologies, generalized mixed topologies, uniformly $\mu$-continuous topologies, inductive limit topologies

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## 1. Preliminaries.

For notation and terminology concerning locally solid Riesz spaces we refer to [1].

Throughout the paper let $(\Omega, \Sigma, \mu)$ be a complete $\sigma$-finite measure space and let $L^0$ denote the corresponding space of equivalence classes of all $\Sigma$-measurable real valued functions. Then $L^0$ is a super Dedekind complete Riesz space under the ordering $u_1 \leq u_2$ whenever $u_1(\omega) \leq u_2(\omega)$ $\mu$-a.e. on $\Omega$.

For $u \in L^0$ let us put

$$\|u\|_\mu = \inf\{\lambda > 0 : \mu(\{\omega \in \Omega : |u(\omega)| > \lambda\}) \leq \lambda\}.$$

It is easy to see that a sequence $(u_n)$ in $L^0$ is convergent to $u \in L^0$ in measure on $\Omega$ (in symbols $u_n \to u$ $(\mu - \Omega)$) iff $\|u_n - u\|_\mu \to 0$. We will denote by $\mathcal{T}_\mu$ the topology on $L^0$ of $\|\cdot\|_\mu$.

For a subset $A$ of $\Omega$ let $\chi_A$ stand for its characteristic function.

Let $[x]$ denote the greatest integer which is less or equal to a real number $x$.

Let $(E, \|\cdot\|_E)$ be an $F$-normed function space, that is $E$ is an ideal of $L^0$ with $\operatorname{supp} E = \Omega$ and $\|\cdot\|_E$ is a complete Riesz $F$-norm. The Köthe dual $E'$ of $E$ is defined by

$$E' = \{v \in L^0 : \int_\Omega |u(\omega)v(\omega)|\, d\mu < \infty \quad \text{for all} \quad u \in E\}.$$

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In case $(E, \|\cdot\|_E)$ is a Banach function space the associated norm $\|\cdot\|_{E'}$ on $E'$ can be defined for $v \in E'$ by

$$\|v\|_{E'} = \sup\Big\{\Big|\int_\Omega u(\omega)v(\omega)\,d\mu\Big| : u \in E,\ \|u\|_E \le 1\Big\}.$$

We will write $A_n \searrow \emptyset$ when $(A_n)$ is a decreasing sequence in $\Sigma$ such that $\mu(A_n \cap A) \to 0$ for every $A \in \Sigma$ with $\mu(A) < \infty$.

We denote by $E_a$ the ideal of elements of absolutely continuous norm in $E$, i.e. $E_a = \{u \in E : \|\chi_{A_n} u\|_E \to 0 \text{ as } A_n \searrow \emptyset\}$.

Let $(X, \|\cdot\|_X)$ be a real Banach space, and let $S_X$ and $B_X$ denote the unit sphere and the closed unit ball in $X$, respectively.

By $L^0(X)$ we will denote the linear space of equivalence classes of all strongly $\Sigma$-measurable functions $f : \Omega \to X$.

For $f \in L^0(X)$ let us put

$$\|f\|_\mu^X = \inf\{\lambda > 0 : \mu(\{\omega \in \Omega : \|f(\omega)\|_X > \lambda\}) \le \lambda\}.$$

We say that a sequence $(f_n)$ in $L^0(X)$ is convergent to $f \in L^0(X)$ in measure on $\Omega$ (in symbols $f_n \to f\ (\mu - \Omega)$) whenever $\mu(\{\omega \in \Omega : \|f_n(\omega) - f(\omega)\|_X > \varepsilon\}) \to 0$ for every $\varepsilon > 0$. It can be seen that a sequence $(f_n)$ in $L^0(X)$ is convergent to $f \in L^0(X)$ in measure on $\Omega$ iff $\|f_n - f\|_\mu^X \to 0$. The topology on $L^0(X)$ of $\|\cdot\|_\mu^X$ will be denoted by $\mathcal{T}_\mu(X)$.

For $f \in L^0(X)$ let

$$\widetilde{f}(\omega) := \|f(\omega)\|_X \quad \text{for} \quad \omega \in \Omega.$$

The linear space $E(X) = \{f \in L^0(X) : \widetilde{f} \in E\}$ provided with the norm $\|f\|_{E(X)} := \|\widetilde{f}\|_E$ is called a Köthe-Bochner space (see [2], [3]).

Now we recall some concepts and terminology concerning locally solid topologies on vector-valued function spaces as set out in [3].

A subset $H$ of $E(X)$ is said to be *solid* whenever $\|f_1(\omega)\|_X \le \|f_2(\omega)\|_X$ $\mu$-a.e. and $f_1 \in E(X)$, $f_2 \in H$ imply $f_1 \in H$.

A pseudonorm $\rho$ on $E(X)$ is said to be *solid* whenever for $f_1, f_2 \in E(X)$, $\|f_1(\omega)\|_X \le \|f_2(\omega)\|_X$ $\mu$-a.e. imply $\rho(f_1) \le \rho(f_2)$.

A linear topology $\tau$ on $E(X)$ is said to be *locally solid* if it has a basis for neighbourhoods of zero consisting of solid sets.

A linear topology $\tau$ on $E(X)$ that is at the same time locally solid and locally convex will be called a *locally convex-solid topology* on $E(X)$.

**Theorem 1.1** (see [3, Theorem 2.2, Theorem 2.3]). *For a linear topology $\tau$ on $E(X)$ the following statements are equivalent:*

(i) *$\tau$ is a locally solid topology (respectively $\tau$ is a locally convex-solid topology);*

(ii) *$\tau$ is generated by some family of solid pseudonorms (respectively seminorms).*

Now we are going to explain the relationship between locally solid topologies on $E$ and $E(X)$ (see [3]).

Let $p$ be a Riesz pseudonorm (respectively seminorm) on $E$, and let

$$\overline{p}(f) := p(\widetilde{f}) \quad \text{for} \quad f \in E(X).$$

Then $\overline{p}$ is a solid pseudonorm (respectively seminorm) on $E(X)$.

Next, fix $x \in S_X$. Given $u \in E$ let us put $\overline{u}(\omega) := u(\omega) \cdot x$ for $\omega \in \Omega$. Then $\overline{u} \in L^0(X)$ and $\|\overline{u}(\omega)\|_X = |u(\omega)|$ for $\omega \in \Omega$, so $\overline{u} \in E(X)$.

Let $\rho$ be a solid pseudonorm (respectively seminorm) on $E(X)$, and let

$$\widetilde{\rho}(u) := \rho(\overline{u}) \quad \text{for} \quad u \in E.$$

Then $\widetilde{\rho}$ is a Riesz pseudonorm (respectively seminorm) on $E$.

**Theorem 1.2** (see [3, Lemma 3.1]). (i) *If $\rho$ is a solid pseudonorm on $E(X)$, then $\overline{\widetilde{\rho}}(f) = \rho(f)$ for $f \in E(X)$.*

(ii) *If $p$ is a Riesz pseudonorm on $E$, then*

$$\widetilde{\overline{p}}(u) = p(u) \quad \textit{for} \quad u \in E.$$

Let $\tau$ be a locally solid topology on $E(X)$ generated by some family $\{\rho_\alpha : \alpha \in \{\alpha\}\}$ of solid pseudonorms defined on $E(X)$. By $\widetilde{\tau}$ we will denote the locally solid topology on $E$ generated by the family $\{\widetilde{\rho}_\alpha : \alpha \in \{\alpha\}\}$ of Riesz pseudonorms on $E$. If $\tau$ is a Hausdorff topology, then so is $\widetilde{\tau}$.

