Quasi-balanced torsion-free groups

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Abstract. An exact sequence $0 \to A \to B \to C \to 0$ of torsion-free abelian groups is quasi-balanced if the induced sequence

$$0 \to \mathbf{Q} \otimes \mathrm{Hom}(X, A) \to \mathbf{Q} \otimes \mathrm{Hom}(X, B) \to \mathbf{Q} \otimes \mathrm{Hom}(X, C) \to 0$$

is exact for all rank-1 torsion-free abelian groups X. This paper sets forth the basic theory of quasi-balanced sequences, with particular attention given to the case in which C is a Butler group. The special case where B is almost completely decomposable gives rise to a descending chain of classes of Butler groups. This chain is a generalization of the chain of Kravchenko classes that arise from balanced sequences. As an application of our results concerning quasi-balanced sequences, the relationship between the two chains in the quasi-category of torsion-free abelian groups is illuminated.

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1. Preliminaries

Throughout we shall deal exclusively with abelian groups, hereafter referred to simply as “groups ”, and those groups that are torsion-free are assumed to be of finite rank. Unexplained notation and terminology will be as in [A1], [A2], [AV] and [F].

There is an extensive literature on the theory of balanced subgroups of torsion-free groups; most notable, from the point of view of this paper, are [AV], [K], [NV1], and [V]. Recall that an exact sequence of torsion-free groups $0 \to A \to B \to C \to 0$ is *balanced* if the induced sequence

$$0 \to \mathrm{Hom}(X, A) \to \mathrm{Hom}(X, B) \to \mathrm{Hom}(X, C) \to 0$$

is exact for every rank-1 torsion-free group X. Thus, a pure subgroup A of a torsion-free group B is a balanced subgroup if the sequence $0 \to A \to B \to B/A \to 0$ is balanced.

It is our purpose here to investigate a generalized version of “balanced” which we call “quasi-balanced”. Recall that if G is a torsion-free group, $\mathbf{Q} \otimes G$ is abbreviated as QG, where of course all tensor products are assumed to be over $\mathbf{Z}$.

Definition 1.1. An exact sequence $0 \to A \to B \to C \to 0$ of torsion-free groups is *quasi-balanced* if the induced sequence

$$0 \to \mathrm{QHom}(X, A) \to \mathrm{QHom}(X, B) \to \mathrm{QHom}(X, C) \to 0$$

is exact for every torsion-free group X of rank 1.

As expected, a pure subgroup A of a torsion-free group B is called quasi-balanced if the exact sequence $0 \to A \to B \to B/A \to 0$ is quasi-balanced. Observe that all balanced sequences and subgroups are also quasi-balanced.

There is a more concrete, though perhaps less elegant way to interpret the notion of quasi-balanced. Specifically, an exact sequence

$$0 \longrightarrow A \longrightarrow B \xrightarrow{\beta} C \longrightarrow 0$$

of torsion-free groups is quasi-balanced if and only if for every rank-1 torsion-free group X and every $f \in \mathrm{Hom}(X, C)$, there exists $g \in \mathrm{Hom}(X, B)$ such that $\beta g = nf$ for some integer $n \neq 0$. This is merely a restatement of the definition. It will be convenient to have several additional methods for recognizing quasi-balanced sequences. Recall that if τ is a type and G is a torsion-free group, then $G(\tau) = \{x \in G : \text{type } x \geq \tau\}$ denotes the τ-socle of G.

Proposition 1.2. *The following statements are equivalent for an exact sequence*

$$E: \quad 0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

of torsion-free groups.

(a) *E is quasi-balanced.*

(b) $0 \longrightarrow \mathrm{Q}A(\tau) \xrightarrow{1\otimes\alpha} \mathrm{Q}B(\tau) \xrightarrow{1\otimes\beta} \mathrm{Q}C(\tau) \longrightarrow 0$ *is exact for all types τ.*

(c) $\mathrm{rank}\ B(\tau) = \mathrm{rank}\ A(\tau) + \mathrm{rank}\ C(\tau)$ *for all types τ.*

(d) $C(\tau)/\beta(B(\tau))$ *is torsion for all types τ.*

Proof: It is easily seen that (b) is equivalent to (c) and that (c) is equivalent to (d). Therefore, we shall only show that (a) and (b) are equivalent.

To see that (a) implies (b), suppose $0 \neq x \in C(\tau)$ and let $\langle x \rangle_*$ denote the pure rank-1 subgroup of C generated by x. If $\iota : \langle x \rangle_* \to C$ is the inclusion map, the fact that E is quasi-balanced implies there exists $g \in \mathrm{Hom}(\langle x \rangle_*, B)$ such that $\beta(g(x)) = mx$ for some integer $m \neq 0$. Setting $y = g(x)$, we have type $x \leq$ type y so that $y \in B(\tau)$. Since $\beta(B(\tau)) \subseteq C(\tau)$ in general, it follows that $1 \otimes \beta$ maps $\mathrm{Q}B(\tau)$ onto $\mathrm{Q}C(\tau)$.