In turn, let $\xi$ be a locally solid topology on $E$ generated by some family $\{p_\alpha : \alpha \in \{\alpha\}\}$ of Riesz pseudonorms on $E$. By $\overline{\xi}$ we will denote the locally solid topology on $E(X)$ generated by the family $\{\overline{p}_\alpha : \alpha \in \{\alpha\}\}$ of solid pseudonorms on $E(X)$. Then $\overline{\xi}$ is a Hausdorff topology, whenever $\xi$ is Hausdorff.

**Theorem 1.3** (see [3, Theorem 3.2]). (i) *For a locally solid topology $\tau$ on $E(X)$ we have: $\overline{\widetilde{\tau}} = \tau$.*

(ii) *For a locally solid topology $\xi$ on $E$ we have: $\widetilde{\overline{\xi}} = \xi$.*

Now we recall some notation and terminology concerning Orlicz spaces (see [5], [6], [11] for more details).

By an Orlicz function we mean a function $\varphi : [0, \infty) \to [0, \infty]$ which is non-decreasing, left continuous, continuous at 0 with $\varphi(0) = 0$ and not identically equal to 0.

A convex Orlicz function is usually called a Young function. For a Young function $\varphi$ we denote by $\varphi^*$ the function complementary to $\varphi$ in the sense of Young, i.e.

$$\varphi^*(s) = \sup\{ts - \varphi(t) : t \geq 0\} \quad \text{for} \quad s \geq 0.$$

Let $\varphi$ and $\psi$ be a pair of Orlicz functions vanishing only at zero (respectively taking only finite values). We say that $\varphi$ *increases essentially more rapidly than $\psi$*

for small $t$ (respectively for large $t$) denoted $\psi \overset{\mathrm{s}}{\prec\prec} \varphi$ (respectively $\psi \overset{\mathrm{l}}{\prec\prec} \varphi$), whenever for any $c > 0$, $\psi(ct)/\varphi(t) \to 0$ as $t \to 0$ (respectively $t \to \infty$). We will write $\psi \prec\prec \varphi$ when $\psi \overset{\mathrm{s}}{\prec\prec} \varphi$ and $\psi \overset{\mathrm{l}}{\prec\prec} \varphi$ hold. For $\varphi$ and $\psi$ being Young functions the condition $\psi \overset{\mathrm{s}}{\prec\prec} \varphi$ (respectively $\psi \overset{\mathrm{l}}{\prec\prec} \varphi$) implies $\varphi^* \overset{\mathrm{s}}{\prec\prec} \psi^*$ (respectively $\varphi^* \overset{\mathrm{l}}{\prec\prec} \psi^*$) (see [5, Lemma 13.1]).

An Orlicz function $\varphi$ determines a functional $m_\varphi : L^0 \to [0, \infty]$ by

$$m_\varphi(u) = \int_\Omega \varphi(|u(\omega)|)\, d\mu.$$

The Orlicz space generated by $\varphi$ is the ideal of $L^0$ defined by

$$L^\varphi = \{u \in L^0 : m_\varphi(\lambda u) < \infty \ \text{ for some } \ \lambda > 0\}.$$

$L^\varphi$ can be equipped with the complete metrizable topology $\mathcal{T}_\varphi$ of the $F$-norm

$$\|u\|_\varphi = \inf\Big\{\lambda > 0 : m_\varphi\big(\frac{u}{\lambda}\big) \le \lambda\Big\}.$$

Let

$$\varphi_0(t) = \begin{cases} 0 & \text{for } \ 0 \le t \le 1 \\ 1 & \text{for } \ t > 1. \end{cases}$$

It is known that $L^{\varphi_0}$ is the largest Orlicz space and consists of all those $u \in L^0$ that are bounded outside of some set of finite measure and $\|u\|_{\varphi_0} = \|u\|_\mu$ for all $u \in L^{\varphi_0}$. (see [11, 0.3.4]).

Moreover one can check that $L^{\varphi_0}$ is the largest linear subspace of $L^0$ such that the functional $\|\cdot\|_\mu$ restricted to $L^{\varphi_0}$ is an $F$-norm.

We will write $\|\cdot\|_\mu$ and $\mathcal{T}_\mu$ instead of $\|\cdot\|_{\varphi_0}$ and $\mathcal{T}_{\varphi_0}$, respectively.

Moreover, if $\varphi$ is a Young function, then the topology $\mathcal{T}_\varphi$ can be generated by the Luxemburg norm:

$$|||u|||_\varphi = \inf\Big\{\lambda > 0 : m_\varphi\big(\frac{u}{\lambda}\big) \le 1\Big\}.$$

For an Orlicz function $\varphi$ let

$$E^\varphi = \{u \in L^0 : m_\varphi(\lambda u) < \infty \ \text{ for all } \ \lambda > 0\}$$

and

$$L^\varphi_a = \{u \in L^\varphi : \|u_{A_n}\|_\varphi \to 0 \ \text{ as } \ A_n \searrow \emptyset\}.$$

It is well known that $E^\varphi = L^\varphi_a$ whenever $\varphi$ takes only finite values. Moreover, for every Young function $\varphi$ the identity $(L^\varphi)' = L^{\varphi^*}$ holds.

Let $M_\varphi : L^0(X) \to [0,\infty]$ be defined by

$$M_\varphi(f) = \int_\Omega \varphi(\|f(\omega)\|_X)\, d\mu.$$

Thus $M_\varphi(f) = m_\varphi(\widetilde{f})$. The Köthe-Bochner space

$$\begin{aligned} L^\varphi(X) &= \{f \in L^0(X) : \widetilde{f} \in L^\varphi\} \\ &= \{f \in L^0(X) : M_\varphi(\lambda f) < \infty \ \text{ for some } \ \lambda > 0\} \end{aligned}$$

is usually called *an Orlicz-Bochner space* and is equipped with the $F$-norm

$$\|f\|_{L^\varphi(X)} = \|\widetilde{f}\|_\varphi \ \text{ for } \ f \in L^\varphi(X).$$

We will denote by $\mathcal{T}_\varphi(X)$ the topology on $L^\varphi(X)$ generated by the $F$-norm $\|\cdot\|_{L^\varphi(X)}$. Moreover, if $\varphi$ is a Young function, then $\mathcal{T}_\varphi(X)$ is generated by the Luxemburg norm: $|||f|||_{L^\varphi(X)} = |||\widetilde{f}|||_\varphi$ for $f \in L^\varphi(X)$. We will write $\|\cdot\|_\mu^X$ and $\mathcal{T}_\mu(X)$ instead of $\|\cdot\|_{L^{\varphi_0}(X)}$ and $\mathcal{T}_{\varphi_0}(X)$, respectively.

## 2. Uniformly $\mu$-continuous topologies on Köthe-Bochner spaces

**Definition 2.1.** (i) A solid pseudonorm $\rho$ on $E(X)$ is said to be *uniformly $\mu$-continuous*, whenever $f_n \in E(X)$, $f_n \to 0$ $(\mu - \Omega)$ with $\sup_n \|f_n\|_{E(X)} < \infty$ imply $\rho(f_n) \to 0$.

(ii) A locally solid topology $\tau$ on $E(X)$ is said to be *uniformly $\mu$-continuous* whenever $f_n \in E(X)$, $f_n \to 0$ $(\mu - \Omega)$ with $\sup_n \|f_n\|_{E(X)} < \infty$ imply $f_n \xrightarrow{\tau} 0$.

In view of [3, Theorem 2.3] a locally solid topology $\tau$ on $E(X)$ is uniformly $\mu$-continuous iff it is generated by some family $\{\rho_\alpha : \alpha \in \{\alpha\}\}$ of uniformly $\mu$-continuous pseudonorms defined on $E(X)$.

It is easy to prove the following lemma.

**Lemma 2.1.** (i) *If $\rho$ is a uniformly $\mu$-continuous pseudonorm on $E(X)$, then $\widetilde{\rho}$ is a uniformly $\mu$-continuous pseudonorm on $E$ (i.e. $u_n \in E$ $u_n \to 0$ $(\mu - \Omega)$ with $\sup_n \|u_n\|_E < \infty$ imply $\widetilde{\rho}(u_n) \to 0$).*

(ii) *If $p$ is a uniformly $\mu$-continuous pseudonorm on $E$, then $\overline{p}$ is a uniformly $\mu$-continuous pseudonorm on $E(X)$.*

From Lemma 2.1 we easily get the following theorem that explains the relationship between uniformly $\mu$-continuous topologies on $E$ and $E(X)$.

**Theorem 2.2.** (i) *If $\tau$ is a uniformly $\mu$-continuous topology on $E(X)$, then $\widetilde{\tau}$ is a uniformly $\mu$-continuous topology on $E$.*

(ii) *If $\xi$ is a uniformly $\mu$-continuous topology on $E$, then $\overline{\xi}$ is a uniformly $\mu$-continuous topology on $E(X)$.*

We shall need the following result.

**Theorem 2.3.** (i) *If $\tau$ is the finest uniformly $\mu$-continuous topology on $E(X)$, then $\widetilde{\tau}$ is the finest uniformly $\mu$-continuous topology on $E$.*

(ii) *If $\xi$ is the finest uniformly $\mu$-continuous topology on $E$, then $\overline{\xi}$ is the finest uniformly $\mu$-continuous topology on $E(X)$.*

Proof: (i) Let $\xi$ be a uniformly $\mu$-continuous topology on $E$. By Theorem 2.2 $\overline{\xi}$ is a uniformly $\mu$-continuous topology on $E(X)$, so $\overline{\xi} \subset \tau$. By [3, Theorem 3.3] and Theorem 1.3 $\xi = \widetilde{\overline{\xi}} \subset \widetilde{\tau}$, as desired.

(ii) Let $\tau$ be a uniformly $\mu$-continuous topology on $E(X)$. By Theorem 2.2 $\widetilde{\tau}$ is a uniformly $\mu$-continuous topology on $E$, so $\widetilde{\tau} \subset \xi$. By [3, Theorem 3.3] and Theorem 1.3 $\tau = \overline{\widetilde{\tau}} \subset \overline{\xi}$, as desired. □

Now we are going to give a description of uniformly $\mu$-continuous topologies on Orlicz-Bochner spaces. We start with the following definition.

**Definition 2.2.** A solid pseudonorm $\rho$ on $E(X)$ is said to be *uniformly summable* whenever the following conditions hold:

For every $r > 0$

$$(*) \qquad \sup\{\rho(\chi_{A(f,\lambda)} f) : f \in E(X), \|f\|_{E(X)} \le r\} \to 0 \quad \text{as} \quad \lambda \to 0_+,$$

where $A(f,\lambda) = \{\omega \in \Omega : \|f(\omega)\|_X \le \lambda \text{ or } \|f(\omega)\|_X > \frac{1}{\lambda}\}$ for $0 < \lambda < 1$ and

$$(**) \qquad \rho(\overline{\chi}_A) \to 0 \quad \text{as} \quad \mu(A) \to 0.$$

**Theorem 2.4.** *Let $\varphi$ be an arbitrary Orlicz function and $\psi$ be a finite valued Orlicz function such that $\psi \prec\prec \varphi$. Then the $F$-norm $\|\cdot\|_{L^\psi(X)}$ (restricted to $L^\varphi(X)$) is uniformly summable on $L^\varphi(X)$.*

Proof: Since $\psi \prec\prec \varphi$, so $L^\varphi \subset L^\psi$ (see [11, 0.2.5, 0.3.5]). Hence $L^\varphi(X) \subset L^\psi(X)$. Let $r > 0$, $\varepsilon > 0$ be given. Choose $\eta > 0$ such that $\eta(r+1) < \varepsilon$ and let $c = \frac{\varepsilon}{r+1}$. Then there exist $0 < t_1 < t_2$ such that $\psi(t) \le \eta\varphi(ct)$ for $0 \le t < t_1$ or $t > t_2$, and choose $\lambda_0 \in (0,1)$ such that $\lambda_0 \le \varepsilon t_1$ and $\frac{1}{\lambda_0} > \varepsilon t_2$. Hence for $f \in L^\varphi(X)$ and $\|f\|_{L^\varphi(X)} \le r$ we have:

$$\begin{aligned} M_\psi\Big(\frac{\chi_{A(f,\lambda)} f}{\varepsilon}\Big) &= \int_{A(f,\lambda)} \psi\Big(\frac{\|f(\omega)\|_X}{\varepsilon}\Big)\, d\mu \le \int_{A(f,\lambda)} \eta\varphi\Big(c\frac{\|f(\omega)\|_X}{\varepsilon}\Big)\, d\mu \\ &\le \int_\Omega \eta\varphi\Big(\frac{\|f(\omega)\|_X}{r+1}\Big)\, d\mu \le \eta(r+1) < \varepsilon \end{aligned}$$

for every $0 < \lambda \le \lambda_0$. It follows that $\|\chi_{A(f,\lambda)} f\|_{L^\psi(X)} \le \varepsilon$ for every $f \in L^\varphi(X)$, $\|f\|_{L^\varphi(X)} \le r$ and $0 < \lambda \le \lambda_0$. This means that for $r > 0$

$$\sup\{\|\chi_{A(f,\lambda)} f\|_{L^\psi(X)} : f \in L^\varphi(X),\ \|f\|_{L^\varphi(X)} \le r\} \to 0 \quad \text{as} \quad \lambda \to 0_+.$$

Now, choose $\delta > 0$ such that $0 < \delta < \frac{\varepsilon}{\psi\left(\frac{1}{\varepsilon}\right)}$. Then $M_\psi\left(\frac{\overline{\chi}_A}{\varepsilon}\right) = \int_A \psi\left(\frac{1}{\varepsilon}\right) d\mu = \mu(A)\cdot\psi\left(\frac{1}{\varepsilon}\right) \le \delta\cdot\psi\left(\frac{1}{\varepsilon}\right) < \varepsilon$ for every $A \in \Sigma$ with $\mu(A) \le \delta$. Hence $\|\overline{\chi}_A\|_{L^\psi(X)} \to 0$ as $\mu(A) \to 0$, and the proof is finished. $\square$

**Remark 2.1.** Let $\varphi$ be an Orlicz function such that $\varphi(u) \to \infty$ as $u \to \infty$. Then $\varphi_0 \prec\prec \varphi$ and it follows that the $F$-norm $\|\cdot\|_\mu^X$ is uniformly summable on $L^\varphi(X)$.