Conversely, suppose X is rank-1 torsion-free of type τ, and $f \in \mathrm{Hom}(X, C)$. If $0 \neq x \in X$ and (b) holds, there exists $y \in B(\tau)$ such that $\beta(y) = mf(x)$ for some integer $m \neq 0$. Select an integer $k \neq 0$ such that the characteristics $\mathrm{h}_B(ky)$

of ky in B and $\mathrm{h}_X(x)$ of x in X satisfy $\mathrm{h}_B(ky) \geq \mathrm{h}_X(x)$. Therefore, there exists $g \in \mathrm{Hom}(X, B)$ such that $g(x) = ky$ and $\beta g = (km)f$. $\square$

Much of our effort in this paper is focused on quasi-balanced sequences of the form

$$(*) \qquad 0 \longrightarrow K \longrightarrow C \longrightarrow G \longrightarrow 0$$

where C is almost completely decomposable. In this case, K, C and G are Butler groups; that is, pure subgroups (or equivalently, homomorphic images) of finite rank completely decomposable groups (see [B] and [A1]). As motivation for our subsequent work, we show below that there are naturally occurring sequences of the form $(*)$ that are quasi-balanced but not balanced. However, we first require some additional notation and terminology.

If G is a torsion-free group and τ is a type, define $G^*(\tau) = \sum\{G(\sigma) : \sigma > \tau\}$ and write $G^{\#}(\tau)$ for the pure subgroup of G generated by $G^*(\tau)$. If T_G denotes the typeset of G, the critical typeset is given by $T'_G = \{\tau \in T_G : G^{\#}(\tau) \neq G(\tau)\}$.

Now assume that G is a Butler group. Then, as shown in [B], $G(\tau) = G_\tau \oplus G^{\#}(\tau)$ for all types τ, where G_τ is τ-homogeneous completely decomposable. Moreover, $\sum\{G_\tau : \tau \in T'_G,\ \tau \geq \sigma\}$ is of finite index in $G(\sigma)$ for all types σ. By definition, a Butler group is a B_0-group if $G^{\#}(\tau) = G^*(\tau)$ for all τ. As shown in [A1], G is a B_0-group if and only if $G(\sigma) = \sum\{G_\tau : \tau \in T'_G,\ \tau \geq \sigma\}$ for all σ.

Example 1.3. Suppose G is a Butler group which is not a B_0-group (as shown in [A1], any almost completely decomposable group which is not completely decomposable is such a group). For each type τ, write $G(\tau) = G_\tau \oplus G^{\#}(\tau)$. Let $\tau_1, \tau_2, \ldots, \tau_n$ be the distinct elements of T'_G (they are finite in number since a Butler group has a finite typeset) and set $C = G_1 \oplus G_2 \oplus \cdots \oplus G_n$ where $G_i = G_{\tau_i}$ $(1 \leq i \leq n)$. Note that C is completely decomposable. Let $\nabla : C \to G$ be the codiagonal map given by $\nabla(g_1, g_2, \ldots, g_n) = g_1 + g_2 + \cdots + g_n$. Now ∇ may not map onto G; but we can always modify C by forming $C \oplus F$, where F is the external direct sum of a finite number of rank-1 free subgroups of G, so that the extended codiagonal map $\nabla : C \oplus F \to G$ does map onto G. Consider the exact sequence

$$(**) \qquad 0 \longrightarrow \mathrm{Ker}\ \nabla \longrightarrow C \oplus F \xrightarrow{\nabla} G \longrightarrow 0.$$

Since G is not a B_0-group, there exists a type σ such that $\nabla((C \oplus F)(\sigma)) = \nabla(C(\sigma)) = \sum\{G_\tau : \tau \in T'_G,\ \tau \geq \sigma\} \neq G(\sigma)$. Thus, $(**)$ is not balanced. However, it is quasi-balanced by Proposition 1.2. Indeed, for every type σ, $\nabla((C \oplus F)(\sigma)) \supseteq \sum\{G_\tau : \tau \in T'_G,\ \tau \geq \sigma\}$ is of finite index in $G(\sigma)$. $\square$

It may seem that Example 1.3 is somewhat artificial in that $G(\sigma)/\nabla((C \oplus F)(\sigma))$ is finite for all types σ, rather than merely torsion as required by Proposition 1.2. However, as shown below in Lemma 3.2, this must always occur whenever G is a Butler group.

2. Quasi-balanced projectives and Schanuel's Lemma

In this section, we develop the machinery needed for studying the generalized Kravchenko classes as defined in Section 4. Highlights of this section include a version of Schanuel's Lemma and its dual for quasi-balanced exact sequences, and an application to the quasi-isomorphism problem for quasi-balanced subgroups of almost completely decomposable groups.

A torsion-free group G is *quasi-balanced projective* if it is projective in the quasi-category of torsion-free abelian groups; that is, the category whose objects are torsion-free abelian groups with morphisms $\mathrm{QHom}(A, B)$ for all A and B. Thus, G is quasi-balanced projective if and only if for every quasi-balanced exact sequence

$$0 \longrightarrow A \longrightarrow B \xrightarrow{\beta} C \longrightarrow 0$$

and $f \in \mathrm{Hom}(G, C)$, there exists $g \in \mathrm{Hom}(G, B)$ such that $\beta g = nf$ for some integer $n \neq 0$. By definition, every torsion-free group of rank 1 is quasi-balanced projective. In our next result we show that a Butler group is quasi-balanced projective if and only if it is almost completely decomposable. Recall that a torsion-free group G is almost completely decomposable if it contains a completely decomposable subgroup of finite index

Proposition 2.1. *Suppose G is a torsion-free group. If G is almost completely decomposable, then G is quasi-balanced projective. The converse holds if G is a Butler group.*

Proof: Suppose $0 \longrightarrow A \longrightarrow B \xrightarrow{\beta} C \longrightarrow 0$ is quasi-balanced exact and $f \in \mathrm{Hom}(G, C)$. Assuming that G is almost completely decomposable, select a positive integer m such that $mG \subseteq X_1 \oplus \cdots \oplus X_k \subseteq G$, where each X_i $(1 \leq i \leq k)$ is a rank-1 pure subgroup of G. For each i, let $\iota_i : X_i \to G$ be the inclusion map, and let $\pi_i : X_1 \oplus \cdots \oplus X_k \to X_i$ denote the natural projection. Observe $\sum_{i=1}^{k} \iota_i \pi_i(mx) = mx$ for all $x \in G$.

Since $f\iota_i \in \mathrm{Hom}(X_i, C)$, for each i there exists a positive integer n_i and $g_i \in \mathrm{Hom}(X_i, B)$ such that $\beta g_i = n_i f \iota_i$. Set $\ell = \mathrm{LCM}(n_1, \ldots, n_k)$ and select positive integers $m_1, \ldots, m_k$ with $m_i \beta g_i = \ell f \iota_i$. Now define

$$g(x) = \sum_{i=1}^{k} m_i g_i \pi_i(mx)$$

for all $x \in G$. Then,

$$\beta g(x) = \sum_{i=1}^{k} m_i \beta g_i \pi_i(mx) = \sum_{i=1}^{k} \ell f \iota_i \pi_i(mx) = \ell f(\sum_{i=1}^{k} \iota_i \pi_i(mx)) = \ell m f(x).$$

Therefore, $\beta g = (\ell m) f$ and G is quasi-balanced projective.

Conversely, if G is a Butler group, there exists a balanced exact sequence

$$0 \longrightarrow K \longrightarrow C \longrightarrow G \longrightarrow 0$$

with C completely decomposable (see [AV]). If G is quasi-balanced projective, the sequence quasi-splits so that G is quasi-isomorphic to a quasi-summand of C. Therefore, G is almost completely decomposable. □

Of fundamental importance for our further work is the following version of Schanuel's Lemma for quasi-balanced sequences. Here and in the sequel we use $\sim$ to denote quasi-isomorphism of torsion-free groups.

Proposition 2.2. *Suppose*

$$0 \longrightarrow K_i \longrightarrow C_i \xrightarrow{\beta_i} G_i \longrightarrow 0$$

is quasi-balanced exact for $i = 1, 2$. If C_1 and C_2 are almost completely decomposable and if $G_1 \sim G_2$, then $C_1 \oplus K_2 \sim K_1 \oplus C_2$.

Proof: Since $G_1 \sim G_2$, there exist monomorphisms $\gamma \in \mathrm{Hom}(G_1, G_2)$ and $\delta \in \mathrm{Hom}(G_2, G_1)$, and a positive integer k such that $\delta\gamma = k1_{G_1}$ and $\gamma\delta = k1_{G_2}$. Observe that $\gamma\beta_1 \in \mathrm{Hom}(C_1, G_2)$ and $\delta\beta_2 \in \mathrm{Hom}(C_2, G_1)$. Hence, by Proposition 2.1 there exist $f \in \mathrm{Hom}(C_1, C_2)$ and $g \in \mathrm{Hom}(C_2, C_1)$ such that $\beta_2 f = m\gamma\beta_1$ and $\beta_1 g = n\delta\beta_2$ for some positive integers m and n. Set

$$H = \{(y_1, y_2) \in C_1 \oplus C_2 : m\gamma\beta_1(y_1) = n\beta_2(y_2) \text{ and } mk\beta_1(y_1) = n\delta\beta_2(y_2)\}.$$

We intend to show that $C_1 \oplus K_2$ and $K_1 \oplus C_2$ are each quasi-isomorphic to H.

Regarding K_2 as a subgroup of C_2, define $\psi : C_1 \oplus K_2 \to C_1 \oplus C_2$ by $\psi(x_1, x_2) = (nx_1, f(x_1) + x_2)$. Observe that ψ is a monomorphism. Moreover,

$$n\beta_2(f(x_1) + x_2) = n\beta_2 f(x_1) = nm\gamma\beta_1(x_1) = m\gamma\beta_1(nx_1), \text{ and}$$
$$n\delta\beta_2(f(x_1) + x_2) = n\delta\beta_2 f(x_1) = n\delta m\gamma\beta_1(x_1) =$$
$$nm\delta\gamma\beta_1(x_1) = nmk\beta_1(x_1) = mk\beta_1(nx_1)$$

so that $\mathrm{Im}\ \psi \subseteq H$.

We now claim that $nH \subseteq \mathrm{Im}\ \psi$. Indeed, suppose $n(y_1, y_2) \in nH$ with $(y_1, y_2) \in H$. Then $n\beta_2(y_2) = m\gamma\beta_1(y_1)$. Also recall that $\beta_2 f = m\gamma\beta_1$. Thus,

$$\beta_2(ny_2 - f(y_1)) = n\beta_2(y_2) - \beta_2 f(y_1) = m\gamma\beta_1(y_1) - m\gamma\beta_1(y_1) = 0$$

and we conclude $(y_1, ny_2 - f(y_1)) \in C_1 \oplus K_2$. Moreover,

$$\psi(y_1, ny_2 - f(y_1)) = (ny_1, f(y_1) + ny_2 - f(y_1)) = n(y_1, y_2)$$

and $nH \subseteq \mathrm{Im}\ \psi$ as claimed. To summarize, ψ is a monomorphism from $C_1 \oplus K_2$ to $C_1 \oplus C_2$ with $nH \subseteq \mathrm{Im}\ \psi \subseteq H$. Since H/nH is finite, $C_1 \oplus K_2 \sim H$.