**Theorem 2.5.** *Let $\varphi$ be an Orlicz function such that $\varphi(u) \to \infty$ as $u \to \infty$. For a solid pseudonorm $\rho$ on $L^\varphi(X)$ the following statements are equivalent:*

(i) *$\rho$ is uniformly summable;*
(ii) *$\rho$ is uniformly $\mu$-continuous.*

Proof: (i) $\Rightarrow$ (ii) Take a sequence $(f_n)$ in $L^\varphi(X)$ such that $f_n \to 0$ $(\mu - \Omega)$ and $\sup_n \|f_n\|_{L^\varphi(X)} \le r$ for some $r > 0$. Fix $\varepsilon > 0$. There exists $\lambda_0 \in (0,1)$ such that $\sup_n \rho(\chi_{A(f_n,\lambda_0)} f_n) < \frac{\varepsilon}{2}$. Moreover, there exists $\delta > 0$ such that

$$\rho(\overline{\chi}_A) < \frac{\varepsilon}{2\left(\left[\frac{1}{\lambda_0}\right]+1\right)} \quad \text{whenever} \quad A \in \Sigma \quad \text{with} \quad \mu(A) \le \delta.$$

Since $f_n \to 0$ $(\mu - \Omega)$, we can find a natural number $k$ such that for all $n \ge k$

$$\mu(\Omega \setminus A(f_n,\lambda_0)) \le \mu(\{\omega \in \Omega : \|f_n(\omega)\|_X > \lambda_0\}) \le \delta.$$

Hence for $n \ge k$

$$\begin{aligned}
\rho(f_n) &= \rho(\chi_{A(f_n,\lambda_0)} f_n + \chi_{\Omega\setminus A(f_n,\lambda_0)} f_n) \le \rho(\chi_{A(f_n,\lambda_0)} f_n) \\
&\qquad + \rho(\chi_{\Omega\setminus A(f_n,\lambda_0)} f_n) \\
&\le \frac{\varepsilon}{2} + \rho\Big(\big([\frac{1}{\chi_0}]+1\big)\overline{\chi}_{\Omega\setminus A(f_n,\lambda_0)}\Big) \le \frac{\varepsilon}{2} + \big([\frac{1}{\lambda_0}]+1\big)\rho(\overline{\chi}_{\Omega\setminus A(f_n,\lambda_0)}) \\
&\le \frac{\varepsilon}{2} + \big([\frac{1}{\lambda_0}]+1\big)\frac{\varepsilon}{2\left(\left[\frac{1}{\lambda_0}\right]+1\right)} \le \varepsilon.
\end{aligned}$$

Thus $\rho(f_n) \to 0$.

(ii) $\Rightarrow$ (i) For $r > 0$ let $B_X^\varphi(r) = \{f \in L^\varphi(X) : \|f\|_{L^\varphi(X)} \le r\}$, $B_X^\rho(r) = \{f \in L^\varphi(X) : \rho(f) \le r\}$, $B_X^\mu(r) = \{f \in L^{\varphi_0}(X) : \|f\|_\mu^X \le r\}$. By (ii) the identity map

$$id : (B_X^\varphi(r), \mathcal{T}_\mu(X)|_{B_X^\varphi(r)}) \to (B_X^\varphi(r), \tau(\rho)|_{B_X^\varphi(r)})$$

is continuous at zero for any $r > 0$, where $\tau(\rho)$ denotes the topology on $L^\varphi(X)$ generated by $\rho$. Let $\varepsilon > 0$, $r > 0$ be given. There exists $\eta > 0$ such that $B_X^\mu(\eta) \cap B_X^\varphi(r) \subset B_X^\rho(\varepsilon)$. Since $\|\cdot\|_\mu^X$ is uniformly summable on $L^\varphi(X)$ (see Remark 2.1) there exists $\lambda_0 \in (0,1)$ such that

$$\sup\{\|\chi_{A(f,\lambda)} f\|_\mu^X : f \in L^\varphi(X),\ \|f\|_{L^\varphi(X)} \le r\} \le \eta \quad \text{whenever} \quad 0 < \lambda \le \lambda_0.$$

Then $\sup\{\rho(\chi_{A(f,\lambda)} f) : f \in L^{\varphi}(X),\ \|f\|_{L^{\varphi}(X)} \le r\} \le \varepsilon$ whenever $0 < \lambda \le \lambda_0$. Hence $\sup\{\rho(\chi_{A(f,\lambda)} f) : f \in L^{\varphi}(X),\ \|f\|_{L^{\varphi}(X)} \le r\} \to 0$ as $\lambda \to 0_+$.

Moreover, there exists $\delta > 0$ such that $\|\overline{\chi}_A\|_{\mu}^{X} \le \eta$ for $A \in \Sigma$ with $\mu(A) \le \delta$. Then $\rho(\overline{\chi}_A) \le \varepsilon$ whenever $A \in \Sigma$ with $\mu(A) \le \delta$. It follows that $\rho(\overline{\chi}_A) \to 0$ as $\mu(A) \to 0$.

Thus $\rho$ is a uniformly summable pseudonorm on $L^{\varphi}(X)$. □

**Theorem 2.6.** *Let $\varphi$ be an Orlicz function such that $\varphi(u) \to \infty$ as $u \to \infty$. For a locally solid topology $\tau$ on $L^{\varphi}(X)$ the following statements are equivalent:*

(i) *$\tau$ is uniformly $\mu$-continuous;*
(ii) *$\tau|_{B_X^{\varphi}(r)} \subset \mathcal{T}_{\mu}(X)|_{B_X^{\varphi}(r)}$ for every $r > 0$;*
(iii) *$\tau$ is generated by some family of uniformly summable pseudonorms.*

Proof: (i) ⇒ (ii) Since $\mathcal{T}_{\mu}(X)$ is a linear metrizable topology, it follows from Definition 2.1 (ii).

(ii) ⇒ (i) Obvious.

(i) ⇒ (iii) Let $\tau$ be defined by the family $\{\rho_{\alpha} : \alpha \in \{\alpha\}\}$ of solid pseudonorms. Then by Definition 2.1 and Theorem 2.5 $\tau$ is generated by the family $\{\rho_{\alpha} : \alpha \in \{\alpha\}\}$ of uniformly summable pseudonorms.

(iii) ⇒ (i) It follows from Theorem 2.5. □

## 3. Generalized mixed topologies on Orlicz-Bochner spaces

In this section we consider some kind of inductive limit topology on Orlicz-Bochner space $L^{\varphi}(X)$.

Let $\varphi$ be an arbitrary Orlicz function, and let

$$F_n^X = B_X^{\varphi}(2^n) \quad \text{and} \quad \mathcal{T}_n(X) = \mathcal{T}_{\mu}(X)|_{F_n^X} \quad \text{for} \quad n \ge 0.$$

It can be seen that the metric bounded sets $F_n^X$ $(n \ge 0)$ are balanced subsets of $L^{\varphi}(X)$. Moreover, the sequence $(F_n^X, \mathcal{T}_n(X))$ $(n \ge 0)$ of balanced topological spaces satisfies the following conditions:

(i) $L^{\varphi}(X) = \bigcup_{n \ge 0} F_n^X$;
(ii) $F_n^X + F_n^X \subset F_{n+1}^X$, and the function

$$F_n^X \times F_n^X \ni (f, g) \mapsto f + g \in F_{n+1}^X$$

is continuous $(n \ge 0)$;
(iii) the function $[-1, 1] \times F_n^X \ni (\lambda, f) \mapsto \lambda \cdot f \in F_n^X$ is continuous $(n \ge 0)$;
(iv) $\mathcal{T}_{n+1}(X)|_{F_n^X} = \mathcal{T}_n(X)$ for $n \ge 0$.