To see that $K_1 \oplus C_2 \sim H$, view K_1 as a subgroup of C_1 and define $\varphi : K_1 \oplus C_2 \to C_1 \oplus C_2$ by $\varphi(x_1, x_2) = (g(x_2) + x_1, kmx_2)$. By an argument similar to the above, one sees that φ is a monomorphism with $kmH \subseteq \mathrm{Im}\ \varphi \subseteq H$. Therefore, $K_1 \oplus C_2 \sim H$ as well. $\square$

At this juncture we note that Proposition 2.2 can be applied to obtain quasi-isomorphism results for quasi-balanced subgroups of almost completely decomposables. Recall that if G is a torsion-free group and X is a rank-1 torsion-free of type τ, the τ-radical of G is defined by $G[\tau] = \bigcap\{\mathrm{Ker}\ f : f \in \mathrm{Hom}(G, X)\}$.

Theorem 2.3. *Suppose K_1 and K_2 are quasi-balanced subgroups of almost completely decomposable groups C_1 and C_2, respectively, and $C_1/K_1 \sim C_2/K_2$.*

(a) *If* rank $K_1(\tau) =$ rank $K_2(\tau)$ *for all types τ, then $K_1 \sim K_2$.*
(b) *If* rank $(K_1/K_1[\tau]) =$ rank $(K_2/K_2[\tau])$ *for all types τ, then $K_1 \sim K_2$.*

Proof: By Proposition 2.2, $C_1 \oplus K_2 \sim K_1 \oplus C_2$. Consequently, rank $C_1(\tau)$ + rank $K_2(\tau)$ = rank $(C_1 \oplus K_2)(\tau)$ = rank $(K_1 \oplus C_2)(\tau)$ = rank $K_1(\tau)$ + rank $C_2(\tau)$ for all types τ. Therefore, if rank $K_1(\tau)$ = rank $K_2(\tau)$ for all τ, then rank $C_1(\tau)$ = rank $C_2(\tau)$ for all τ. Since C_1 and C_2 are almost completely decomposable, we conclude that $C_1 \sim C_2$. Hence $K_1 \sim K_2$ and (a) is established.

As for (b), observe that

$$C_1[\tau] \oplus K_2[\tau] = (C_1 \oplus K_2)[\tau] \sim (K_1 \oplus C_2)[\tau] = K_1[\tau] \oplus C_2[\tau].$$

Hence,

$$C_1/C_1[\tau] \oplus K_2/K_2[\tau] \sim K_1/K_1[\tau] \oplus C_2/C_2[\tau].$$

If rank $(K_1/K_1[\tau])$ = rank $(K_2/K_2[\tau])$ for all types τ, then

$$\mathrm{rank}\ (C_1/C_1[\tau]) = \mathrm{rank}\ (C_2/C_2[\tau])$$

for all τ. Since C_1 and C_2 are almost completely decomposable, it now follows that $C_1 \sim C_2$. Therefore, as above, $K_1 \sim K_2$. $\square$

Corollary 2.4. *Suppose K_1 and K_2 are quasi-balanced subgroups of an almost completely decomposable group C. If $C/K_1 \sim C/K_2$, then $K_1 \sim K_2$.*

Proof: From Proposition 1.2 we have rank $K_1(\tau)$ = rank $C(\tau)$ − rank $(C/K_1)(\tau)$ = rank $C(\tau)$ − rank $(C/K_2)(\tau)$ = rank $K_2(\tau)$ for all types τ. Therefore, $K_1 \sim K_2$ by Theorem 2.3(a). $\square$

In our study of quasi-balanced exactness, we shall also find it necessary to deal with the dual setting.

Definition 2.5. An exact sequence

$$0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow C \longrightarrow 0$$

of torsion-free abelian groups is called *quasi-cobalanced* if for every rank-1 torsion-free abelian group X and $f \in \mathrm{Hom}(A, X)$, there exists $g \in \mathrm{Hom}(B, X)$ such that $g\alpha = nf$ for some integer $n \neq 0$.

If the integer n in Definition 2.5 can be taken to be 1 for all X and f, the sequence is called *cobalanced*. Of course every cobalanced sequence is quasi-cobalanced. By dualizing the definition of quasi-balanced projective, we obtain the notion of quasi-cobalanced injective and the following dual of Proposition 2.1.

Proposition 2.6. *An almost completely decomposable group is quasi-cobalanced injective. Conversely, a quasi-cobalanced injective Butler group is almost completely decomposable.*

The next result is the dual of Proposition 2.2.

Proposition 2.7. *Suppose*

$$0 \longrightarrow K_i \xrightarrow{\alpha_i} C_i \xrightarrow{\beta_i} G_i \longrightarrow 0$$

is quasi-cobalanced exact for $i = 1, 2$. If C_1 and C_2 are almost completely decomposable and if $K_1 \sim K_2$, then $C_1 \oplus G_2 \sim G_1 \oplus C_2$.

Proof: Since $K_1 \sim K_2$, there exist monomorphisms $\gamma : K_1 \to K_2$ and $\delta : K_2 \to K_1$ such that $\delta\gamma = k1_{K_1}$ and $\gamma\delta = k1_{K_2}$ for some integer $k \neq 0$. Observe that $\alpha_2\gamma \in \mathrm{Hom}(K_1, C_2)$ and $\alpha_1\delta \in \mathrm{Hom}(K_2, C_1)$. By Proposition 2.6, C_1 and C_2 are quasi-cobalanced injectives. Thus, there exist $f \in \mathrm{Hom}(C_2, C_1)$ and $g \in \mathrm{Hom}(C_1, C_2)$ together with nonzero integers m and n such that $f\alpha_2 = m\alpha_1\delta$ and $g\alpha_1 = n\alpha_2\gamma$. Now define

$$\begin{aligned} H = \{(y_1, y_2) \in C_1 \oplus C_2 : y_1 = n\alpha_1(a_1) + f\alpha_2(a_2) \ \text{ and} \\ y_2 = -g\alpha_1(a_1) - mk\alpha_2(a_2) \ \text{ for some } \ a_1 \in K_1, a_2 \in K_2\} \end{aligned}$$

and let H_* denote the purification of H in $C_1 \oplus C_2$.

Define maps $\psi : (C_1 \oplus C_2)/H_* \to C_1 \oplus G_2$ and $\varphi : (C_1 \oplus C_2)/H_* \to G_1 \oplus C_2$ by $\psi((y_1, y_2) + H_*) = (kmy_1 + f(y_2), \beta_2(y_2))$ and $\varphi((y_1, y_2) + H_*) = (\beta_1(y_1), ny_2 + g(y_1))$. It is easily verified that ψ and φ are well-defined monomorphisms. Moreover, $km(C_1 \oplus G_2) \subseteq \mathrm{Im}\ \psi \subseteq C_1 \oplus G_2$ and $n(G_1 \oplus C_2) \subseteq \mathrm{Im}\ \varphi \subseteq G_1 \oplus C_2$. It now follows that $C_1 \oplus G_2 \sim (C_1 \oplus C_2)/H_* \sim G_1 \oplus C_2$. □

To conclude this section, we record the following technical result for later use.

Corollary 2.8. *Suppose*

$$E_i: \quad 0 \longrightarrow K_i \longrightarrow C_i \longrightarrow G_i \longrightarrow 0$$

is quasi-cobalanced exact for $i = 1, 2$ and E_1 is quasi-balanced. If C_1 and C_2 are almost completely decomposable and $K_1 \sim K_2$, then E_2 is also quasi-balanced.

Proof: By Proposition 1.2 and the fact that E_1 is quasi-balanced, rank $C_1(\tau) =$ rank $K_1(\tau)+$rank $G_1(\tau)$ for all types τ. Moreover, since $K_1 \sim K_2$, rank $K_1(\tau) =$ rank $K_2(\tau)$ for all τ. By Proposition 2.7 we have $C_1 \oplus G_2 \sim G_1 \oplus C_2$ so that

$$\text{rank } C_1(\tau) + \text{rank } G_2(\tau) = \text{rank } G_1(\tau) + \text{rank } C_2(\tau)$$

for all τ. Therefore,

$$\begin{gathered}\text{rank } C_2(\tau) = \text{rank } C_1(\tau) + \text{rank } G_2(\tau) - \text{rank } G_1(\tau) = \\ \text{rank } K_1(\tau) + \text{rank } G_2(\tau) = \text{rank } K_2(\tau) + \text{rank } G_2(\tau)\end{gathered}$$

and we conclude that E_2 is quasi-balanced by Proposition 1.2. □

3. Almost balanced Butler groups

Closely related to quasi-balanced exactness is the following uniform version.

Definition 3.1. An exact sequence

$$E: \quad 0 \longrightarrow A \longrightarrow B \xrightarrow{\beta} C \longrightarrow 0$$

of torsion-free groups is *almost balanced* if there exists an integer $n \neq 0$ with the following property: For every rank-1 torsion-free X and $f \in \text{Hom}(X, B)$, there exists $g \in \text{Hom}(X, B)$ with $\beta g = nf$.

Observe that the sequence E of Definition 3.1 is almost balanced if and only if there exists an integer $n \neq 0$ such that the cokernel of the induced map β^* : $\text{Hom}(X, B) \rightarrow \text{Hom}(X, C)$ is bounded by n for all rank-1 torsion-free X.

Clearly every balanced exact sequence is almost balanced and every almost balanced sequence is quasi-balanced. In this section we show that almost balanced and quasi-balanced coincide for Butler groups, and we conclude with an example to demonstrate that, in general, the two notions are distinct. We first require a lemma which should clarify the remark made after Example 1.3.

Lemma 3.2. *Suppose*

$$E: \quad 0 \longrightarrow A \longrightarrow B \xrightarrow{\beta} C \longrightarrow 0$$

is an exact sequence of torsion-free groups. If C is a Butler group, then E is quasi-balanced if and only if $C(\tau)/\beta(B(\tau))$ is finite for all types τ.

PROOF: If $C(\tau)/\beta(B(\tau))$ is finite for all τ, then E is quasi-balanced by Proposition 1.2. Conversely, suppose E is quasi-balanced and τ is a type such that $C(\tau) \neq 0$. Since $C(\tau)$ is pure in C, $C(\tau)$ is also a Butler group. Consequently, there exist rank-1 pure subgroups $C_1, C_2, \ldots, C_k$ of $C(\tau)$ such that $C(\tau) = C_1 + C_2 + \cdots + C_k$. Observe that type $C_i \geq \tau$ for $1 \leq i \leq k$.