Thus the space $L^{\varphi}(X)$ with the system $\{(F_n^X, \mathcal{T}_n(X)) : n \ge 0\}$ comes under the definition of the strict inductive limit of balanced topological spaces (in the sense of Turpin; see [11, Definition 1.1.1]).

**Definition 3.1.** Let $\varphi$ be an Orlicz function and let $(\varepsilon_n)$ be a sequence of positive numbers. The family of all sets of the form:

$$(*) \qquad \bigcup_{N=0}^{\infty}\Big(\sum_{n=0}^{N} B_X^{\varphi}(2^n)\cap B_X^{\mu}(\varepsilon_n)\Big)$$

forms a base of neighbourhoods of zero for a linear topology $\mathcal{T}_I^{\varphi}(X)$ on $L^{\varphi}(X)$ that will be called *generalized mixed topology.* $\mathcal{T}_I^{\varphi}(X)$ is exactly the strict inductive limit topology of balanced topological spaces $\{(B_X^{\varphi}(2^n),\mathcal{T}_\mu(X)|_{B_X^{\varphi}(2^n)}) : n\geq 0\}$ in the sense of Turpin [11, Chapter I].

Using the solid decomposition property (see [3, Lemma 1.1]) it is easy to verify that the sets of the form $(*)$ are solid, so $\mathcal{T}_I^{\varphi}(X)$ is locally solid.

According to [11, Theorem 1.1.6] $\mathcal{T}_I^{\varphi}(X)$ is the finest of all linear topologies $\tau$ on $L^{\varphi}(X)$, which satisfy the condition

$$(1) \qquad \tau|_{B_X^{\varphi}(2^n)}\subset \mathcal{T}_\mu(X)|_{B_X^{\varphi}(2^n)} \quad \text{for} \quad n\geq 0.$$

Moreover, in view of [11, Theorem 1.1.8] we have

$$(2) \qquad \mathcal{T}_I^{\varphi}(X)|_{B_X^{\varphi}(2^n)} = \mathcal{T}_\mu(X)|_{B_X^{\varphi}(2^n)} \quad \text{for} \quad n\geq 0.$$

Since $\mathcal{T}_\mu(X)|_{L^{\varphi}(X)}\subset \mathcal{T}_\varphi(X)$ we have $\mathcal{T}_I^{\varphi}(X)\subset\mathcal{T}_\varphi(X)$; hence $\mathcal{T}_\mu(X)|_{L^{\varphi}(X)}\subset \mathcal{T}_I^{\varphi}(X)\subset\mathcal{T}_\varphi(X)$.

*Henceforth, we assume in this section that $\varphi(u)\to\infty$ as $u\to\infty$.*

**Theorem 3.1.** *The topology $\mathcal{T}_I^{\varphi}(X)$ is the finest uniformly $\mu$-continuous topology on $L^{\varphi}(X)$.*

Proof: It follows from (1) and Theorem 2.6. □

The generalized mixed topology $\mathcal{T}_I^{\varphi}$ on Orlicz spaces $L^{\varphi}$ has been studied in [11], [8], [9], [10]. Now we will extend the study of the generalized mixed topology to the Orlicz-Bochner spaces.

**Theorem 3.2.** *The space $(L^{\varphi}(X),\mathcal{T}_I^{\varphi}(X))$ is complete.*

Proof: First we show that the balls $B_X^{\varphi}(2^n)$ are closed subsets of $(L^{\varphi_0}(X),\mathcal{T}_\mu(X))$. Indeed, let $(f_k)$ be a sequence in $B_X^{\varphi}(2^n)$ and let $f\in L^{\varphi_0}(X)$ be such that $f_k\to f$ for $\mathcal{T}_\mu(X)$. This means that $\mu(\{\omega\in\Omega : \|f_k(\omega)-f(\omega)\|_X > \varepsilon\})\to 0$ for any $\varepsilon>0$. Hence $\mu(\{\omega\in\Omega : \big|\|f_k(\omega)\|_X-\|f(\omega)\|_X\big|>\varepsilon\})\to 0$ for every $\varepsilon>0$. Thus $\widetilde{f}_k\to\widetilde{f}$ for $\mathcal{T}_\mu$ in $L^{\varphi_0}$. It is known that the balls $B_\varphi(2^n)$ are closed subsets of $(L^{\varphi_0},\mathcal{T}_\mu)$ (see [11, 0.3.6]). But $\widetilde{f}_k\in B_\varphi(2^n)$ $(k=1,2,\ldots)$, $\widetilde{f}\in L^{\varphi_0}$, so we get $\widetilde{f}\in B_\varphi(2^n)$. It follows that $f\in B_X^{\varphi}(2^n)$.

Since the spaces $(B_X^{\varphi}(2^n),\mathcal{T}_\mu(X)|_{B_X^{\varphi}(2^n)})$ $(n\geq 0)$ are complete, by [11, Theorem 1.1.10] the space $(L^{\varphi}(X),\mathcal{T}_I^{\varphi}(X))$ is complete.

□

**Theorem 3.3.** *For a subset $Z \subset L^\varphi(X)$ the following statements are equivalent:*

(i) $\sup\{\|f\|_{L^\varphi(X)} : f \in Z\} < \infty$;

(ii) *$Z$ is bounded for $\mathcal{T}_I^\varphi(X)$.*

Proof: Observe that the balls $B_X^\varphi(2^n)$ are bounded subsets of $(L^\varphi(X), \mathcal{T}_\mu(X)|_{L^\varphi(X)})$. In fact, fix an $r > 0$, let $f_n \in B_X^\varphi(r)$ $(n = 1, 2, \dots)$ and let $\lambda_n \to 0$. For $\varepsilon > 0$ let $\Omega_n(\varepsilon) = \{\omega \in \Omega : \|\lambda_n f_n(\omega)\|_X > \varepsilon\}$. Then we have

$$\mu(\Omega_n(\varepsilon)) \cdot \varphi\Big(\frac{\varepsilon}{r|\lambda_n|}\Big) \le \int_{\Omega_n(\varepsilon)} \varphi\Big(\frac{\|f_n(\omega)\|_X}{r}\Big)\, d\mu \le M_\varphi\Big(\frac{f_n}{r}\Big) \le r.$$

Since $\varphi(u) \to \infty$ as $u \to \infty$ we get $\mu(\Omega_n(\varepsilon)) \to 0$ and this means that $\lambda_n f_n \to 0$ for $\mathcal{T}_\mu(X)$.