If $\nabla : \bigoplus C_i \to C(\tau)$ is the map give by $\nabla(c_1, c_2, \ldots, c_k) = \sum c_i \in C(\tau) \subseteq C$, Proposition 2.1 and E quasi-balanced imply that there exists $g \in \mathrm{Hom}(\bigoplus C_i, B)$ with $\beta g = n\nabla$ for some integer $n \neq 0$. Note Im $g \subseteq B(\tau)$. Thus, if $c = c_1 + c_2 + \cdots + c_k \in C(\tau)$ with $c_i \in C_i$ for all i, then

$$nc = n\nabla(c_1, c_2, \ldots, c_k) = \beta g(c_1, c_2, \ldots, c_k) \in \beta(B(\tau))$$

and so $nC(\tau) \subseteq \beta(B(\tau)) \subseteq C(\tau)$. Since $C(\tau)/nC(\tau)$ is finite, the result follows. □

Theorem 3.3. *Suppose the exact sequence*

$$E: \quad 0 \longrightarrow A \longrightarrow B \xrightarrow{\beta} C \longrightarrow 0$$

of torsion-free groups is quasi-balanced. If C is a Butler group, then E is almost balanced.

PROOF: We claim that the induced map

$$\beta^* : \mathrm{Hom}(X, B) \longrightarrow \mathrm{Hom}(X, C)$$

has bounded (and hence finite) cokernel for every rank-1 torsion-free X. Indeed, suppose X is rank-1 torsion-free of type τ and consider the commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}(X.A) & \longrightarrow & \mathrm{Hom}(X,B) & \xrightarrow{\beta^*} & \mathrm{Hom}(X,C) \\
& & \downarrow{\scriptstyle \iota_A} & & \downarrow{\scriptstyle \iota_B} & & \downarrow{\scriptstyle \iota_C} \\
0 & \longrightarrow & \mathrm{Hom}(X,A)\otimes X & \longrightarrow & \mathrm{Hom}(X,B)\otimes X & \xrightarrow{\beta^*\otimes 1} & \mathrm{Hom}(X,C)\otimes X \\
& & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\
0 & \longrightarrow & A(\tau) & \longrightarrow & B(\tau) & \xrightarrow{\beta} & C(\tau)
\end{array}$$

with exact rows, where ι_A, ι_B, ι_C are the natural embeddings, and the vertical isomorphisms are the respective evaluation maps. Set $T = \mathrm{Coker}\ \beta^*$ and observe T is a torsion group since E is quasi-balanced. Furthermore, $T_p = 0$ if $pX = X$, from which it follows that $T \cong T \otimes X$. From the diagram, we obtain

$$T \cong (\mathrm{Hom}(X,C) \otimes X)/(\mathrm{Im}\ (\beta^*) \otimes X) \cong$$
$$(\mathrm{Hom}(X,C) \otimes X)/(\mathrm{Im}\ (\beta^* \otimes 1)) \cong C(\tau)/\beta(B(\tau)).$$

By Lemma 3.2, for each τ in the typeset of C, there is an integer n_τ such that $C(\tau)/\beta B(\tau)$ is bounded by n_τ. Because C has a finite typeset, $n = \Pi_\tau n_\tau$ is well-defined and bounds $C(\tau)/\beta B(\tau)$ for all types τ. From the observation above, T is bounded by n as well, and therefore E is almost-balanced. $\square$

To conclude this section, we now present the promised example to show the existence of quasi-balanced sequences which are not almost balanced. If n is a nonnegative integer or the symbol ∞, we write $\bar{\tau}(n)$ for the type containing the characteristic $(n, n, n, \dots)$.

Example 3.4. Let H be a rank-2 torsion-free group which is homogeneous of type $\bar{\tau}(0)$ and cohomogeneous of cotype $\bar{\tau}(\infty)$. With Y a rank-1 torsion-free group of type $\bar{\tau}(1)$, regard H as a subgroup of $G = H \otimes Y$. Let F be a full free subgroup of H with $H/F \cong \oplus_p \mathbf{Z}(p^\infty)$. Then $G/F \cong H/F \oplus T$, with $T = \oplus_p \mathbf{Z}(p)$. Write $F = \langle x_1 \rangle \oplus \langle x_2 \rangle$ and note that each $\langle x_i \rangle$ is a pure subgroup of H. Set $C = X_1 \oplus X_2$ where X_i is the purification of $\langle x_i \rangle$ in G. Then $C \cap H = F$ so that the projection of G/F onto T still maps onto T when restricted to C/F. Hence, $G = H + C$.