Moreover the balls $B_X^\varphi(2^n)$ are also closed in $(L^\varphi(X), \mathcal{T}_\mu(X)|_{L^\varphi(X)})$. In view of (1) and (2) $\mathcal{T}_I^\varphi(X)$ is the finest of all linear topologies $\tau$ on $L^\varphi(X)$ such that $\tau|_{B_X^\varphi(2^n)} = \mathcal{T}_\mu(X)|_{B_X^\varphi(2^n)}$ $(n = 0, 1, 2, \dots)$. Hence by [11, Corollary 1.1.12] the equivalence (i) $\Leftrightarrow$ (ii) holds. □

**Theorem 3.4.** *For a subset $Z \subset L^\varphi(X)$ the following statements are equivalent:*

(i) *$Z$ is relatively compact for $\mathcal{T}_I^\varphi(X)$;*

(ii) *$Z$ is relatively compact for $\mathcal{T}_\mu(X)|_{L^\varphi(X)}$ and* $\sup\{\|f\|_{L^\varphi(X)} : f \in Z\} < \infty$.

Proof: follows from Theorem 3.3 and (2). □

**Definition 3.2.** A sequence $(f_n)$ in $L^\varphi(X)$ is said to be $\gamma_\varphi^X$-convergent to $f \in L^\varphi(X)$, in symbols $f_n \xrightarrow{\gamma_\varphi} f$, whenever

$$f_n \to f \ (\mu - \Omega) \ \text{ and } \ \sup_n \|f_n\|_{L^\varphi(X)} < \infty.$$

**Theorem 3.5.** *For a sequence $(f_n)$ in $L^\varphi(X)$ the following statements are equivalent:*

(i) *$f_n \to 0$ for $\mathcal{T}_I^\varphi(X)$;*

(ii) $f_n \xrightarrow{\gamma_\varphi} 0$.

*Moreover, $\mathcal{T}_I^\varphi(X)$ is the finest of all linear topologies $\tau$ on $L^\varphi(X)$ which satisfy the condition:*

$$(+) \qquad f_n \xrightarrow{\gamma_\varphi} 0 \ \textit{ implies } \ f_n \to 0 \ \textit{ for } \ \tau.$$

Proof: The equivalence (i) $\Leftrightarrow$ (ii) follows immediately from Theorem 3.3 and (2). Now let $\tau$ be a linear topology on $L^\varphi(X)$ for which the condition (+) holds. Then $\tau|_{B_X^\varphi(r)} \subset \mathcal{T}_\mu(X)|_{B_X^\varphi(r)}$ for $r > 0$, because $\mathcal{T}_\mu(X)$ is a linear metrizable topology. Hence by (1) we get $\tau \subset \mathcal{T}_I^\varphi(X)$. □

**Definition 3.3.** Let $(Y,\eta)$ be a linear topological space. A linear mapping $T : L^\varphi(X) \to Y$ is said to be $\gamma_\varphi$-linear, whenever

$$f_n \xrightarrow{\gamma_\varphi} 0 \ \text{ implies } \ T(f_n) \to 0 \ \text{ for } \ \eta.$$

Then following theorem gives a characterization of $\gamma_\varphi$-linear operators on $L^\varphi(X)$.

**Theorem 3.6.** *For a linear topological space $(Y,\eta)$ and a linear mapping $T : L^\varphi(X) \to Y$ the following statements are equivalent:*

(i) *$T$ is $(\mathcal{T}_I^\varphi(X),\eta)$-continuous;*
(ii) *$T$ is $\gamma_\varphi$-linear;*
(iii) *for every $r>0$, the restriction $T|_{B_X^\varphi(r)}$ is $(\mathcal{T}_\mu(X)|_{B_X^\varphi(r)},\eta)$-continuous.*

Proof: (i) $\Rightarrow$ (ii) It follows from Theorem 3.5.

(ii) $\Rightarrow$ (iii) Obvious.

(iii) $\Rightarrow$ (i) Let $W$ be a neighbourhood of zero in $Y$ for $\eta$. Since $\eta$ is a linear topology, there exists a sequence $(W_n : n \geq 0)$ of neighbourhoods of zero for $\eta$ such that $\sum_{n=0}^N W_n \subset W$ for every $N \geq 0$. By (iii) we can find a sequence $(\varepsilon_n : n \geq 0)$ of positive numbers such that $T(B_X^\varphi(2^n) \cap B_X^\mu(\varepsilon_n)) \subset W_n$ for $n \geq 0$. Thus for $N \geq 0$ we have

$$T\Big(\sum_{n=0}^N (B_X^\varphi(2^n) \cap B_X^\mu(\varepsilon_n))\Big) \subset \sum_{n=0}^N W_n \subset W,$$

so

$$T\Big(\bigcup_{N=0}^\infty \Big(\sum_{n=0}^N (B_X^\varphi(2^n) \cap B_X^\mu(\varepsilon_n))\Big)\Big) \subset \bigcup_{N=0}^\infty T\Big(\sum_{n=0}^N (B_X^\varphi(2^n) \cap B_X^\mu(\varepsilon_n))\Big) \subset W.$$

It follows that $T$ is $(\mathcal{T}_I^\varphi(X),\eta)$-continuous. □

**Theorem 3.7.** *Assume that $(\Omega,\Sigma,\mu)$ is an atomless measure space or that $\mu$ is the counting measure on $\mathbb{N}$. If $(L^\varphi(X),\mathcal{T}_\varphi(X))$ is a locally bounded space then for a subset $Z$ of $L^\varphi(X)$ the following statements are equivalent:*

(i) *$Z$ is bounded for $\mathcal{T}_I^\varphi(X)$;*
(ii) $\sup\{\|f\|_{L^\varphi(X)} : f \in Z\} < \infty$;
(iii) *$Z$ is bounded for $\mathcal{T}_\varphi(X)$.*

Proof: (i) $\Leftrightarrow$ (ii) See Theorem 3.3.

(ii) $\Rightarrow$ (iii) In view of [11, 0.3.10.2] $\sup\{\|f\|_{L^\varphi(X)} : f \in Z\} < \infty$ iff $Z$ is additively bounded (see [11, 0.3.10.1]), so arguing as in the proof of [9, Lemma 2.5] we obtain that $Z$ is bounded for $\mathcal{T}_\varphi(X)$.

(iii) $\Rightarrow$ (i) Obvious. □

The next theorem compares the topology $\mathcal{T}_I^\varphi(X)$ with the mixed topology $\gamma[\mathcal{T}_\varphi(X),\mathcal{T}_\mu(X)|_{L^\varphi(X)}]$ in the sense of Wiweger (see [12]).

**Theorem 3.8.** *Assume that* $(\Omega, \Sigma, \mu)$ *is an atomless measure space or that* $\mu$ *is the counting measure on* $\mathbb{N}$. *If* $(L^\varphi(X), \mathcal{T}_\varphi(X))$ *is a locally bounded space, then the generalized mixed topology* $\mathcal{T}_I^\varphi(X)$ *coincides with the mixed topology* $\gamma[\mathcal{T}_\varphi(X), \mathcal{T}_\mu(X)|_{L^\varphi(X)}]$.

PROOF: In view of Theorem 3.7 it follows from [12, 2.2.1, 2.2.2]. □

An Orlicz function $\varphi$ continuous for all $u \geq 0$, taking only finite values, vanishing only at zero and not bounded is usually called a $\varphi$-function. By $\Phi$ we will denote the collection of all $\varphi$-functions.

A Young function $\varphi$ vanishing only at zero and taking only finite values is called an $N$-function whenever $\frac{\varphi(t)}{t} \to 0$ as $t \to 0$ and $\frac{\varphi(t)}{t} \to \infty$ as $t \to \infty$. By $\Phi_N$ we will denote the collection of all $N$-functions.