Let $\beta : H \oplus C \to G$ be given by $\beta(h, c) = h + c$. So, we have an exact sequence

$$E: \quad 0 \longrightarrow \text{Ker } \beta \longrightarrow H \oplus C \xrightarrow{\beta} G \longrightarrow 0.$$

Suppose X is a rank-1 torsion-free group with type $X > \bar{\tau}(0)$. If type $X \not\leq \bar{\tau}(1)$ then $\text{Hom}(X, G) = 0$. Therefore to show that E is quasi-balanced, it is enough to show that the cokernel of $\beta^* : \text{Hom}(X, H \oplus C) \to \text{Hom}(X, G)$ is torsion whenever type $X = \bar{\tau}(1)$. In this case, observe that $\text{Hom}(X, G) \cong H$ and $\text{Hom}(X, H \oplus C) = \text{Hom}(X, C) \cong F$. Hence, Coker $\beta^* \cong H/F$ is an unbounded torsion group. Therefore, E is quasi-balanced but not almost balanced. $\square$

4. Generalized Kravchenko classes

In this final section, we study classes of Butler groups defined in terms of almost balanced sequences. These classes will be seen to generalize the Kravchenko classes originally introduced in [K]. For each integer $n \geq 0$, the n-th *Kravchenko class* $\mathcal{K}(n)$ is defined inductively as follows: $\mathcal{K}(0)$ denotes the class of all Butler groups, and for $n \geq 1$, $\mathcal{K}(n)$ is defined to be the class of all torsion-free groups K that appear as the first term in a balanced exact sequence $0 \to K \to C \to G \to 0$, where $G \in \mathcal{K}(n-1)$ and C is completely decomposable. Clearly $\mathcal{K}(n) \subseteq \mathcal{K}(n-1)$ for all $n \geq 1$ and each $\mathcal{K}(n)$ consists entirely of Butler groups. Our reference for the theory of Kravchenko classes is [NV1].

As mentioned above, we generalize and expand the classes $\mathcal{K}(n)$ by considering almost balanced sequences as opposed to balanced ones. For each torsion-free G, define $\mathcal{C}_G$ to be the class of all torsion-free groups K that appear as the first term in an almost balanced sequence $0 \to K \to C \to G \to 0$ with C almost completely decomposable. (It may be of interest to observe that the groups $K_1, K_2 \in \mathcal{C}_G$ are quasi-isomorphic if and only if rank $K_1(\tau) =$ rank $K_2(\tau)$ for all types τ. This is an immediate consequence of Theorem 2.3(a).) As above, we take $\overline{\mathcal{K}}(0) = \mathcal{K}(0)$ to be the class of all Butler groups. If $n \geq 1$, we define the n-th *generalized Kravchenko class* $\overline{\mathcal{K}}(n)$ by

$$\overline{\mathcal{K}}(n) = \bigcup\{\mathcal{C}_G : G \in \overline{\mathcal{K}}(n-1)\}.$$

For each n, it is clear that $\mathcal{K}(n) \subseteq \overline{\mathcal{K}}(n)$ and that each $\overline{\mathcal{K}}(n)$ is a class of Butler groups. Moreover, if $n \geq 1$, $\overline{\mathcal{K}}(n) \subseteq \overline{\mathcal{K}}(n-1)$.

It is well known that the category of Butler groups with quasi-homomorphisms is categorically equivalent to the category of representations of finite distributive lattices. Under this equivalence, a quasi-balanced sequence of Butler groups corresponds to a balanced sequence of representations. This observation will be exploited in an up-coming paper [NV2], but will not be the focus of our paper.

In [NV1] it is shown that the class $\mathcal{K}(n)$ is not closed under quasi-isomorphism whenever $n \geq 1$. However, for the classes $\overline{\mathcal{K}}(n)$ we have the following.

Proposition 4.1. *For each $n \geq 0$, $\overline{\mathcal{K}}(n)$ is closed under quasi-isomorphism.*

Proof: It is well known that the class of Butler groups is closed under quasi-isomorphism (for example, see [A1]). So, we may assume $n \geq 1$. Suppose H is a torsion-free group with $H \sim K$ for some $K \in \overline{\mathcal{K}}(n)$. Without loss we may assume $H \subseteq K$ and K/H is finite. By definition, there exists an almost balanced exact sequence

$$0 \longrightarrow K \longrightarrow C \longrightarrow G \longrightarrow 0$$

with C almost completely decomposable and $G \in \overline{\mathcal{K}}(n-1)$. Regarding K and H as subgroups of C, C/H has finite torsion subgroup K/H. Thus, there exists a subgroup D of C such that $H \subseteq D$ and $C/H = (K/H) \oplus (D/H)$, with D/H torsion-free. From this we conclude that C/D is finite so that $D \sim C$ and D is almost completely decomposable. Moreover, $D/H \sim C/K \cong G \in \overline{\mathcal{K}}(n-1)$. By induction on n, $D/H \in \overline{\mathcal{K}}(n-1)$. Finally, by utilizing Proposition 1.2, the exact sequence

$$0 \longrightarrow H \longrightarrow D \longrightarrow D/H \longrightarrow 0$$

is easily seen to be quasi-balanced, and hence is almost balanced by Theorem 3.3. Therefore, $H \in \overline{\mathcal{K}}(n)$. □

It follows from results in [NV1] that the classes $\mathcal{K}(n)$ are closed under the formation of direct sums and summands, but are not closed under the formation of quasi-summands whenever $n \geq 1$. In contrast, our next result demonstrates that the class $\overline{\mathcal{K}}(n)$ is closed under the formation of quasi-summands for all n. Moreover, $\overline{\mathcal{K}}(n)$ is precisely the class of quasi-summands of groups appearing in $\mathcal{K}(n)$.