Let $\Phi_1$ be the set of all Orlicz functions $\varphi$ vanishing only at zero and such that $\varphi(t) \to \infty$ as $t \to \infty$. Denote by

$\Phi_{11} = \{\varphi \in \Phi_1 : \varphi(t) < \infty \text{ for } t \geq 0\}$,

$\Phi_{12} = \{\varphi \in \Phi_1 : \varphi \text{ jumps to } \infty\}$.

Then $\Phi_1 = \Phi_{11} \cup \Phi_{12}$.

**Theorem 3.9.** *Let* $\varphi \in \Phi_{1i}$ $(i = 1, 2)$. *Then the topology* $\mathcal{T}_I^\varphi(X)$ *is generated by the family of solid* $F$*-norms:*

$$\{\|\cdot\|_{L^\psi(X)} : \psi \in \Psi_{1i}^\varphi\},$$

*where* $\Psi_{11}^\varphi = \{\psi \in \Phi : \psi \prec\!\prec \varphi\}$, $\Psi_{12}^\varphi = \{\psi \in \Phi : \psi \overset{s}{\prec\!\prec} \varphi\}$.

*Moreover, the following identities hold:*

$$L^\varphi(X) = \bigcap\{L^\psi(X) : \psi \in \Psi_{1i}^\varphi\} = \bigcap\{E^\psi(X) : \psi \in \Psi_{1i}^\varphi\}. \tag{3}$$

PROOF: Let $\varphi \in \Phi_{1i}$ $(i = 1, 2)$. Then $\mathcal{T}_I^\varphi$ is the finest uniformly $\mu$-continuous topology on $L^\varphi$ (see [10, Theorem 2.4]) and is generated by the family $\{\|\cdot\|_\psi : \psi \in \Psi_{1i}^\varphi\}$ (see [10, Theorem 4.5, Theorem 3.8]). Then the topology $\overline{\mathcal{T}_I^\varphi}$ on $L^\varphi(X)$ is generated by the family $\{\|\cdot\|_{L^\psi(X)} : \psi \in \Psi_{1i}^\varphi\}$ of solid $F$-norms and by Theorem 2.3 $\overline{\mathcal{T}_I^\varphi}$ is the finest uniformly $\mu$-continuous topology on $L^\varphi(X)$. By Theorem 3.1 $\overline{\mathcal{T}_I^\varphi} = \mathcal{T}_I^\varphi(X)$, and we are done.

The identities (3) follow from [10, Theorem 3.1]. □

Let $\Phi_1^c$ be the set of all Young functions $\varphi$ vanishing only at zero and such that $\frac{\varphi(t)}{t} \to \infty$ as $t \to \infty$. Denote by

$\Phi_{11}^c = \{\varphi \in \Phi_1^c : \varphi(t) < \infty \text{ for } t \geq 0 \text{ and } \frac{\varphi(t)}{t} \to 0 \text{ as } t \to 0\}$,

$\Phi_{12}^c = \{\varphi \in \Phi_1^c : \varphi \text{ jumps to } \infty \text{ and } \frac{\varphi(t)}{t} \to 0 \text{ as } t \to 0\}$,

$\Phi^c_{13} = \{\varphi \in \Phi^c_1 : \varphi(t) < \infty$ for $t \geq 0$ and $\frac{\varphi(t)}{t} \to a$ as $t \to 0$ for some $a > 0\}$,

$\Phi^c_{14} = \{\varphi \in \Phi^c_1 : \varphi$ jumps to $\infty$ and $\frac{\varphi(t)}{t} \to a$ as $t \to 0$ for some $a > 0\}$.

Then $\Phi^c_1 = \bigcup_{i=1}^4 \Phi^c_{1i}$ and the sets $\Phi^c_{1i}$ $(i = 1, 2, 3, 4)$ are pairwise disjoint. It can be seen that $\Phi^c_{11} = \Phi_N$.

**Theorem 3.10.** *Let $\varphi \in \Phi^c_{1i}$ $(i = 1, 2, 3, 4)$. Then the topology $\mathcal{T}^\varphi_I(X)$ is generated by the family of solid norms*

$$\{\|\!| \cdot |\!\|_{L^\psi(X)} : \psi \in \Psi^\varphi_{1i}(N)\},$$

*where* $\Psi^\varphi_{11}(N) = \{\psi \in \Phi_N : \psi \prec\!\prec \varphi\}$, $\Psi^\varphi_{12}(N) = \{\psi \in \Phi_N : \psi \overset{\mathrm{s}}{\prec\!\prec} \varphi\}$, $\Psi^\varphi_{13}(N) = \{\psi \in \Phi_N : \psi \overset{1}{\prec\!\prec} \varphi\}$, $\Psi^\varphi_{14}(N) = \Phi_N$.
*Moreover, the following identities hold:*

$$L^\varphi(X) = \bigcap\{L^\psi(X) : \psi \in \Psi^\varphi_{1i}(N)\} = \bigcap\{E^\psi(X) : \psi \in \Psi^\varphi_{1i}(N)\}. \tag{4}$$

Proof: Let $\varphi \in \Phi^c_{1i}$ $(i = 1, 2, 3, 4)$. Then $\mathcal{T}^\varphi_I$ is the finest uniformly $\mu$-continuous topology on $L^\varphi$ (see [10, Theorem 2.4]) and is generated by the family $\{\|\!| \cdot |\!\|_\psi : \psi \in \Psi^\varphi_{1i}(N)\}$ (see [10, Theorem 3.12 and Theorem 4.5]). Then the topology $\overline{\mathcal{T}^\varphi_I}$ on $L^\varphi(X)$ is generated by the family $\{\|\!| \cdot |\!\|_{L^\psi(X)} : \psi \in \Psi^\varphi_{1i}(N)\}$ of solid norms, and by Theorem 2.3 $\overline{\mathcal{T}^\varphi_I}$ is the finest uniformly $\mu$-continuous topology on $L^\varphi(X)$. By Theorem 3.1 $\overline{\mathcal{T}^\varphi_I} = \mathcal{T}^\varphi_I(X)$, as desired.

The identities (4) follow from [10, Theorem 3.2]. □

As an application of Theorem 3.10 we get a characterization of uniformly $\mu$-continuous seminorms on $L^\varphi(X)$.