Theorem 4.2. *Suppose K is a torsion-free group and n is a nonnegative integer. Then $K \in \overline{\mathcal{K}}(n)$ if and only if K is a quasi-summand of some $H \in \mathcal{K}(n)$. In particular, the class $\overline{\mathcal{K}}(n)$ is closed under the formation of quasi-summands.*

Proof: Suppose that $K \in \overline{\mathcal{K}}(n)$. By induction on n, we show that K is a quasi-summand of a group in the class $\mathcal{K}(n)$. Since $\overline{\mathcal{K}}(0) = \mathcal{K}(0)$, we may assume $n \geq 1$. Select an almost balanced sequence

$$E: \quad 0 \longrightarrow K \longrightarrow C \longrightarrow G \longrightarrow 0$$

with C almost completely decomposable and $G \in \overline{\mathcal{K}}(n-1)$. By induction, there exists $G_0 \in \mathcal{K}(n-1)$ such that $G_0 \sim G \oplus G_1$, for some torsion-free group G_1. Since G_0 and G_1 are Butler groups, there exist balanced exact sequences

$$E_0: \quad 0 \longrightarrow K_0 \longrightarrow C_0 \longrightarrow G_0 \longrightarrow 0$$
$$E_1: \quad 0 \longrightarrow K_1 \longrightarrow C_1 \longrightarrow G_1 \longrightarrow 0$$

with C_0 and C_1 completely decomposable (see [AV]). Thus, there is an almost balanced sequence

$$E \oplus E_1: \quad 0 \longrightarrow K \oplus K_1 \longrightarrow C \oplus C_1 \longrightarrow G \oplus G_1 \longrightarrow 0$$

with $C \oplus C_1$ almost completely decomposable. From Proposition 2.2 applied to the sequences E_0 and $E \oplus E_1$, we obtain

$$K \oplus K_1 \oplus C_0 \sim K_0 \oplus C \oplus C_1$$

Therefore, K is a quasi-summand of $K_0 \oplus C \oplus C_1$. Noting that $K_0 \in \mathcal{K}(n)$, we find that $H = K_0 \oplus C \oplus C_1$ belongs to $\mathcal{K}(n)$ since C and C_1 are completely decomposable.

Conversely, suppose $K_0 \in \mathcal{K}(n)$ and $K_0 \sim K_1 \oplus K_2$. We show that $K_1 \in \overline{\mathcal{K}}(n)$ by induction on n. The result is clear if $n = 0$, so assume $n \geq 1$. Since K_0, K_1 and K_2 are all Butler groups, there exist cobalanced exact sequences

$$E_0: \quad 0 \longrightarrow K_0 \longrightarrow C_0 \longrightarrow G_0 \longrightarrow 0$$
$$E_1: \quad 0 \longrightarrow K_1 \longrightarrow C_1 \xrightarrow{\beta_1} G_1 \longrightarrow 0$$
$$E_2: \quad 0 \longrightarrow K_2 \longrightarrow C_2 \xrightarrow{\beta_2} G_2 \longrightarrow 0$$

with C_0, C_1 and C_2 completely decomposable (see [AV]). Moreover, since $K_0 \in \mathcal{K}(n)$, it follows from Corollary 1.9 of [NV1] that E_0 is balanced and $G_0 \in \mathcal{K}(n-1)$. Also note that

$$E_1 \oplus E_2: \quad 0 \longrightarrow K_1 \oplus K_2 \longrightarrow C_1 \oplus C_2 \xrightarrow{\beta_1 \oplus \beta_2} G_1 \oplus G_2 \longrightarrow 0$$

is cobalanced. Hence, by Proposition 2.7, $G_1 \oplus G_2$ is a quasi-summand $C_1 \oplus C_2 \oplus G_0 \in \mathcal{K}(n-1)$. By induction, both G_1 and G_2 are in $\overline{\mathcal{K}}(n-1)$. Also, Corollary 2.8 implies that $E_1 \oplus E_2$ is quasi-balanced. Proposition 1.2 now shows that

$$(G_1 \oplus G_2)(\tau)/(\beta_1 \oplus \beta_2)((C_1 \oplus C_2)(\tau)) \cong G_1(\tau)/\beta_1(C_1(\tau)) \oplus G_2(\tau)/\beta_2(C_2(\tau))$$

is torsion for all types τ. Consequently, $G_1(\tau)/\beta_1(C_1(\tau))$ is torsion for all τ so that E_1 is quasi-balanced by Proposition 1.2. Therefore, E_1 is almost balanced by Theorem 3.3 so that $K_1 \in \overline{\mathcal{K}}(n)$. □

We can reformulate Theorem 4.2 as

Corollary 4.3. *For each integer $n \geq 0$, $\overline{\mathcal{K}}(n)$ is the smallest class of Butler groups that contains $\mathcal{K}(n)$ and is closed under the formation of quasi-summands.*

Since $\mathcal{K}(n) \supseteq \mathcal{K}(n+1)$, it follows from Theorem 4.2 that $\overline{\mathcal{K}}(n) \supseteq \overline{\mathcal{K}}(n+1)$. The main result from [V] (Theorem 1) can successfully be modified to characterize the classes $\overline{\mathcal{K}}(n)$; replace pure, equal, and balanced with almost-pure, almost-equal, and almost-balanced. As this process is laborious, and at any rate appears in [NV2], we will not amplify our remarks. It follows from this observation that Example 2 of [V] provides a group G which belongs to $\overline{\mathcal{K}}(n)$ but not $\overline{\mathcal{K}}(n+1)$. Furthermore, the intersection of the classes $\overline{\mathcal{K}}(n)$ is the class of almost completely decomposable groups. Thus, as with the classes $\mathcal{K}(n)$, we have a properly decreasing sequence of classes with a tractable limit.

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