**Theorem 3.11.** *Let $\varphi \in \Phi^c_{1i}$ $(i = 1, 2, 3, 4)$. Then for a solid seminorm $\rho$ on $L^\varphi(X)$ the following statements are equivalent:*

(i) *$\rho$ is uniformly $\mu$-continuous;*
(ii) *there exist $\psi \in \Psi^\varphi_{1i}(N)$ and a number $a > 0$ such that*

$$\rho(f) \leq a\|\!| f |\!\|_{L^\psi(X)} \quad \textit{for all} \quad f \in L^\varphi(X).$$

Proof: (i) $\Rightarrow$ (ii) Since $\mathcal{T}^\varphi_I(X)$ is the finest uniformly $\mu$-continuous topology on $L^\varphi(X)$ (see Theorem 3.1), in view of Theorem 3.10 and [4, Chapter 4, § 18(4)] there exist $\psi_1, \ldots, \psi_n \in \Psi^\varphi_{1i}(N)$ and a number $a > 0$ such that

$$\rho(f) \leq a \max(\|\!| f |\!\|_{L^{\psi_1}(X)}, \ldots, \|\!| f |\!\|_{L^{\psi_n}(X)}) \quad \text{for all} \quad f \in L^\varphi(X).$$

Let $\psi(u)=\max(\psi_1(u),\ldots,\psi_n(u))$ for $u\ge 0$. Then $\psi\in\Psi_{1i}^{\varphi}(N)$ and $|||f|||_{L^{\psi_i}(X)}\le |||f|||_{L^{\psi}(X)}$ for $i=1,\ldots,n$ and all $f\in L^{\varphi}(X)$, so

$$\rho(f)\le a|||f|||_{L^{\psi}(X)} \quad \text{for all} \quad f\in L^{\varphi}(X).$$

(ii) $\Rightarrow$ (i) It is obvious, because for each $\psi\in\Psi_{1i}^{\varphi}(X)$, $|||\cdot|||_{L^{\psi}(X)}$ is a uniformly $\mu$-continuous norm on $L^{\varphi}(X)$. □

To present the general form of $\mathcal{T}_I^{\varphi}(X)$-continuous linear functionals on $L^{\varphi}(X)$ we recall the terminology concerning some spaces of $X$-weak measurable functions (see [2]).

Given a function $g:\Omega\to X^*$ and $x\in X$ we denote by $g_x$ the real function on $\Omega$ defined by $g_x(\omega)=g(\omega)(x)$. A function $g$ is said to be $X$-weak measurable if the functions $g_x$ are measurable for each $x\in X$. We say that two $X$-weak measurable functions $g_1$, $g_2$ are equivalent whenever $g_1(\omega)(x)=g_2(\omega)(x)$ $\mu$-a.e. for all $x\in X$.

By $L^0(X^*,X)$ we denote the linear space of equivalence classes of all $X$-weak measurable functions $g:\Omega\to X^*$. It is known that the set $\{|g_x|:x\in B_X\}$ is order bounded in $L^0$ for every $g\in L^0(X^*,X)$.

The function $\vartheta: L^0(X^*,X)\to L^0$ defined by

$$\vartheta(g)=\sup\{|g_x|:x\in B_X\} \quad \text{for} \quad g\in L^0(X^*,X)$$

is called an abstract norm.

It is known that for $f\in L^0(X)$, $g\in L^0(X^*,X)$ the function $\langle f,g\rangle:\Omega\to R$ defined by $\langle f,g\rangle(\omega)=\langle f(\omega),g(\omega)\rangle=g(\omega)(f(\omega))$ is measurable and

$$|\langle f,g\rangle(\omega)|\le \|f(\omega)\|_X\cdot\vartheta(g)(\omega) \quad \mu\text{-a.e.}$$

For an ideal $I$ of $L^0$ let

$$I(X^*,X)=\{g\in L^0(X^*,X):\vartheta(g)\in I\}.$$

**Theorem 3.12.** *Let $\varphi\in\Phi_{1i}^c$ $(i=1,2,3,4)$. Then for a linear functional $F$ on $L^{\varphi}(X)$ the following statements are equivalent:*

(i) *$F$ is continuous for $\mathcal{T}_I^{\varphi}(X)$;*
(ii) *$F$ is $\gamma_{\varphi}$-linear;*
(iii) *there exists a unique $g\in E^{\varphi^*}(X^*,X)$ such that*

$$F(f)=F_g(f)=\int_{\Omega}\langle f(\omega),g(\omega)\rangle\,d\mu \quad \textit{for} \quad f\in L^{\varphi}(X).$$

PROOF: (i) $\Leftrightarrow$ (ii) The equivalence follows from Theorem 3.6.
(i) $\Rightarrow$ (iii) Let $\varphi \in \Phi^c_{1i}$ $(i = 1, 2, 3, 4)$. In view of Theorem 3.10 (see also the proof of Theorem 3.11) there exist $\psi \in \Psi^{\varphi}_{1i}(N)$ and $r > 0$ such that $F$ is bounded on $B^{(\psi)}_X(r) \cap L^{\varphi}(X)$, where $B^{(\psi)}_X(r) = \{f \in L^{\psi}(X) : |||f|||_{L^{\psi}(X)} \le r\}$. This means that $F$ is continuous on the linear subspace $(L^{\varphi}(X), \mathcal{T}_{\psi}(X)|_{L^{\varphi}(X)})$ of the normed space $(E^{\psi}(X), \mathcal{T}_{\psi}(X)|_{E^{\psi}(X)})$. Hence by the Hahn-Banach extension theorem there exists a $\mathcal{T}_{\psi}(X)|_{E^{\psi}(X)}$-continuous linear functional $\overline{F}$ on $E^{\psi}(X)$ such that $\overline{F}(f) = F(f)$ for $f \in L^{\varphi}(X)$. Since $E^{\psi} = L^{\psi}_a$, we get $E^{\psi}(X) = L^{\psi}_a(X)$. By [2, Corollary 4.1] there exists a unique $g \in (L^{\psi}_a)'(X^*, X)$ such that

$$\overline{F}(f) = \int_{\Omega} \langle f(\omega), g(\omega)\rangle \, d\mu \quad \text{for} \quad f \in L^{\psi}_a(X).$$

But $(L^{\psi}_a)' = L^{\psi^*}$ (see [6, p.56]), so by [10, Corollary 3.5] we get $L^{\psi^*} \subset E^{\varphi^*}$. Finally, there exists a unique $g \in E^{\varphi^*}(X^*, X)$ such that

$$\overline{F}(f) = \int_{\Omega} \langle f(\omega), g(\omega)\rangle \, d\mu \quad \text{for} \quad f \in L^{\psi}_a(X).$$

Hence

$$F(f) = F_g(f) = \int_{\Omega} \langle f(\omega), g(\omega)\rangle \, d\mu \quad \text{for} \quad f \in L^{\varphi}(X).$$

(iii) $\Rightarrow$ (i) Let $\varphi \in \Phi^c_{1i}$ $(i = 1, 2, 3, 4)$. According to [10, Corollary 3.5] there exists $\psi \in \Psi^{\varphi}_{1i}(N)$ such that $g \in L^{\psi^*}(X^*, X)$. Then $L^{\varphi}(X) \subset E^{\psi}(X) \subset L^{\psi}(X)$. Moreover, by [2, Theorem 1.1] using the Hölder's inequality we get for $f \in L^{\varphi}(X)$

$$\begin{aligned} |F_g(f)| &\le \int_{\Omega} |\langle f(\omega), g(\omega)\rangle| \, d\mu \le \int_{\Omega} \|f(\omega)\|_X \cdot \vartheta(g)(\omega) \, d\mu \\ &\le 2|||\widetilde{f}|||_{\psi} \cdot |||\vartheta(g)|||_{\psi^*} = 2|||f|||_{L^{\psi}(X)} \cdot |||\vartheta(g)|||_{\psi^*}. \end{aligned}$$

This means that $F_g$ is $\mathcal{T}_{\psi}(X)|_{L^{\varphi}(X)}$-continuous, so $F_g$ is $\mathcal{T}^{\varphi}_I(X)$-continuous, because $\mathcal{T}_{\psi}(X)|_{L^{\varphi}(X)} \subset \mathcal{T}^{\varphi}_I(X)$ by Theorem 3.10.

Thus the proof is complete. □

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